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Estimates for the norms of products of sines and cosines

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ABSTRACT

In this paper we prove asymptotic formulas for the L^p norms of $P_n(\theta) = \prod_{k=1}^n (1 - e^{ik\theta})$ and $Q_n(\theta) = \prod_{k=1}^n (1 + e^{ik\theta})$. These products can be expressed using $\prod_{k=1}^n \sin\left(\frac{k\theta}{2}\right)$ and $\prod_{k=1}^n \cos\left(\frac{k\theta}{2}\right)$ respectively. We prove an estimate for P_n at a point near where its maximum occurs. Finally, we give an asymptotic formula for the maximum of the Fourier coefficients of Q_n.

1. Introduction

Euler's pentagonal number theorem is the expansion

$$\prod_{k=1}^{\infty} (1 - z^k) = \sum_{k=-\infty}^{\infty} (-1)^k z^{k(3k-1)/2},$$

for $|z| < 1$. Euler's discovery and proof of it are explained in detail in [1]. The coefficients in the power series expansion of $\prod_{k=1}^{\infty}(1 - z^k)$ have a combinatorial interpretation that can be used to prove the pentagonal number theorem [10, pp. 286–287, Section 19.11]. One can see that

$$\prod_{k=1}^{\infty} (1 + z^k) = \sum_{k=0}^{\infty} q(k) z^k,$$

where $q(k)$ is the number of ways to write k as a sum of distinct positive integers.

In this paper we are concerned with the behavior on the unit circle of the partial products of the above infinite products. (The distribution of the zeros of the partial sums of the above infinite series is studied in [4].) Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We define $P_n : \mathbb{T} \to \mathbb{C}$ by

$$P_n(\theta) = \prod_{k=1}^{n} (1 - e^{ik\theta}),$$

and we define $Q_n : \mathbb{T} \to \mathbb{C}$ by

$$Q_n(\theta) = \prod_{k=1}^{n} (1 + e^{ik\theta}).$$

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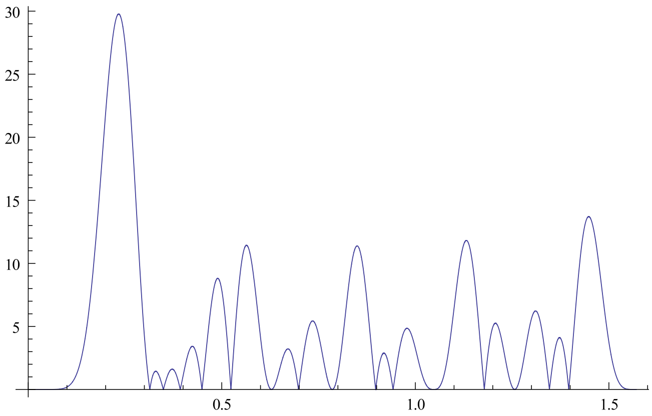

Fig. 1. $\prod_{k=1}^{10} 2|\sin(k\theta)|$ for $0 \le \theta \le \frac{\pi}{2}$.

One can check that

$$P_n(\theta) = (-2i)^n e^{\frac{iN\theta}{2}} \prod_{k=1}^{n} \sin\left(\frac{k\theta}{2}\right), \quad N = \frac{n(n+1)}{2}, \tag{1}$$

and that

$$Q_n(\theta) = 2^n e^{\frac{iN\theta}{2}} \prod_{k=1}^{n} \cos\left(\frac{k\theta}{2}\right), \quad N = \frac{n(n+1)}{2}. \tag{2}$$

For $f : \mathbb{T} \to \mathbb{C}$, we define the Fourier coefficients of f by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta.$$

For $1 \le p < \infty$, we define the L^p norm of f by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^p d\theta \right)^{1/p},$$

and we define the ℓ^p norm of $\hat{f}$ by

$$\|\hat{f}\|_p = \left(\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^p \right)^{1/p}.$$

We deal with P_n in Section 2 and we deal with Q_n in Section 3. We give combinatorial interpretations of their Fourier coefficients, prove asymptotic formulas for their L^p norms, present some other approaches for bounding their norms, and give an asymptotic formula for the ℓ^∞ norm of the Fourier coefficients of Q_n. We also prove an estimate for P_n at a point near where its maximum occurs. In Section 4 we discuss what remains to be shown about these products.

2. Norms of the trigonometric polynomials P_n

The Fourier coefficients of P_n have a combinatorial interpretation. One can see that

$$\widehat{P_n}(k) = e_{n,k} - o_{n,k},$$

where $e_{n,k}$ is the number of ways in which k can be written as a sum of an even number of positive integers that are distinct and each $\le n$, and $o_{n,k}$ is the number of ways in which k can be written as a sum of an odd number of positive integers that are distinct and each $\le n$. For example, one can check that $6+5+2+1$, $6+4+3+1$, $5+4+3+2$ are the only ways to write 14 as a sum of an even number of positive integers that are distinct and each ≤ 6, so $e_{6,14} = 3$, and that $6+5+3$ is the only way to write 14 as a sum of an odd number of positive integers that are distinct and each ≤ 6, so $o_{6,14} = 1$. Thus $\widehat{P_6}(14) = 2$.

We see from (1) that $|P_n(\theta)| = \prod_{k=1}^{n} 2|\sin(\frac{k\theta}{2})|$. In Fig. 1 we plot $\prod_{k=1}^{10} 2|\sin(k\theta)|$ for $0 \le \theta \le \frac{\pi}{2}$.

Of course, $P_n(0) = 0$. Aside from $\theta = 0$ we can explicitly evaluate $P_n(\theta)$ for certain other θ. If $\gcd(n+1, h) = 1$, then $z^{n+1} - 1 = \prod_{k=1}^{n+1}(z - e^{\frac{2\pi ihk}{n+1}})$. Since $z^{n+1} - 1 = (z-1)(z^n + \cdots + z + 1)$ we get $z^n + \cdots + z + 1 = \prod_{k=1}^{n}(z - e^{\frac{2\pi ihk}{n+1}})$ and setting $z = 1$ gives

$$P_n\left(\frac{2\pi h}{n+1}\right) = n+1$$

for each h such that $\gcd(n+1, h) = 1$. In particular this gives us $\|P_n\|_\infty \geq n+1$.

Wright [22], using work of Sudler [18], proves the following theorem giving an asymptotic formula for $\|\widehat{P_n}\|_\infty$.

Theorem 1 (*Wright*)**.** *We have*

$$\|\widehat{P_n}\|_\infty \sim \frac{Be^{Kn}}{n},$$

where B and K are defined by

$$K = \log 2 + \max_{0<w<1}\left(\frac{1}{w}\int_0^w \log\sin(\pi t)dt\right)$$

and

$$B = 2e^K\left(1 - \frac{1}{4}e^{2K}\right)^{-1/4}.$$

The constant K in Theorem 1 is defined using the integral $\int_0^w \log\sin(\pi t)dt$, and in the proof of Theorem 3 we deal with $\int_0^{\frac{3\pi}{4n}} \log\sin x dx$. Milnor in the appendix to [15] shows how to use the integrals $-\int_0^\theta \log|2\sin u|du$ to compute hyperbolic volumes.

Using the fact that

$$\|\widehat{P_n}\|_\infty \leq \|P_n\|_1 \leq \|P_n\|_\infty \leq \|\widehat{P_n}\|_1 \leq (N+1)\|\widehat{P_n}\|_\infty,$$

we obtain from Theorem 1 that $\lim_{n\to\infty}\|P_n\|_\infty^{1/n} = e^K$. Freiman and Halberstam [8] give a different proof of this.

In fact, the method of Wright's proof [22] can be used to estimate the L^p norms of P_n for $1 \leq p \leq \infty$, and we do this in the following.

