The Wiener algebra and Wiener's lemma

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1 Introduction

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For $f \in L^1(\mathbb{T})$ we define

$$\|f\|_{L^1(\mathbb{T})} = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|dt.$$

For $f, g \in L^1(\mathbb{T})$, we define

$$(f * g)(t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\tau)g(t-\tau)d\tau, \qquad t \in \mathbb{T}.$$

$f * g \in L^1(\mathbb{T})$, and satisfies Young's inequality

$$\|f * g\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})}.$$

With convolution as the operation, $L^1(\mathbb{T})$ is a commutative Banach algebra.

For $f \in L^1(\mathbb{T})$, we define $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-ikt}dt, \qquad k \in \mathbb{Z}.$$

We define $c_0(\mathbb{Z})$ to be the collection of those $F : \mathbb{Z} \to \mathbb{C}$ such that $|F(k)| \to 0$ as $|k| \to \infty$. For $f \in L^1(\mathbb{T})$, the Riemann-Lebesgue lemma tells us that $\hat{f} \in c_0(\mathbb{Z})$.

We define $\ell^1(\mathbb{Z})$ to be the set of functions $F : \mathbb{Z} \to \mathbb{C}$ such that

$$\|F\|_{\ell^1(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} |F(k)|.$$

For $F, G \in \ell^1(\mathbb{Z})$, we define

$$(F * G)(k) = \sum_{j\in\mathbb{Z}} F(j)G(k-j).$$

$F * G \in \ell^1(\mathbb{Z})$, and satisfies Young's inequality

$$\|F * G\|_{\ell^1(\mathbb{Z})} \leq \|F\|_{\ell^1(\mathbb{Z})} \|G\|_{\ell^1(\mathbb{Z})}.$$

$\ell^1(\mathbb{Z})$ is a commutative Banach algebra, with unity

$$F(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0. \end{cases}$$

For $f \in L^1(\mathbb{T})$ and $n \geq 0$ we define $S_n(f) \in C(\mathbb{T})$ by

$$S_n(f)(t) = \sum_{|k| \leq n} \hat{f}(k) e^{ikt}, \qquad t \in \mathbb{T}.$$

For $0 < \alpha < 1$, we define $\mathrm{Lip}_\alpha(\mathbb{T})$ to be the collection of those functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\sup_{t \in \mathbb{T}, h \neq 0} \frac{|f(t+h) - f(t)|}{|h|^\alpha} < \infty.$$

For $f \in \mathrm{Lip}_\alpha(\mathbb{T})$, we define

$$\|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} = \|f\|_{C(\mathbb{T})} + \sup_{t \in \mathbb{T}, h \neq 0} \frac{|f(t+h) - f(t)|}{|h|^\alpha}.$$

2 Total variation

For $f : \mathbb{T} \to \mathbb{C}$, we define

$$\mathrm{var}(f) = \sup \left\{ \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : n \geq 1, 0 = t_0 < \cdots < t_n = 2\pi \right\}.$$

If $\mathrm{var}(f) < \infty$ then we say that f is of **bounded variation**, and we define $BV(\mathbb{T})$ to be the set of functions $\mathbb{T} \to \mathbb{C}$ of bounded variation. We define

$$\|f\|_{BV(\mathbb{T})} = \sup_{t \in \mathbb{T}} |f(t)| + \mathrm{var}(f).$$

This is a norm on $BV(\mathbb{T})$, with which $BV(\mathbb{T})$ is a Banach algebra.[1]

Theorem 1. If $f \in BV(\mathbb{T})$, then

$$|\hat{f}(n)| \leq \frac{\mathrm{var}(f)}{2\pi |n|}, \qquad n \in \mathbb{Z}, n \neq 0.$$

Proof. Integrating by parts,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} dt = -\frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{-int}}{-in} df(t) = \frac{1}{2\pi i n} \int_{\mathbb{T}} e^{-int} df(t),$$

hence

$$|\hat{f}(n)| \leq \frac{1}{2\pi |n|} \mathrm{var}(f).$$

□

[1] N. L. Carothers, *Real Analysis*, p. 206, Theorem 13.4.