Theorem 2. *Let* $g(w) = \log 2 + \frac{1}{w}\int_0^w \log\sin(\pi t)dt$, *let* w_0 *be the (unique)* $w \in (0,1)$ *at which the maximum of* g *occurs, let* $K = g(w_0)$, *let* $B = 2e^K\left(1 - \frac{1}{4}e^{2K}\right)^{-1/4}$, *and let* $C = \sqrt{-\frac{1}{2}g''(w_0)}$. *For each* $1 \leq p < \infty$ *we have*

$$\|P_n\|_p \sim C_1 C_2^{\frac{1}{p}} n^{\frac{1}{2}} n^{-\frac{3}{2p}} \exp(nK),$$

where $C_1 = \frac{BC}{2\sqrt{\pi}}$ *and* $C_2 = \frac{2\sqrt{\pi}}{C\sqrt{p}}$, *and for* $p = \infty$ *we have*

$$\|P_n\|_\infty \sim C_1 n^{\frac{1}{2}} \exp(nK).$$

Proof. For $\theta \in [0, \frac{1}{2}]$, we define $\Pi_n(\theta) = \prod_{k=1}^n 2|\sin(\pi k\theta)|$. Let $\gamma = n^{-\frac{17}{12}}$ (the exponent is the arithmetic mean of $-\frac{4}{3}$ and $-\frac{3}{2}$; we will just use that it is strictly between these two numbers), and let $J = [\frac{w_0}{n} - \gamma, \frac{w_0}{n} + \gamma]$. We first estimate $\int_{[0,\frac{1}{2}]\setminus J} \Pi_n(\theta)^p d\theta$ and then estimate $\int_J \Pi_n(\theta)^p d\theta$.

We shall separately estimate $\Pi_n(\theta)$ for $\theta \in [0, \frac{1}{n+1}] \setminus J$ and for $\theta \in [\frac{1}{n+1}, \frac{1}{2}]$. Since Π_n is a product of functions that are each increasing on $[0, \frac{1}{2n}]$, an upper bound for $\Pi_n(\theta)$ on $[\frac{1}{2n}, \frac{1}{n+1}] \setminus J$ is an upper bound for $\Pi_n(\theta)$ on $[0, \frac{1}{n+1}] \setminus J$. Let $\theta \in [\frac{1}{2n}, \frac{1}{n+1}]$. We have

$$\log \Pi_n(\theta) = \log\prod_{k=1}^n 2\sin(\pi k\theta) = n\log 2 + \sum_{k=1}^n G(k),$$

where $G(y) = \log\sin(\pi\theta y)$. The Euler–Maclaurin summation formula [7, p. 99, Eq. (3)] gives us

$$\sum_{k=1}^n G(k) = \int_0^n G(y)dy - \int_0^1 G(y)dy + \frac{1}{2}G(n) + \frac{1}{2}G(1) + R_n,$$

where $|R_n| \leq \frac{1}{2}\int_1^n |G'(y)|dy$.

If $0 \le y \le 1$ then, as $\frac{1}{2n} \le \theta \le \frac{1}{n+1}$,

$$G(y) = \log(\pi\theta y) + \log\frac{\sin(\pi\theta y)}{\pi\theta y} = \log(\pi\theta y) + \log(1 + O(\theta^2 y^2)) = \log(\pi\theta y) + O(\theta^2),$$

and it follows that

$$\int_0^1 G(y)dy = \log(\pi\theta) - 1 + O(\theta^2) = O(\log n).$$

Because $\frac{1}{2n} \le \theta \le \frac{1}{n+1}$, we get

$$G(n) = O(\log n) \quad \text{and} \quad G(1) = O(\log n).$$

Set $y_0 = \frac{1}{2\theta}$. If $1 \le y \le y_0$ then $0 < \pi\theta y \le \frac{\pi}{2}$ and hence $G'(y) \ge 0$, and if $y_0 \le y \le n$ then $\frac{\pi}{2} \le y < \pi$ and hence $G'(y) \le 0$. Thus, as $G(y_0) = 0$,

$$\begin{aligned}\int_1^n |G'(y)|dy &= \int_1^{y_0} G'(y)dy - \int_{y_0}^n G'(y)dy \\ &= -G(n) - G(1) \\ &= O(\log n).\end{aligned}$$

Therefore for $\theta \in [\frac{1}{2n}, \frac{1}{n+1}]$ we have

$$\begin{aligned}\log \Pi_n(\theta) &= n\log 2 + \frac{1}{\theta}\int_0^{n\theta} \log\sin(\pi y)dy + O(\log n) \\ &= ng(n\theta) + O(\log n).\end{aligned}$$

Let $n \ge 4$. Then $w_0 \in (\frac{1}{2}, \frac{n}{n+1})$, and by Taylor's theorem there is some $\xi \in (\frac{1}{2}, \frac{n}{n+1})$ such that if $\frac{1}{2} \le w \le \frac{n}{n+1}$ then

$$g(w) = g(w_0) + g'(w_0)(w - w_0) + \frac{g''(\xi)}{2}(w - w_0)^2 = K + \frac{g''(\xi)}{2}(w - w_0)^2.$$

One can show that if $\frac{1}{2} \le w < 1$ then $g''(w) \le -4$ (we just use that there is some $A < 0$ such that if $\frac{1}{2} \le w < 1$ then $g''(w) \le A$). Hence if $\frac{1}{2n} \le \theta \le \frac{1}{n+1}$ then

$$g(n\theta) \le K - 2(n\theta - w_0)^2 = K - 2n^2\left(\theta - \frac{w_0}{n}\right)^2.$$

Now let $\theta \in [\frac{1}{2n}, \frac{1}{n+1}] \setminus J$. Thus

$$ng(n\theta) \le nK - 2n^3\gamma^2 = nK - 2n^{\frac{1}{6}}.$$

Therefore, if $\theta \in [\frac{1}{2n}, \frac{1}{n+1}] \setminus J$ then

$$\Pi_n(\theta) = o\left(\frac{e^{nK}}{n}\right).$$

On the other hand, Sudler [18, p. 4, Theorem III] proves that if $\theta \in [\frac{1}{n+1}, \frac{1}{2}]$ then

$$\Pi_n(\theta) < 2n^3 \cdot 2^{\frac{n}{4}} = o\left(\frac{e^{nK}}{n}\right).$$

Altogether, we have for $\theta \in [0, \frac{1}{2}] \setminus J$ that

$$\Pi_n(\theta) = o\left(\frac{e^{nK}}{n}\right).$$

We now estimate $\int_J \Pi_n(\theta)^p d\theta$. Let $\theta \in [0, \frac{1+w_0}{2n}]$. We have

$$\begin{aligned}\log \Pi_n(\theta) &= \log\prod_{k=1}^n 2\sin(\pi k\theta) \\ &= n\log 2 + \log\Gamma(n+1) + \sum_{k=1}^n F(k),\end{aligned}$$

where

$$F(y) = \log\sin(\pi\theta y) - \log y.$$

The Euler–Maclaurin summation formula [6, p. 303, Eq. (7.2.4)] gives us

$$\sum_{k=1}^{n} F(k) = \int_0^n F(y)dy + \underbrace{\frac{1}{2}F(n) + \frac{1}{2}F(1) + \frac{1}{12}F'(n) - \frac{1}{12}F'(1) - \int_0^1 F(y)dy}_{M_n} + R_n$$

where $|R_n| \le \frac{2}{(2\pi)^2}\int_1^n |F'''(y)|dy$. We have

$$F'(y) = \pi\theta\cot(\pi\theta y) - \frac{1}{y} \quad \text{and} \quad F'''(y) = 2(\pi\theta)^3\csc^3(\pi\theta y)\cos(\pi\theta y) - \frac{2}{y^3}.$$