3 Absolutely convergent Fourier series

Suppose that $f \in L^1(\mathbb{T})$ and that $\hat{f} \in \ell^1(\mathbb{Z})$. For $n \geq m$,

$$\|S_n(f) - S_m(f)\|_{C(\mathbb{T})} = \sup_{t\in\mathbb{T}} \left| \sum_{m<|k|\leq n} \hat{f}(k)e^{ikt} \right| \leq \sum_{m<|k|\leq n} |\hat{f}(k)|,$$

and because $\hat{f} \in \ell^1(\mathbb{Z})$ it follows that $S_n(f)$ converges to some $g \in C(\mathbb{T})$. We check that $f(t) = g(t)$ for almost all $t \in \mathbb{T}$.

We define $A(\mathbb{T})$ to be the collection of those $f \in C(\mathbb{T})$ such that $\hat{f} \in \ell^1(\mathbb{Z})$, and we define

$$\|f\|_{A(\mathbb{T})} = \left\|\hat{f}\right\|_{\ell^1(\mathbb{Z})}.$$

$A(\mathbb{T})$ is a commutative Banach algebra, with unity $t \mapsto 1$, and the Fourier transform is an isomorphism of Banach algebras $\mathscr{F} : A(\mathbb{T}) \to \ell^1(\mathbb{Z})$. We call $A(\mathbb{T})$ the **Wiener algebra**. The inclusion map $A(\mathbb{T}) \subset C(\mathbb{T})$ has norm 1.

Theorem 2. If $f : \mathbb{T} \to \mathbb{C}$ is absolutely continuous, then

$$\hat{f}(k) = o(k^{-1}), \qquad |k| \to \infty.$$

Proof. Because f is absolutely continuous, the fundamental theorem of calculus tells us that $f' \in L^1(\mathbb{T})$. Doing integration by parts, for $k \in \mathbb{Z}$ we have

$$\begin{aligned}\mathscr{F}(f')(k) &= \frac{1}{2\pi}\int_{\mathbb{T}} f'(t)e^{-ikt}dt \\ &= \frac{1}{2\pi} f(t)e^{-ikt}\Big|_0^{2\pi} - \frac{1}{2\pi}\int_{\mathbb{T}} f(t)(-ike^{-ikt})dt \\ &= ik\mathscr{F}(f)(k).\end{aligned}$$

The Riemann-Lebesgue lemma tells us that $\mathscr{F}(f')(k) = o(1)$, so

$$\mathscr{F}(f)(k) = o\left(\frac{1}{k}\right), \qquad |k| \to \infty.$$

□

Theorem 3. If $f : \mathbb{T} \to \mathbb{C}$ is absolutely continuous and $f' \in L^2(\mathbb{T})$, then

$$\|f\|_{A(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} + \left(2\sum_{k=1}^{\infty} k^{-2}\right)^{1/2} \|f'\|_{L^2(\mathbb{T})}.$$

Proof. First,

$$|\hat{f}(0)| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} f(t)dt\right| \leq \|f\|_{L^1(\mathbb{T})}.$$

Next, because f is absolutely continuous, by the fundamental theorem of calculus we have $f' \in L^1(\mathbb{T})$, and for $k \in \mathbb{Z}$,

$$\mathscr{F}(f')(k) = ik\mathscr{F}(f)(k).$$

Using the Cauchy-Schwarz inequality, and since $\mathscr{F}(f')(0) = 0$,

$$\begin{aligned}
\|f\|_{A(\mathbb{T})} &= |\hat{f}(0)| + \sum_{k\neq 0} |\hat{f}(k)| \\
&= |\hat{f}(0)| + \sum_{k\neq 0} |k|^{-1} |\mathscr{F}(f')(k)| \\
&\leq \|f\|_{L^1(\mathbb{T})} + \left(\sum_{k\neq 0} |k|^{-2}\right)^{1/2} \left(\sum_{k\neq 0} |\mathscr{F}(f')(k)|^2\right)^{1/2} \\
&= \|f\|_{L^1(\mathbb{T})} + \left(2\sum_{k=1}^{\infty} k^{-2}\right)^{1/2} \|\mathscr{F}(f')\|_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

By Parseval's theorem we have $\|\mathscr{F}(f')\|_{\ell^2(\mathbb{Z})} = \|f'\|_{L^2(\mathbb{T})}$, completing the proof. □

We now prove that if $\alpha > \frac{1}{2}$, then $\mathrm{Lip}_\alpha(\mathbb{T}) \subset A(\mathbb{T})$, and the inclusion map is a bounded linear operator.[2]

Theorem 4. If $\alpha > \frac{1}{2}$, then $\mathrm{Lip}_\alpha(\mathbb{T}) \subset A(\mathbb{T})$, and for any $f \in \mathrm{Lip}_\alpha(\mathbb{T})$ we have

$$\|f\|_{A(\mathbb{T})} \leq c_\alpha \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})},$$

with

$$c_\alpha = 1 + 2^{1/2} \left(\frac{2\pi}{3}\right)^\alpha \frac{1}{1 - 2^{\frac{1}{2}-\alpha}}.$$

Proof. For $f : \mathbb{T} \to \mathbb{C}$ and $h \in \mathbb{R}$, we define

$$f_h(t) = f(t-h), \qquad t \in \mathbb{T},$$

which satisfies, for $n \in \mathbb{Z}$,

$$\begin{aligned}
\mathscr{F}(f_h)(n) &= \frac{1}{2\pi} \int_{\mathbb{T}} f(t-h) e^{-int} dt \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-in(t+h)} dt \\
&= e^{-inh} \mathscr{F}(f)(n).
\end{aligned}$$

Thus

$$\mathscr{F}(f_h - f)(n) = (e^{-inh} - 1)\hat{f}(n), \qquad n \in \mathbb{Z}. \tag{1}$$

[2]Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 34, Theorem 6.3.

For $m \geq 0$ and for $n \in \mathbb{Z}$ such that $2^m \leq |n| < 2^{m+1}$, let

$$h_m = \frac{2\pi}{3} \cdot 2^{-m}.$$

Then

$$\frac{2\pi}{3} = 2^m \cdot \frac{2\pi}{3} \cdot 2^{-m} \leq |nh_m| < 2^{m+1} \cdot \frac{2\pi}{3} \cdot 2^{-m} = \frac{4\pi}{3}.$$

If $n > 0$ this implies that

$$\frac{\pi}{3} \leq \frac{nh_m}{2} < \frac{2\pi}{3}$$

and so

$$|e^{-inh_m} - 1| = 2\sin\frac{nh_m}{2} \geq 2\sin\frac{\pi}{3} = \sqrt{3},$$

and if $n < 0$ this implies that

$$-\frac{2\pi}{3} < \frac{nh_m}{2} \leq -\frac{\pi}{3}$$

and so

$$|e^{-inh_m} - 1| \geq \sqrt{3}.$$

This gives us

$$\begin{aligned}\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)|^2 &\leq \sum_{2^m \leq |n| < 2^{m+1}} 3|\hat{f}(n)|^2 \\ &\leq \sum_{2^m \leq |n| < 2^{m+1}} |e^{-inh_m} - 1|^2 |\hat{f}(n)|^2 \\ &\leq \sum_{n \in \mathbb{Z}} |e^{-inh_m} - 1|^2 |\hat{f}(n)|^2.\end{aligned}$$