We estimate the terms in M_n as follows. As $\theta = O(n^{-1})$ we have

$$F(1) = \log\sin(\pi\theta) = \log(\pi\theta) + O(n^{-2}).$$

Next, set $a = \frac{\pi}{2}(1 + w_0)$. Since $a < \pi$, for $x \in [0, a]$ we have $x\cot(x) - 1 = O(x^2)$. It follows that

$$nF'(n) = \pi\theta n\cot(\pi\theta n) - 1 = O(n^2\theta^2).$$

We also have

$$F'(1) = \pi\theta\cot(\pi\theta) - 1 = O(\theta^2).$$

Finally, for $0 \le y \le 1$ we have

$$F(y) = \log(\pi\theta) + \log\frac{\sin(\pi\theta y)}{\pi\theta y} = \log(\pi\theta) + \log(1 + O(\theta^2)) = \log(\pi\theta) + O(\theta^2).$$

Putting these estimates together we obtain

$$\begin{aligned} M_n &= \frac{1}{2}\log\sin(n\pi\theta) - \frac{1}{2}\log n + \frac{1}{2}\log(\pi\theta) + O(n^{-2}) + O(n\theta^2) + O(\theta^2) - \log(\pi\theta) + O(\theta^2) \\ &= \frac{1}{2}\log\sin(n\pi\theta) - \frac{1}{2}\log n - \frac{1}{2}\log(\pi\theta) + O(n^{-1}). \end{aligned}$$

We now bound the R_n term. Take $a = \frac{\pi}{2}(1 + w_0)$, and as $a < \pi$ we have for $x \in [0, a]$ that $x\csc(x) - 1 = O(x^2)$. Hence for $1 \le y \le n$ we have

$$F'''(y) = \frac{2}{y^3}\cos(\pi\theta y)\left(1 + O(\theta^2 y^2)\right) - \frac{2}{y^3} = O(\theta^2 y^{-1}),$$

and so

$$|R_n| = O\left(\int_1^n \theta^2 y^{-1} dy\right) = O(\theta^2\log n) = O(n^{-2}\log n).$$

From the asymptotic expansion for $\log\Gamma(n+1)$ [6, p. 306, Eq. (7.6.5)] we get $\log\Gamma(n+1) = n\log n - n + \frac{1}{2}\log(2\pi n) + O(n^{-1})$. Therefore, for $\theta \in [0, \frac{1+w_0}{2n}]$ we have

$$\begin{aligned} \log\Pi_n(\theta) &= n\log 2 + n\log n - n + \frac{1}{2}\log(2\pi n) + O(n^{-1}) + \frac{1}{\theta}\int_0^{n\theta}\log\sin(\pi y)dy - n\log n + n \\ &\quad + \frac{1}{2}\log\sin(n\pi\theta) - \frac{1}{2}\log n - \frac{1}{2}\log(\pi\theta) + O(n^{-1}) + O(n^{-2}\log n) \\ &= ng(n\theta) + \frac{1}{2}\log(2\sin(n\pi\theta)) - \frac{1}{2}\log\theta + O(n^{-1}). \end{aligned}$$

As $g'(w_0) = 0$, we have

$$\begin{aligned} g(w) &= g(w_0) + g'(w_0)(w - w_0) + \frac{1}{2}g''(w_0)(w - w_0)^2 + O((w - w_0)^3) \\ &= K - C^2(w - w_0)^2 + O((w - w_0)^3). \end{aligned}$$

Thus if $\theta \in J$ then, as $n^4\gamma^3 = n^{-\frac{1}{4}}$ (this is where we use that the exponent of γ is strictly less than $-\frac{4}{3}$), we have

$$ng(n\theta) = nK - n^3C^2\left(\theta - \frac{w_0}{n}\right)^2 + O(n^4\gamma^3) = nK - n^3C^2\left(\theta - \frac{w_0}{n}\right)^2 + o(1).$$

Also,

$$\frac{\sin(\pi w)}{w} = \frac{\sin(\pi w_0)}{w_0} + O(w - w_0),$$

and so if $\theta \in J$ then

$$\begin{aligned}\log(2\sin(n\pi\theta)) &= \log(2n) + \log\theta + \log\left(\frac{\sin(n\pi\theta)}{n\theta}\right)\\ &= \log(2n) + \log\theta + \log\left(\frac{\sin(\pi w_0)}{w_0} + O(n\theta - w_0)\right)\\ &= \log\left(2n\frac{\sin(\pi w_0)}{w_0}\right) + \log(\theta) + O(n\gamma).\end{aligned}$$

Because $e^K = 2\sin(\pi w_0)$ (which follows from $g'(w_0) = 0$), it follows that $\frac{\sin(\pi w_0)}{w_0} = \frac{B^2C^2}{8\pi}$.

We put together the above to get for $\theta \in J$ that

$$\log \Pi_n(\theta) = nK - n^3C^2\left(\theta - \frac{w_0}{n}\right)^2 + \frac{1}{2}\log\left(n\frac{B^2C^2}{4\pi}\right) + o(1).$$

Using this estimate for $\log \Pi_n(\theta)$ for $\theta \in J$ we obtain

$$\begin{aligned}\int_J \Pi_n(\theta)^p d\theta &= \int_J \exp(p\log \Pi_n(\theta))d\theta\\ &= \exp(pnK)\left(n\frac{B^2C^2}{4\pi}\right)^{p/2}(1+o(1))\int_{\frac{w_0}{n}-\gamma}^{\frac{w_0}{n}+\gamma}\exp\left(-pn^3C^2\left(\theta - \frac{w_0}{n}\right)^2\right)d\theta.\end{aligned}$$

Doing the change of variable $v = \sqrt{p}Cn^{\frac{3}{2}}\left(\theta - \frac{w_0}{n}\right)$ and setting $V = \sqrt{p}n^{\frac{3}{2}}C\gamma = \sqrt{p}Cn^{\frac{1}{12}}$ we get

$$\begin{aligned}\int_J \Pi_n(\theta)^p d\theta &= \exp(pnK)\left(n\frac{B^2C^2}{4\pi}\right)^{p/2}(1+o(1))\int_{-V}^{V}\exp(-v^2)\frac{dv}{C\sqrt{p}n^{\frac{3}{2}}}\\ &= \exp(pnK)\left(n\frac{B^2C^2}{4\pi}\right)^{p/2}\frac{1}{C\sqrt{p}n^{\frac{3}{2}}}(1+o(1))\int_{-V}^{V}\exp(-v^2)dv.\end{aligned}$$

As [6, p. 97, Eq. (10.8.4)]

$$\int_{-V}^{V}\exp(-v^2)dv = \sqrt{\pi} + O\left(\frac{\exp(-V^2)}{V}\right) = \sqrt{\pi}\,(1+o(1)),$$

we get (this is where we use that the exponent of γ is strictly greater than $-\frac{3}{2}$)

$$\int_J \Pi_n(\theta)^p d\theta = \exp(pnK)\left(n\frac{B^2C^2}{4\pi}\right)^{p/2}\frac{\sqrt{\pi}}{C\sqrt{p}n^{\frac{3}{2}}}(1+o(1)).$$

We have obtained estimates for $\int_{[0,\frac{1}{2}]\setminus J}\Pi_n(\theta)^p d\theta$ and for $\int_J \Pi_n(\theta)^p d\theta$, which we now use to estimate $\|P_n\|_p$. Put $\phi = \frac{\exp(nK)}{n}$. For $1 \le p < \infty$ we have

$$\begin{aligned}\|P_n\|_p^p &= 2\int_{\left[0,\frac{1}{2}\right]}\Pi_n(\theta)^p d\theta\\ &= 2\int_{\left[0,\frac{1}{2}\right]\setminus J}\Pi_n(\theta)^p d\theta + 2\int_J \Pi_n(\theta)^p d\theta\\ &= o(\phi^p) + 2\phi^p n^{\frac{3p}{2}}B^pC^p2^{-p}\pi^{-\frac{p}{2}}\pi^{\frac{1}{2}}C^{-1}n^{-\frac{3}{2}}p^{-\frac{1}{2}}(1+o(1))\\ &= 2\phi^p n^{\frac{3p}{2}}B^pC^p2^{-p}\pi^{-\frac{p}{2}}\pi^{\frac{1}{2}}C^{-1}n^{-\frac{3}{2}}p^{-\frac{1}{2}}(1+o(1)),\end{aligned}$$

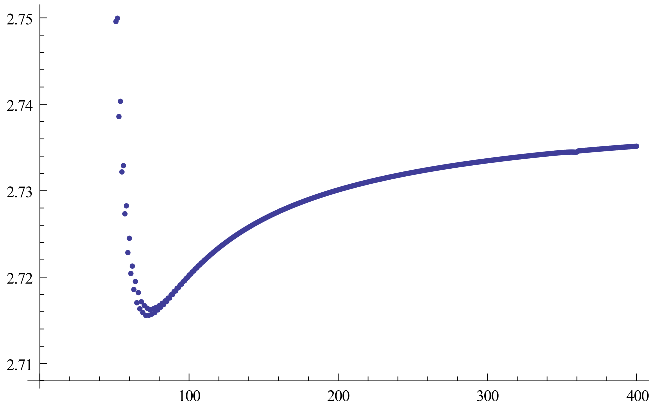

Fig. 2. $\frac{\|P_n\|_1}{e^{nK}n^{-1}}$, for $n = 1, \ldots, 400$.