Using (1) and Parseval's theorem we have

$$\sum_{n \in \mathbb{Z}} |e^{-inh_m} - 1|^2 |\hat{f}(n)|^2 = \|\mathscr{F}(f_{h_m} - f)\|_{\ell^2(\mathbb{Z})}^2 = \|f_{h_m} - f\|_{L^2(\mathbb{T})}^2,$$

and thus

$$\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)|^2 \leq \|f_{h_m} - f\|_{L^2(\mathbb{T})}^2.$$

Furthermore, for $g \in L^\infty(\mathbb{T})$ we have $\|g\|_{L^2(\mathbb{T})} \leq \|g\|_{L^\infty(\mathbb{T})}$, so

$$\begin{aligned}\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)|^2 &\leq \|f_{h_m} - f\|_{L^\infty(\mathbb{T})}^2 \\ &\leq \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})}^2 \cdot h_m^{2\alpha} \\ &= \left(\frac{2\pi}{3 \cdot 2^m}\right)^{2\alpha} \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})}^2.\end{aligned}$$

By the Cauchy-Schwarz inequality, because there are $\leq 2^{m+1}$ nonzero terms in $\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)|$,

$$\begin{aligned}
\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)| &\leq (2^{m+1})^{1/2} \left(\sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)|^2 \right)^{1/2} \\
&\leq 2^{\frac{m+1}{2}} \left(\frac{2\pi}{3 \cdot 2^m} \right)^{\alpha} \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} \\
&= 2^{m\left(\frac{1}{2}-\alpha\right)} \cdot 2^{1/2} \left(\frac{2\pi}{3} \right)^{\alpha} \cdot \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} .
\end{aligned}$$

Then, since $\alpha > \frac{1}{2}$,

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} |\hat{f}(n)| &= |\hat{f}(0)| + \sum_{m=0}^{\infty} \sum_{2^m \leq |n| < 2^{m+1}} |\hat{f}(n)| \\
&\leq |\hat{f}(0)| + \sum_{m=0}^{\infty} 2^{m\left(\frac{1}{2}-\alpha\right)} \cdot 2^{1/2} \left(\frac{2\pi}{3} \right)^{\alpha} \cdot \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} \\
&= |\hat{f}(0)| + 2^{1/2} \left(\frac{2\pi}{3} \right)^{\alpha} \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} \sum_{m=0}^{\infty} 2^{m\left(\frac{1}{2}-\alpha\right)} \\
&= |\hat{f}(0)| + 2^{1/2} \left(\frac{2\pi}{3} \right)^{\alpha} \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} \frac{1}{1 - 2^{\frac{1}{2}-\alpha}}
\end{aligned}$$

As

$$|\hat{f}(0)| \leq \|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})} \leq \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} ,$$

we have for all $f \in \mathrm{Lip}_\alpha(\mathbb{T})$ that

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)| \leq c_\alpha \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} ,$$

completing the proof. □

We now prove that if $\alpha > 0$, then $BV(\mathbb{T}) \cap \mathrm{Lip}_\alpha(\mathbb{T}) \subset A(\mathbb{T})$.[3]

Theorem 5. If $\alpha > 0$ and $f \in BV(\mathbb{T}) \cap \mathrm{Lip}_\alpha(\mathbb{T})$, then

$$\|f_h - f\|_{L^2(\mathbb{T})}^2 \leq \frac{1}{2\pi} h^{1+\alpha} \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} \mathrm{var}(f), \qquad h > 0.$$

and $f \in A(\mathbb{T})$.

[3] Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 35, Theorem 6.4.