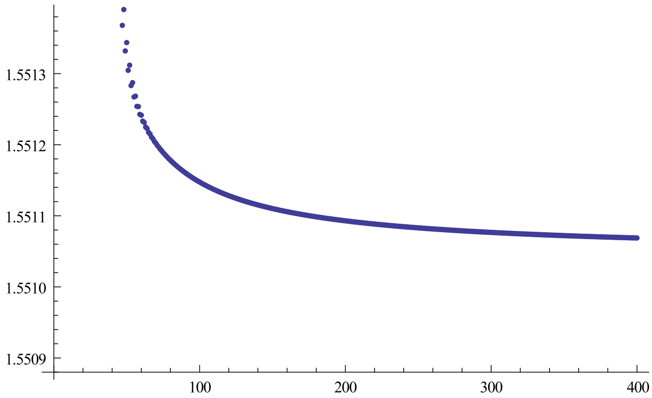

Fig. 3. $\frac{\|P_n\|_2}{e^{nK}n^{-1/4}}$, for $n = 1, \ldots, 400$.

and hence

$$\|P_n\|_p = 2^{\frac{1}{p}} \exp(nK) n^{\frac{1}{2}} BC 2^{-1} \pi^{-\frac{1}{2}} \pi^{\frac{1}{2p}} C^{-\frac{1}{p}} n^{-\frac{3}{2p}} p^{-\frac{1}{2p}} (1 + o(1)).$$

Taking $p \to \infty$ gives

$$\|P_n\|_\infty = \exp(nK) n^{\frac{1}{2}} BC 2^{-1} \pi^{-\frac{1}{2}} (1 + o(1)). \quad \square$$

Doing integration by parts one can show that w_0 is the unique zero $w \in (0, 1)$ of $\int_0^w t \cot(\pi t) dt$. We compute that $w_0 = 0.7912265710\ldots$, from which we get $K = 0.1986176152\ldots$, so $e^K = 1.219715476\ldots$ and $B = 2.740222990\ldots$. We also compute that $C = 1.606193491\ldots$.

We show in Fig. 2 a plot of $\frac{\|P_n\|_1}{e^{Kn}n^{-1}}$ for $n = 1, \ldots, 400$ and in Fig. 3 a plot of $\frac{\|P_n\|_2}{e^{Kn}n^{-1/4}}$ for $n = 1, \ldots, 400$. We have from Theorem 2 that

$$\|P_n\|_1 \sim Bn^{-1}e^{nK} = 2.740222990\ldots \cdot n^{-1}e^{nK}$$

and

$$\|P_n\|_2 \sim BC^{\frac{1}{2}} 2^{-\frac{3}{4}} \pi^{-\frac{1}{4}} n^{-\frac{1}{4}} e^{nK} = 1.551046691\ldots \cdot n^{-\frac{1}{4}} e^{nK}.$$

Using the pentagonal number theorem we can deduce that $\|P_n\|_1 \to \infty$ as $n \to \infty$ from a general result on exponential sums. Littlewood's conjecture, proved in [14], is that there is a constant H such that if the first M nonzero Fourier coefficients of an L^1 function f each has absolute value ≥ 1, then $\|f\|_1 \geq H \log M$. The case of the Dirichlet kernel shows us that $H \leq \frac{4}{\pi^2}$, since $\|D_n\|_1 = \frac{4}{\pi^2} \log n + O(1)$. Of course all the nonzero Fourier coefficients of P_n have absolute value ≥ 1 (namely, they are integers), and one can show using the pentagonal number theorem that P_n has $\geq \frac{3}{2}\sqrt{n}$ nonzero Fourier coefficients with absolute value ≥ 1, hence

$$\|P_n\|_1 \geq H \log\left(\frac{3}{2}\sqrt{n}\right).$$

The L^∞ norm of $\prod_{k=1}^{n}\sin(k\theta)$ is discussed by Carley and Li [3]. They observe that the maximum of $\prod_{k=1}^{n}\sin(k\theta)$ occurs around $\theta = \frac{3\pi}{4n}$. Using the Euler–Maclaurin summation formula, they show that

$$\prod_{k=1}^{n}\sin\left(\frac{3\pi k}{4n}\right) \geq C\sqrt{n}\exp\left(-\frac{5}{6}n\log 2\right),$$

for some $C > 0$. Thus

$$\left|P_n\left(\frac{3\pi}{2n}\right)\right| \geq C\sqrt{n}\exp\left(\frac{1}{6}n\log 2\right). \tag{3}$$

We shall improve on the lower bound given in (3). Let $A = \frac{2G}{3\pi}$, where

$$G = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}(-1)^n = 0.9159655942\ldots$$

is Catalan's constant.

Theorem 3. *For some* $C_0 > 0$,

$$\left|P_n\left(\frac{3\pi}{2n}\right)\right| \leq n^{C_0}e^{An}$$

and

$$\left|P_n\left(\frac{3\pi}{2n}\right)\right| \geq n^{-C_0}e^{An}.$$

Proof. Let $f(x) = \log|\sin x|$. Let $l = \lfloor\frac{2n}{3}\rfloor$. We have

$$\begin{aligned}
\left|\sum_{k=1}^{n} f\left(\frac{3\pi k}{4n}\right)\cdot\frac{3\pi}{4n} - \int_0^{\frac{3\pi}{4}} f(x)dx\right| &\leq \sum_{k=1}^{n}\left|f\left(\frac{3\pi k}{4n}\right)\cdot\frac{3\pi}{4n} - \int_{(k-1)\frac{3\pi}{4n}}^{k\frac{3\pi}{4n}} f(x)dx\right| \\
&\leq f\left(\frac{3\pi}{4n}\right)\frac{3\pi}{4n} - \int_0^{\frac{3\pi}{4n}} f(x)dx + \sum_{k=2}^{l}\left(f\left(k\frac{3\pi}{4n}\right) - f\left((k-1)\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n} \\
&\quad + \left(f\left(\frac{\pi}{2}\right) - f\left(l\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n} + \left(f\left(\frac{\pi}{2}\right) - f\left((l+1)\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n} \\
&\quad + \sum_{k=l+2}^{n}\left(f\left((k-1)\frac{3\pi}{4n}\right) - f\left(k\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n}.
\end{aligned}$$

We will estimate these lines separately. For the first line, because $\sin x \leq x$ for all $x \geq 0$ and because $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \frac{\pi}{2}]$, we have

$$\begin{aligned}
\frac{3\pi}{4n}\log\sin\frac{3\pi}{4n} - \int_0^{\frac{3\pi}{4n}}\log\sin x dx &\leq \frac{3\pi}{4n}\log\frac{3\pi}{4n} - \int_0^{\frac{3\pi}{4n}}\log\frac{2}{\pi}x dx \\
&= \frac{3\pi}{4n}\log\frac{3\pi}{4n} - \frac{3\pi}{4n}\log\frac{2}{\pi} - \frac{3\pi}{4n}\log\frac{3\pi}{4n} + \frac{3\pi}{4n} \\
&= \frac{3\pi}{4n}\left(1 - \log\frac{2}{\pi}\right) \\
&= O\left(\frac{1}{n}\right).
\end{aligned}$$

For the second line, because $f'(x) = \cot x$ we have

$$\begin{aligned}
\sum_{k=2}^{l}\left(f\left(k\frac{3\pi}{4n}\right)-f\left((k-1)\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n} &= \left(f\left(l\frac{3\pi}{4n}\right)-f\left(\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n}\\
&= \frac{3\pi}{4n}\int_{\frac{3\pi}{4n}}^{l\frac{3\pi}{4n}}\cot x dx\\
&\le \frac{3\pi}{4n}\int_{\frac{3\pi}{4n}}^{\frac{\pi}{2}}\cot x dx\\
&= \frac{3\pi}{4n}\left(f\left(\frac{\pi}{2}\right)-f\left(\frac{3\pi}{4n}\right)\right)\\
&= -\frac{3\pi}{4n}\log\sin\frac{3\pi}{4n}\\
&= O\left(\frac{\log n}{n}\right).
\end{aligned}$$

For the third line, $|f\left((l+1)\frac{3\pi}{4n}\right)| \le |f\left(l\frac{3\pi}{4n}\right)|$. Moreover, $\lfloor\frac{2n}{3}\rfloor \ge \frac{n}{2}$ for $n \ge 2$, so $|f\left(l\frac{3\pi}{4n}\right)| \le |\log\sin\frac{3\pi}{8}|$. Therefore the third line is $O(\frac{1}{n})$.

For the fourth line, because $f'(x) = \cot x$ we have

$$\begin{aligned}
\sum_{k=l+2}^{n}\left(f\left((k-1)\frac{3\pi}{4n}\right)-f\left(k\frac{3\pi}{4n}\right)\right)\frac{3\pi}{4n} &= -\frac{3\pi}{4n}\left(f\left(\frac{3\pi}{4}\right)-f\left((l+1)\frac{3\pi}{4n}\right)\right)\\
&= -\frac{3\pi}{4n}\int_{(l+1)\frac{3\pi}{4n}}^{\frac{3\pi}{4}}\cot x dx\\
&\le -\frac{3\pi}{4n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\cot x dx\\
&= O\left(\frac{1}{n}\right).
\end{aligned}$$

The sum of the four lines is $O\left(\frac{\log n}{n}\right)$, and thus there is some $C_0 > 0$ such that

$$\left|\sum_{k=1}^{n}\log\sin\left(\frac{3\pi k}{4n}\right)-\frac{4n}{3\pi}\int_0^{\frac{3\pi}{4}}\log\sin x dx\right| \le C_0\log n.$$

One can check that $\log|\sin x|$ has the Fourier series

$$-\log 2-\sum_{n=1}^{\infty}\frac{1}{n}(1+(-1)^n)\cos(nx).$$

If $f \in L^1(\mathbb{T})$ has the Fourier series $\sum a_k e^{ikx}$, then $\int_a^b f(x)dx = \sum a_k\int_a^b e^{ikx}dx$ [20, Section 13.5]. For $f(x) = \log|\sin x|$, $a = 0$, and $b = \frac{3\pi}{4}$, we have

$$\begin{aligned}
\int_0^{\frac{3\pi}{4}}\log\sin x dx &= -\frac{3\pi\log 2}{4}-\sum_{n=1}^{\infty}\frac{1}{n^2}(1+(-1)^n)\sin\frac{3\pi n}{4}\\
&= -\frac{3\pi\log 2}{4}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\cdot 2\sin\frac{3\pi\cdot 2n}{4}\\
&= -\frac{3\pi\log 2}{4}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{3\pi n}{2}\\
&= -\frac{3\pi\log 2}{4}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\sin\frac{3\pi\cdot(2n+1)}{2}
\end{aligned}$$

$$= -\frac{3\pi\log 2}{4} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}(-1)^{n+1}$$
$$= -\frac{3\pi\log 2}{4} + \frac{G}{2}.$$

Therefore

$$\left|\sum_{k=1}^{n}\log\sin\left(\frac{3\pi k}{4n}\right) - (A-\log 2)n\right| \le C_0\log n$$

for $A = \frac{2G}{3\pi}$. Taking exponentials, it follows that

$$\prod_{k=1}^{n}\sin\left(\frac{3\pi k}{4n}\right) \le n^{C_0}e^{(A-\log 2)n}$$

and

$$\prod_{k=1}^{n}\sin\left(\frac{3\pi k}{4n}\right) \ge n^{-C_0}e^{(A-\log 2)n}.$$

Thus by (1) we get $|P_n\left(\frac{3\pi}{2n}\right)| \le n^{C_0}e^{An}$ and $|P_n\left(\frac{3\pi}{2n}\right)| \ge n^{-C_0}e^{An}$. □

This shows that $\|P_n\|_\infty \ge n^{-C_0}e^{An}$. We compute that $e^A = 1.214550362\ldots$.

Lubinsky [13, Theorem 1.1] proves that if $\epsilon > 0$, then for almost all θ we have

$$|\log|P_n(\theta)|| = O((\log n)(\log\log n)^{1+\epsilon}),$$

but that this is false if $\epsilon = 0$. If θ has bounded partial quotients, Lubinsky shows that $\log|P_n(\theta)| = O(\log n)$ [13, Theorem 1.3]. However, almost all θ do not have a continued fraction expansion with bounded partial quotients [10, p. 166, Theorem 196].

3. Norms of the trigonometric polynomials Q_n

One can see that the Fourier coefficient $\widehat{Q_n}(j)$ is equal to the number of ways to write j as a sum of distinct positive integers each $\le n$. For example, the partitions of 9 into distinct parts each ≤ 6 are $1+2+6$, $1+3+5$, $2+3+4$, $2+7$, $3+6$, $4+5$, and thus $\widehat{Q_7}(9) = 6$.

Various results have been proved about the number of partitions of j as a sum of integers each $\ge n$ and the number of partitions of j as a sum of distinct integers each $\ge n$ for n small relative to j; see for example Szekeres [19], Freiman and Pitman [9], and Mosaki [16].

By (2), we can express $Q_n(\theta)$ using $\prod_{k=1}^{n}\cos\left(\frac{k\theta}{2}\right)$. The product $\prod_{k=1}^{n}\cos(k\theta)$ has the following probabilistic interpretation. Let X_k be independent Bernoulli ± 1 random variables. One can check that the characteristic function of $\sum_{k=1}^{n} kX_k$ is $\prod_{k=1}^{n}\cos(k\theta)$. Unfortunately, to use the central limit theorem we would first have to normalize the sum by dividing it by $n^{3/2}$, and the characteristic function of $\sum_{k=1}^{n}\frac{k}{n^{3/2}}X_k$ is $\prod_{k=1}^{n}\cos\frac{k\theta}{n^{3/2}}$, not $\prod_{k=1}^{n}\cos(k\theta)$.

We see from (2) that $|Q_n(\theta)| = \prod_{k=1}^{n}2|\cos\left(\frac{k\theta}{2}\right)|$. In Fig. 4 we plot $\prod_{k=1}^{10}2|\cos(k\theta)|$ for $0 \le \theta \le \frac{\pi}{2}$ (here the ordinate of 0 is 1024).

Of course $Q_n(0) = 2^n$, so $\|Q_n\|_\infty = 2^n$. Aside from $\theta = 0$ we can explicitly evaluate $Q_n(\theta)$ for certain other θ. If gcd $(n+1, h) = 1$, then $z^{n+1} - 1 = \prod_{k=1}^{n+1}(z - e^{\frac{2\pi ihk}{n+1}})$. Since $z^{n+1} - 1 = (z-1)(z^n + \cdots + z + 1)$, we get $z^n + \cdots + z + 1 = \prod_{k=1}^{n}(z - e^{\frac{2\pi ihk}{n+1}})$, and setting $z = -1$ yields

$$Q_n\left(\frac{2\pi h}{n+1}\right) = \frac{1+(-1)^n}{2}$$

for each h such that $\gcd(n+1, h) = 1$.