Proof. For $N \geq 1$ and $h = \frac{2\pi}{N}$,

$$\begin{aligned}
\|f_h - f\|_{L^2(\mathbb{T})}^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f_h(t) - f(t)|^2 dt \\
&= \frac{1}{2\pi}\sum_{j=1}^N \int_{(j-1)h}^{jh} |f_h(t) - f(t)|^2 dt \\
&= \frac{1}{2\pi}\sum_{j=1}^N \int_0^h |f_{jh}(t) - f_{(j-1)h}(t)|^2 dt \\
&= \frac{1}{2\pi}\int_0^h \sum_{j=1}^N |f_{jh}(t) - f_{(j-1)h}(t)|^2 dt \\
&\leq \frac{1}{2\pi}\|f_h - f\|_{L^\infty(\mathbb{T})} \int_0^h \sum_{j=1}^N |f_{jh}(t) - f_{(j-1)h}(t)| dt \\
&\leq \frac{1}{2\pi}\|f_h - f\|_{L^\infty(\mathbb{T})} \int_0^h \mathrm{var}(f) dt.
\end{aligned}$$

As $f \in \mathrm{Lip}_\alpha(\mathbb{T})$, $\|f_h - f\|_{L^\infty(\mathbb{T})} \leq h^\alpha \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})}$, hence

$$\|f_h - f\|_{L^2(\mathbb{T})}^2 \leq \frac{1}{2\pi} h^{1+\alpha} \|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} \mathrm{var}(f).$$

□

4 Wiener's lemma

For $k \geq 1$, using the product rule $(fg)' = f'g + fg'$ we check that $C^k(\mathbb{T})$ is a Banach algebra with the norm

$$\|f\|_{C^k(\mathbb{T})} = \sum_{j=0}^k \left\| f^{(j)} \right\|_{C(\mathbb{T})}.$$

If $f \in C^k(\mathbb{T})$ and $f(t) \neq 0$ for all $t \in \mathbb{T}$, then the quotient rule tells us that

$$\left(f^{-1}\right)'(t) = -\frac{f'(t)}{f(t)^2},$$

using which we get $\frac{1}{f} \in C^k(\mathbb{T})$. That is, if $f \in C^k(\mathbb{T})$ does not vanish then $f^{-1} = \frac{1}{f} \in C^k(\mathbb{T})$.

If B is a commutative unital Banach algebra, a **multiplicative linear functional** on B is a nonzero algebra homomorphism $B \to \mathbb{C}$, and the collection Δ_B of multiplicative linear functionals on B is called the **maximal ideal space** of B. The **Gelfand transform** of $f \in B$ is $\Gamma(f) : \Delta_B \to \mathbb{C}$ defined by

$$\Gamma(f)(h) = h(f), \qquad h \in \Delta_B.$$

It is a fact that $f \in B$ is invertible if and only if $h(f) \neq 0$ for all $h \in \Delta_B$, i.e., $f \in B$ is invertible if and only if $\Gamma(f)$ does not vanish.

We now prove that if $f \in A(\mathbb{T})$ and does not vanish, then f is invertible in $A(\mathbb{T})$. We call this statement **Wiener's lemma**.[4]

Theorem 6 (Wiener's lemma)**.** If $f \in A(\mathbb{T})$ and $f(t) \neq 0$ for all $t \in \mathbb{T}$, then $1/f \in A(\mathbb{T})$.

Proof. Let $w : A(\mathbb{T}) \to \mathbb{C}$ be a multiplicative linear functional. The fact that w is a multiplicative linear functional implies that $\|w\| = 1$. Define $u(t) = e^{it}$, $t \in \mathbb{T}$, for which $\|u\|_{A(\mathbb{T})} = 1$. We define $\lambda = w(u)$, which satisfies

$$|\lambda| \leq \|w\| \, \|u\|_{A(\mathbb{T})} = 1$$

and because $\left\|u^{-1}\right\|_{A(\mathbb{T})} = 1$ we have $\lambda^{-1} = w(u^{-1})$ and

$$|\lambda^{-1}| \leq \|w\| \left\|u^{-1}\right\|_{A(\mathbb{T})} = 1,$$

hence $|\lambda| = 1$. Then there is some $t_w \in \mathbb{T}$ such that $\lambda = e^{it_w}$. For $n \in \mathbb{Z}$,

$$w(u^n) = \lambda^n = e^{int_w}.$$

If $P(t) = \sum_{|n| \leq N} a_n e^{int}$ is a trigonometric polynomial, then

$$w(P) = w\left(\sum_{|n| \leq N} a_n u^n\right) = \sum_{|n| \leq N} a_n w(u)^n = \sum_{|n| \leq N} a_n e^{int_w} = P(t_w). \qquad (2)$$