For all $1 \le p \le \infty$ we have $\|Q_n\|_p \le \|Q_n\|_\infty = 2^n$. On the other hand, let $1 \le p \le q \le \infty$. One can show that there is some $C > 0$ such that if f satisfies $\hat{f}(j) = 0$ for $|j| > N$ then $\|f\|_q \le CN^{\frac{1}{p}-\frac{1}{q}}\|f\|_p$ [12, p. 123, Exercise 1.8]. (In fact one can take $C = 5$.) Since $\|Q_n\|_\infty = 2^n$, we get for $1 \le p \le \infty$ that $\|Q_n\|_p \ge \frac{1}{C}2^nN^{-1/p}$.

We can do better than this. Following Wright's method of proving Theorem 1, which we used in our proof of Theorem 2, we get in the following theorem an asymptotic formula for $\|Q_n\|_p$.

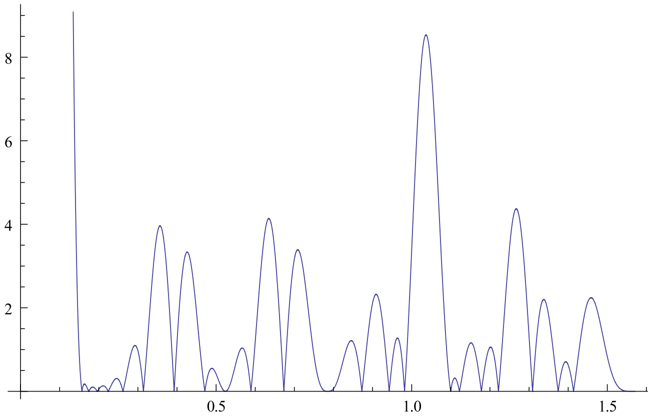

Fig. 4. $\prod_{k=1}^{10} 2|\cos(k\theta)|$ for $0 \le \theta \le \frac{\pi}{2}$.

Theorem 4. *For* $1 \le p < \infty$ *we have*

$$\|Q_n\|_p \sim \left(\frac{6}{p\pi}\right)^{\frac{1}{2p}} 2^n n^{-\frac{3}{2p}}.$$

Proof. Let $\Psi_n(\theta) = \prod_{k=1}^{n} 2|\cos(\pi k\theta)|$. We can check that

$$\|Q_n\|_p = \left(2\int_0^{1/2} \Psi_n(\theta)^p d\theta\right)^{1/p}.$$

Let $\gamma = n^{-4/3}$. We shall estimate $\Psi_n(\theta)$ separately for $0 \le \theta \le \gamma$ and for $\gamma \le \theta \le \frac{1}{2}$.

Let $0 \le \theta \le \gamma$. We define $F(y)$, depending on θ, by $F(y) = \log\cos(\pi\theta y)$. Then

$$\log \Psi_n(\theta) = n\log 2 + \sum_{k=1}^{n} \log\cos(\pi k\theta) = n\log 2 + \sum_{k=1}^{n} F(k).$$

By the Euler–Maclaurin summation formula [6, p. 303, Eq. (7.2.4)] we have

$$\sum_{k=1}^{n} F(k) = \int_0^n F(y)dy + \underbrace{\frac{1}{2}F(n) + \frac{1}{2}F(1) + \frac{1}{12}F'(n) - \frac{1}{12}F'(1) - \int_0^1 F(y)dy}_{M_n} + R_n$$

where $|R_n| \le \frac{2}{(2\pi)^2}\int_1^n |F'''(y)|dy$. First, doing a change of variables, because $\theta \le \gamma = n^{-4/3}$ and because $\log\cos x = -\frac{x^2}{2} + O(x^4)$, we have

$$\begin{aligned}\int_0^n F(y)dy &= \frac{1}{\theta}\int_0^{n\theta} \log\cos(\pi z)dz \\ &= \frac{1}{\theta}\int_0^{n\theta}\left(-\frac{\pi^2 z^2}{2} + O(z^4)\right)dz \\ &= -\frac{\pi^2}{6}n^3\theta^2 + O(n^5\theta^4) \\ &= -\frac{\pi^2}{6}n^3\theta^2 + O(n^{-1/3}).\end{aligned}$$

Second, using $\theta \le \gamma = n^{-4/3}$, $\log\cos x = -\frac{x^2}{2} + O(x^4)$, and $\tan x = O(x)$, we have

$$\begin{aligned}M_n &= \frac{1}{2}\log\cos(\pi\theta n) + \frac{1}{2}\log\cos(\pi\theta) - \frac{\pi\theta}{12}\tan(\pi\theta n) + \frac{\pi\theta}{12}\tan(\pi\theta) - \int_0^1 \log\cos(\pi\theta y)dy \\ &= O(n^{-2/3}) + O(n^{-8/3}) + O(n^{-5/3}) + O(n^{-8/3}) + O(n^{-8/3}) \\ &= O(n^{-2/3}).\end{aligned}$$

Third, $F'''(y) = -2\pi^3\theta^3 \sec^2(\pi\theta y)\tan(\pi\theta y)$, which yields $|R_n| = O(n^{-10/3})$. Putting these three pieces together gives

$$\log \Psi_n(\theta) = n\log 2 - \frac{\pi^2}{6}n^3\theta^2 + O(n^{-1/3})$$

and thus

$$\Psi_n(\theta) = 2^n \exp\left(-\frac{\pi^2}{6}n^3\theta^2\right)\exp(O(n^{-1/3})) = 2^n \exp\left(-\frac{\pi^2}{6}n^3\theta^2\right)(1+O(n^{-1/3})).$$

Therefore, making the change of variables $\phi = \sqrt{\frac{p}{6}}\pi n^{3/2}\theta$ and because $\int_0^V e^{-\phi^2}d\phi \sim \frac{\sqrt{\pi}}{2} - \frac{\exp(-V^2)}{2V}$ as $V\to\infty$ [6, p. 97, Eq. (10.8.4)], we have

$$\begin{aligned}\int_0^\gamma \Psi_n(\theta)^p d\theta &= 2^{pn}\int_0^\gamma \exp\left(-p\frac{\pi^2}{6}n^3\theta^2\right)d\theta\cdot(1+O(n^{-1/3}))\\ &= 2^{pn}\sqrt{\frac{6}{p}}\pi^{-1}n^{-3/2}\int_0^{n^{1/6}\sqrt{\frac{p}{6}}\pi} e^{-\phi^2}d\phi\cdot(1+O(n^{-1/3}))\\ &= 2^{pn}\sqrt{\frac{3}{2p\pi}}n^{-3/2}\cdot(1+O(n^{-1/3}))(1+O(n^{-1/6}))\\ &= 2^{pn}\sqrt{\frac{3}{2p\pi}}n^{-3/2}\cdot(1+O(n^{-1/6})).\end{aligned}$$

Now we bound $\Psi_n(\theta)$ for $\gamma\le\theta\le\frac{1}{2}$. We have, for $\Psi_n(\theta)\ne 0$,

$$\Psi_n(\theta) = \exp(\log\Psi_n(\theta)) = 2^n\exp\left(\sum_{k=1}^n \log|\cos(\pi k\theta)|\right).$$

Using the inequality $\log x \le x-1$ for $x>0$ and the identity $\cos(2x) = 2\cos^2 x - 1$, we get for all x with $\cos x\ne 0$ that

$$\log|\cos x| = \frac{1}{2}\log(\cos^2 x) \le \frac{1}{2}(\cos^2 x - 1) = \frac{1}{4}(-1+\cos(2x)).$$

Hence, for $\Psi_n(\theta)\ne 0$,

$$\Psi_n(\theta) \le 2^n \exp\left(\frac{1}{4}\sum_{k=1}^n(-1+\cos(2\pi k\theta))\right);$$

but of course this inequality is true when $\Psi_n(\theta)=0$, hence the inequality is true for all θ. Let

$$H_n(\theta) = \sum_{k=1}^n(-1+\cos(2\pi k\theta)).$$

We first deal with the interval $\gamma\le\theta\le\frac{1}{2\pi n}$. For $0\le x\le 1$ one has $\cos x\le 1-\frac{x^2}{2}$ (using the Taylor series for $\cos x$, which is an alternating series), so for $\gamma\le\theta\le\frac{1}{2\pi n}$ we have

$$H_n(\theta) \le \sum_{k=1}^n -\frac{(2\pi k\theta)^2}{2} = -2\pi^2\theta^2\sum_{k=1}^n k^2 = -2\pi^2\theta^2\frac{2n^3+3n^2+n}{6} \le -\frac{2\pi^2\theta^2 n^3}{3},$$

so $H_n(\theta)\le -\frac{2\pi^2 n^{1/3}}{3}$.