For $g \in A(\mathbb{T})$, if $\epsilon > 0$, then there is some N such that $\|g - S_N(g)\|_{A(\mathbb{T})} < \epsilon$. Using (2) and the fact that $\|g\|_{C(\mathbb{T})} \leq \|g\|_{A(\mathbb{T})}$,

$$\begin{aligned}
|w(g) - g(t_w)| &\leq |w(g) - w(S_N(g))| + |w(S_N(g)) - S_N(g)(t_w)| \\
&\quad + |S_N(g)(t_w) - g(t_w)| \\
&= |w(g - S_N(g))| + |S_N(g)(t_w) - f(t_w)| \\
&\leq \|w\| \, \|g - S_N(g)\|_{A(\mathbb{T})} + \|S_N(g) - g\|_{C(\mathbb{T})} \\
&\leq \|w\| \, \|g - S_N(g)\|_{A(\mathbb{T})} + \|g - S_N(g)\|_{A(\mathbb{T})} \\
&< 2\epsilon.
\end{aligned}$$

Because this is true for all $\epsilon > 0$, it follows that $w(g) = g(t_w)$.

Let Δ be the maximal ideal space of $A(\mathbb{T})$. Then for $w \in \Delta$ there is some $t_w \in \mathbb{T}$ such that $w(f) = f(t_w)$, hence, because $f(t) \neq 0$ for all $t \in \mathbb{T}$,

$$\Gamma(f)(w) = w(f) = f(t_w) \neq 0.$$

That is, $\Gamma(f)$ does not vanish, and therefore f is invertible in $A(\mathbb{T})$. It is then immediate that $f^{-1}(t) = \frac{1}{f(t)}$ for all $t \in \mathbb{T}$, completing the proof. □

[4] Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 239, Theorem 2.9.

The above proof of Wiener's lemma uses the theory of the commutative Banach algebras. The following is a proof of the theorem that does not use the Gelfand transform.[5]

Proof. Because $f \in A(\mathbb{T})$, f^* defined by $f^*(t) = \overline{f(t)}$, $t \in \mathbb{T}$, belongs to $A(\mathbb{T})$. Let

$$g = \frac{|f|^2}{\|f\|_{C(\mathbb{T})}^2} = \frac{ff^*}{\|f\|_{C(\mathbb{T})}^2} \in A(\mathbb{T}),$$

which satisfies $0 < g(t) \leq 1$ for all $t \in \mathbb{T}$. As $\frac{1}{f} = \frac{f^*}{|f|^2} = \frac{f^*}{\|f\|_{C(\mathbb{T})}^2 g}$, to show that $1/f \in A(\mathbb{T})$ it suffices to show that $\frac{1}{g} \in A(\mathbb{T})$.

Because g is continuous and $g(t) \neq 0$ for all $t \in \mathbb{T}$,

$$\delta = \inf_{t \in \mathbb{T}} g(t) > 0;$$

if $\delta = 1$ then $g = 1$, and indeed $\frac{1}{g} \in A(\mathbb{T})$. Otherwise, $\|g - 1\|_{C(\mathbb{T})} = 1 - \delta < 1$. This implies that g is invertible in the Banach algebra $C(\mathbb{T})$ and that $g^{-1} = \sum_{j=0}^{\infty}(1-g)^j$ in $C(\mathbb{T})$. Let $h = 1 - g \in A(\mathbb{T})$.

For $\epsilon > 0$, there is some N such that $\|h - S_N(h)\|_{A(\mathbb{T})} < \epsilon$. Now, if P is a trigonometric polynomial of degree M then using the Cauchy-Schwarz inequality and Parseval's theorem,

$$\begin{aligned} \|P\|_{A(\mathbb{T})} &= \left\|\hat{P}\right\|_{\ell^1(\mathbb{Z})} \\ &\leq (2M+1)^{1/2} \left\|\hat{P}\right\|_{\ell^2(\mathbb{Z})} \\ &= (2M+1)^{1/2} \|P\|_{L^2(\mathbb{T})} \\ &\leq (2M+1)^{1/2} \|P\|_{L^\infty(\mathbb{T})}. \end{aligned}$$

Furthermore, for $j \geq 1$, P^j is a trigonometric polynomial of degree jM. The binomial theorem tells us, with $P = S_N(h)$ and $r = h - P$,

$$h^k = (P + r)^k = \sum_{j=0}^{k} \binom{k}{j} P^j r^{k-j},$$

[5] Karlheinz Gröchenig, *Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance and Its Applications*, p. 180, §5.2.4, in Brigitte Forster and Peter Massopust, eds., *Four Short Courses on Harmonic Analysis*, pp. 175–234.

and using this and $\|P^j\|_{A(\mathbb{T})} \leq (2jN+1)^{1/2} \|P^j\|_{L^\infty(\mathbb{T})}$,

$$\begin{aligned}
\|h^k\|_{A(\mathbb{T})} &\leq \sum_{j=0}^{k} \binom{k}{j} \|P^j\|_{A(\mathbb{T})} \|r^{k-j}\|_{A(\mathbb{T})} \\
&\leq \sum_{j=0}^{k} \binom{k}{j} \|P^j\|_{A(\mathbb{T})} \|h - S_N(h)\|_{A(\mathbb{T})}^{k-j} \\
&\leq \sum_{j=0}^{k} \binom{k}{j} (2jN+1)^{1/2} \|P^j\|_{L^\infty(\mathbb{T})} \epsilon^{k-j} \\
&\leq (2kN+1)^{1/2} \sum_{j=0}^{k} \binom{k}{j} \|P\|_{L^\infty(\mathbb{T})}^j \epsilon^{k-j} \\
&= (2kN+1)^{1/2} (\|P\|_{L^\infty(\mathbb{T})} + \epsilon)^k.
\end{aligned}$$

Because

$$\begin{aligned}
\|P\|_{L^\infty(\mathbb{T})} &\leq \|h - S_N(h)\|_{L^\infty(\mathbb{T})} + \|h\|_{L^\infty(\mathbb{T})} \\
&\leq \|h - S_N(h)\|_{A(\mathbb{T})} + \|h\|_{L^\infty(\mathbb{T})} \\
&< \epsilon + \|h\|_{L^\infty(\mathbb{T})},
\end{aligned}$$

we have

$$\|h^k\|_{A(\mathbb{T})} \leq (2kN+1)^{1/2} (\|h\|_{L^\infty(\mathbb{T})} + 2\epsilon)^k = (2kN+1)^{1/2} (1 - \delta + 2\epsilon)^k.$$

Take some $\epsilon < \frac{\delta}{2}$, so that $1 - \delta + 2\epsilon < 1$. Then with $N = N(\epsilon)$,

$$\sum_{k=0}^{\infty} \|h^k\|_{A(\mathbb{T})} \leq \sum_{k=0}^{\infty} (2kN+1)^{1/2} (1-\delta+2\epsilon)^k = \sqrt{2N} \Phi\left(1 - \delta + 2\epsilon, -\frac{1}{2}, \frac{1}{2N}\right) < \infty,$$

where Φ is the Lerch transcendent. This implies that the the series $\sum_{k=0}^{\infty} h^k$ converges in $A(\mathbb{T})$. We check that $\sum_{k=0}^{\infty} h^k$ is the inverse of $1 - h$, namely, $g = 1 - h$ is invertible in $A(\mathbb{T})$, proving the claim. □

5 Spectral theory

Suppose that A is a commutative Banach algebra with unity 1. We define $U(A)$ to be the collection of those $f \in A$ such that f is invertible in A. It is a fact that $U(A)$ is an open subset of A. We define

$$\sigma_A(f) = \{\lambda \in \mathbb{C} : f - \lambda \notin U(A)\},$$

called the **spectrum of** f. It is a fact that $\sigma_A(f)$ is a nonempty compact subset of $\mathbb{C}$.