We now deal with the interval $\frac{1}{2\pi n}\le\theta\le\frac{1}{2n}$. Since $-1+\cos x = -2\sin^2\left(\frac{x}{2}\right)$, we have

$$H_n(\theta) = -2\sum_{k=1}^n \sin^2(\pi k\theta).$$

Using that $\sin^2 x$ is nondecreasing for $0 \le x \le \frac{\pi}{2}$ we have

$$\begin{aligned}\left|\sum_{k=1}^{n}\pi\theta\sin^2(\pi k\theta) - \int_0^{\pi n\theta}\sin^2 x dx\right| &= \left|\sum_{k=1}^{n}\left(\pi\theta\sin^2(\pi k\theta) - \int_{(k-1)\pi\theta}^{k\pi\theta}\sin^2 x dx\right)\right| \\ &\le \sum_{k=1}^{n}\left|\pi\theta\sin^2(\pi k\theta) - \int_{(k-1)\pi\theta}^{k\pi\theta}\sin^2 x dx\right| \\ &\le \sum_{k=1}^{n}\left|\pi\theta\sin^2(\pi k\theta) - \pi\theta\sin^2((k-1)\pi\theta)\right| \\ &= \pi\theta\sum_{k=1}^{n}\left|\sin(\pi\theta)\sin((2k-1)\pi\theta)\right| \\ &\le \pi\theta\sum_{k=1}^{n}\pi\theta \\ &\le \frac{\pi^2}{4n}.\end{aligned}$$

Therefore

$$\sum_{k=1}^{n}\pi\theta\sin^2(\pi k\theta) \ge \int_0^{\pi n\theta}\sin^2 x dx - \frac{\pi^2}{4n}.$$

But $\int_0^{\pi n\theta}\sin^2 x dx \ge \int_0^{1/2}\sin^2 x dx = \frac{1}{4}(1-\sin(1))$, because $\theta \ge \frac{1}{2\pi n}$, so

$$\sum_{k=1}^{n}\sin^2(\pi k\theta) \ge \frac{1}{4\pi\theta}(1-\sin(1)) - \frac{\pi}{4n\theta} \ge \frac{n}{2\pi}(1-\sin(1)) - \frac{\pi^2}{2}.$$

So for $\frac{1}{2\pi n} \le \theta \le \frac{1}{2n}$ we have

$$H_n(\theta) \le -\frac{n}{\pi}(1-\sin(1)) + \pi^2.$$

Finally we deal with the interval $\frac{1}{2n} \le \theta \le \frac{1}{2}$. Using $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, the formula for a finite geometric series, and then $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, one can check that

$$H_n(\theta) = -n - \frac{1}{2} + \frac{1}{2}\frac{\sin((2n+1)\pi\theta)}{\sin(\pi\theta)}.$$

For $0 \le x \le \frac{\pi}{2}$ we have $\sin x \ge \frac{2}{\pi}x$, so for $\frac{1}{2n} \le \theta \le \frac{1}{2}$ we have

$$H_n(\theta) \le -n - \frac{1}{2} + \frac{1}{4\theta} \le -\frac{n}{2} - \frac{1}{2}.$$

Putting together the bounds we have for $\gamma \le \theta \le \frac{1}{2\pi n}$, $\frac{1}{2\pi n} \le \theta \le \frac{1}{2n}$, and $\frac{1}{2n} \le \theta \le \frac{1}{2}$, we get

$$\Psi_n(\theta) = O\left(2^n \exp\left(-\frac{\pi^2 n^{1/3}}{6}\right)\right).$$

In summary, we have shown that

$$\begin{aligned}2\int_0^{1/2}\Psi_n(\theta)^p d\theta &= 2^{pn}\sqrt{\frac{6}{p\pi}}n^{-3/2}\cdot(1+O(n^{-1/6})) + O\left(2^{pn}\exp\left(-p\frac{\pi^2 n^{1/3}}{6}\right)\right) \\ &= 2^{pn}\sqrt{\frac{6}{p\pi}}n^{-3/2}\cdot(1+O(n^{-1/6})). \quad \square\end{aligned}$$

Following Pribitkin's [5], which gives an upper bound on the number of partitions of j with at most n parts, Bidar [2] gives an upper bound on $\widehat{Q_n}(j)$ involving the dilogarithm function Li_2. However, take n to be even, and let $j = \lfloor \frac{n(n+1)}{4} \rfloor$. We compute that the exponential term in Bidar's upper bound for $\widehat{Q_n}(j)$ is e^{Ln}, with

$$L \geq \frac{\pi}{2\sqrt{3}} - \frac{\sqrt{3}}{2\pi}\mathrm{Li}_2\left(\exp\left(-\frac{\pi}{\sqrt{3}}\right)\right) = 0.8599790113\dots.$$

But $\log 2 = 0.6931471805\dots$. Thus here Bidar's bound is worse than the bound $\widehat{Q_n}(j) \leq \|Q_n\|_1 \leq \|Q_n\|_\infty = 2^n$.

In the following theorem we show that for j sufficiently close to $\frac{n(n+1)}{4}$ the Fourier coefficient $\widehat{Q_n}(j)$ is close to $2^n\sqrt{\frac{6}{\pi}}n^{-3/2}$ and use this to get $\|\widehat{Q_n}\|_\infty \sim 2^n\sqrt{\frac{6}{\pi}}n^{-3/2}$. We use the bounds on $\Psi_n(\theta) = \prod_{k=1}^n 2|\cos(\pi k\theta)|$ that we established in our proof of Theorem 4.

Theorem 5. *We have*

$$\|\widehat{Q_n}\|_\infty \sim 2^n\sqrt{\frac{6}{\pi}}n^{-3/2}.$$

Proof. We can check that

$$\widehat{Q_n}(j) = 2\int_0^{1/2}\cos(\pi(N-2j)\theta)\prod_{k=1}^n 2\cos(\pi k\theta)d\theta, \quad N = \frac{n(n+1)}{2}.$$

Following the proof of Theorem 4, with $\Psi_n(\theta) = \prod_{k=1}^n 2|\cos(\pi k\theta)|$ and $\gamma = n^{-4/3}$, we get

$$\widehat{Q_n}(j) = 2\int_0^{\gamma}\cos(\pi(N-2j)\theta)\Psi_n(\theta)d\theta + O\left(2^n\exp\left(-\frac{\pi^2 n^{1/3}}{6}\right)\right).$$

We have from our proof of Theorem 4 that

$$\int_0^{\gamma}\Psi_n(\theta)d\theta = 2^n\sqrt{\frac{3}{2\pi}}n^{-3/2}\cdot(1+O(n^{-1/6})).$$