If $A \subset B$ are Banach algebras with unity 1, we say that A **is inverse-closed in** B if $f \in A$ and $f^{-1} \in B$ together imply that $f^{-1} \in A$.[6]

Lemma 7. Suppose that $A \subset B$ are Banach algebras with unity 1. The following are equivalent:

1. A is inverse-closed in B.
2. $\sigma_A(f) = \sigma_B(f)$ for all $f \in A$.

Proof. Assume that A is inverse-closed in B and let $f \in A$. If $\lambda \notin \sigma_A(f)$ then $f - \lambda \in U(A) \subset U(B)$, hence $\lambda \notin \sigma_B(f)$. Therefore $\sigma_B(f) \subset \sigma_A(f)$. If $\lambda \notin \sigma_B(f)$ then $f - \lambda \in U(B)$. That is, $(f - \lambda)^{-1} \in B$. Because A is inverse-closed in B and $f - \lambda \in A$, we get $(f - \lambda)^{-1} \in A$. Thus $\lambda \notin \sigma_A(f)$, and therefore $\sigma_A(f) \subset \sigma_B(f)$. We thus have obtained $\sigma_A(f) = \sigma_B(f)$.

Assume that for all $f \in A$, $\sigma_A(f) = \sigma_B(f)$. Suppose that $f \in A$ and $f^{-1} \in B$. That is, $f \in U(B)$, so $0 \notin \sigma_B(f)$. Then $0 \notin \sigma_A(f)$, meaning that $f \in U(A)$. □

$A(\mathbb{T}) \subset C(\mathbb{T})$ are Banach algebras with unity 1. Wiener's lemma states that $A(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$. It is apparent that for $f \in C(\mathbb{T})$, $\sigma_{C(\mathbb{T})}(f) = f(\mathbb{T}) \subset \mathbb{C}$. Therefore, Lemma 7 tells us for $f \in A(\mathbb{T})$ that $\sigma_{A(\mathbb{T})}(f) = f(\mathbb{T})$.

The **Wiener-Lévy theorem** states that if $f \in A(\mathbb{T})$, $\Omega \subset \mathbb{C}$ is an open set containing $f(\mathbb{T})$, and $F : \Omega \to \mathbb{C}$ is holomorphic, then $F \circ f \in A(\mathbb{T})$.[7] In particular, if $f \in A(\mathbb{T})$ does not vanish, then $\Omega = \mathbb{C} \setminus \{0\}$ is an open set containing $f(\mathbb{T})$ and $F(z) = \frac{1}{z}$ is a holomorphic function on Ω, and hence $F \circ f(t) = \frac{1}{f(t)}$ belongs to $A(\mathbb{T})$, which is the statement of Wiener's lemma.

[6] Karlheinz Gröchenig, *Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance and Its Applications*, p. 183, §5.2.5, in Brigitte Forster and Peter Massopust, eds., *Four Short Courses on Harmonic Analysis*, pp. 175–234.

[7] Karlheinz Gröchenig, *Wiener's Lemma: Theme and Variations. An Introduction to Spectral Invariance and Its Applications*, p. 187, Theorem 5.16, in Brigitte Forster and Peter Massopust, eds., *Four Short Courses on Harmonic Analysis*, pp. 175–234; Walter Rudin, *Fourier Analysis on Groups*, Chapter 6; N. K. Nikolski (ed.), *Functional Analysis I*, p. 235; V. P. Havin and N. K. Nikolski (eds.), *Commutative Harmonic Analysis II*, p. 240, §7.7.