Using this and the inequality $\cos(x) \geq 1 - \frac{x^2}{2}$ for $0 \leq x \leq 1$ we have for $|N-2j| = o(n^{4/3})$ that

$$\begin{aligned}\widehat{Q_n}(j) &= 2\int_0^{\gamma}\Psi_n(\theta)d\theta + o\left(\int_0^{\gamma}\Psi_n(\theta)d\theta\right) + O\left(2^n\exp\left(-\frac{\pi^2 n^{1/3}}{6}\right)\right)\\ &= 2^n\sqrt{\frac{6}{\pi}}n^{-3/2}(1+o(1)).\end{aligned}$$

But by Theorem 4 we have $\|\widehat{Q_n}\|_\infty \leq \|Q_n\|_1 \sim 2^n\sqrt{\frac{6}{\pi}}n^{-3/2}$. It follows that $\|\widehat{Q_n}\|_\infty \sim 2^n\sqrt{\frac{6}{\pi}}n^{-3/2}$. □

In the above proof we showed that $\widehat{Q_n}(j)$ is $2^n\sqrt{\frac{6}{\pi}}n^{-3/2}(1+o(1))$ for $|N-2j| = o(n^{4/3})$ and that for other j, $\widehat{Q_n}(j)$ is upper bounded by $2^n\sqrt{\frac{6}{\pi}}n^{-3/2}(1+o(1))$, but we did not establish whether $\widehat{Q_n}(j)$ is close to $2^n\sqrt{\frac{6}{\pi}}n^{-3/2}$ for other j or is substantially smaller. Generally, a sequence $a_0, \dots, a_N$ is said to be *symmetric* if $a_k = a_{N-k}$ for all $0 \leq k \leq N$, and is said to be *unimodal* if there is some m such that $a_0 \leq a_1 \leq \dots \leq a_m$ and $a_N \leq a_{N-1} \leq \dots \leq a_m$. If $a_0, \dots, a_N$ is symmetric and unimodal then for $m = \lfloor \frac{N}{2} \rfloor$ the term a_m is equal to the maximum of the sequence. For $N = \frac{n(n+1)}{2}$, there is a bijection between the set of partitions of j into distinct parts each $\leq n$ and the set of partitions of $N-j$ into distinct parts each $\leq n$: for each partition we take the positive integers $\leq n$ not in this partition. Thus $\widehat{Q_n}(j) = \widehat{Q_n}(N-j)$, i.e. the sequence $\widehat{Q_n}(j)$ is symmetric. Hughes and Van der Jeugt [11] show using the representation theory of Lie algebras that the sequence $\widehat{Q_n}(j)$ is unimodal, and survey how to use these methods to prove the unimodality of other sequences. The unimodality of $\widehat{Q_n}(j)$ can also be proved without using Lie algebraic methods [17].

4. Conclusions

It remains to determine the asymptotic behavior of the ℓ^p norms of $\widehat{P_n}$ and $\widehat{Q_n}$ for $1 \leq p < \infty$. Let $N = \frac{n(n+1)}{2}$ and let $L = \frac{(2N-k)w_0}{n} - \frac{1}{4}n$, with w_0 as defined in Theorem 2. Wright's proof [22] of our Theorem 1 shows that if $k = \frac{N}{2} + o(n^{3/2})$

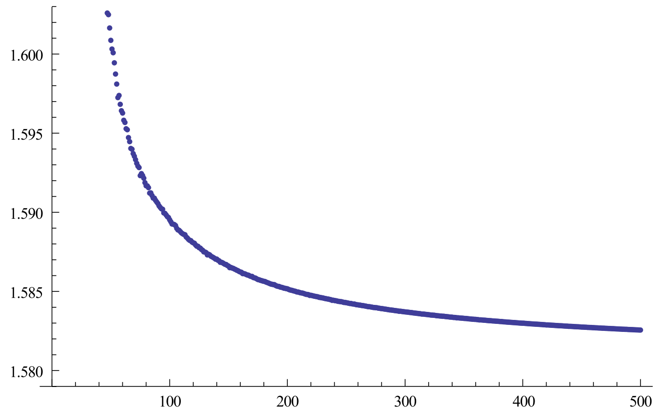

Fig. 5. $\frac{\|\widehat{P_n}\|_1}{e^{Kn}n^{1/2}}$, for $n = 1, \ldots, 500$.

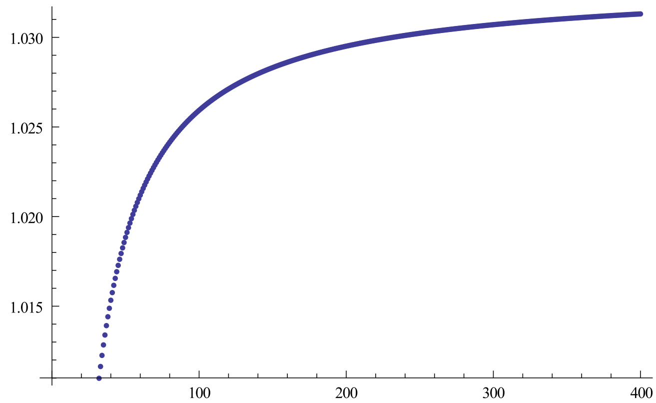

Fig. 6. $\frac{\|\widehat{Q_n}\|_3}{2^n n^{-1}}$, for $n = 1, \ldots, 400$.

then

$$\widehat{P_n}(k) = \frac{Be^{Kn}}{n}\cos(2\pi L) + o\left(\frac{e^{Kn}}{n}\right),$$

with K and B as defined in Theorem 1. Furthermore, Wright [21] proves a result that specializes to the following. Take C as defined in Theorem 2. If $m = k - \frac{N}{2} = o(n^{5/3})$ then

$$\widehat{P_n}(k) = \frac{B}{n}\exp\left(Kn - \frac{\pi^2 m^2}{C^2 n^2}\right)\left(\cos\left(\frac{n\pi}{2} + 2\pi m n^{-1} w_0\right) + o(1)\right).$$

If $n^{3/2}$ of the Fourier coefficients of P_n have magnitude on the order of $\frac{e^{Kn}}{n}$ and the other Fourier coefficients of P_n are relatively negligible, then $\|\widehat{P_n}\|_p$ would have order of magnitude

$$e^{Kn} n^{\frac{3}{2p}-1}. \tag{4}$$

For $p = 2$ we have from Parseval's theorem that $\|\widehat{P_n}\|_2 = \|P_n\|_2$, and by Theorem 2 that $\|P_n\|_2 \sim 2^{-3/4}\pi^{-1/4}BC^{1/2}e^{Kn}n^{-1/4}$, which is consistent with $\|\widehat{P_n}\|_p$ having order of magnitude (4). In Fig. 5 we plot $\frac{\|\widehat{P_n}\|_1}{e^{Kn}n^{1/2}}$ for $n = 1, \ldots, 500$.

Since Q_n has nonnegative Fourier coefficients, $Q_n(0) = \|\widehat{Q_n}\|_1$, and so $\|\widehat{Q_n}\|_1 = 2^n$. If n^α of the Fourier coefficients of Q_n have magnitude on the order of $2^n n^{-3/2}$ (which from Theorem 5 is the order of magnitude of $\|\widehat{Q_n}\|_\infty$), then the identity $\|\widehat{Q_n}\|_1 = 2^n$ implies that $\alpha = \frac{3}{2}$. Then $\|\widehat{Q_n}\|_p$ would have order of magnitude

$$2^n n^{\frac{3}{2p}-\frac{3}{2}}. \tag{5}$$

By Theorem 4, we have $\|Q_n\|_2 \sim \left(\frac{3}{\pi}\right)^{\frac{1}{4}} 2^n n^{-3/4}$, and so by Parseval's theorem, $\|\widehat{Q_n}\|_2 \sim \left(\frac{3}{\pi}\right)^{\frac{1}{4}} 2^n n^{-3/4}$, which is consistent with $\|\widehat{Q_n}\|_p$ having order of magnitude (5). In Fig. 6 we plot $\frac{\|\widehat{Q_n}\|_3}{2^n n^{-1}}$ for $n = 1, \ldots, 400$.

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