Norms of trigonometric polynomials

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Theorem 1. *Let $1 \leq p \leq q \leq \infty$. If $\hat{f}(j) = 0$ for $|j| > n+1$ then*

$$\|f\|_q \leq 5(n+1)^{\frac{1}{p}-\frac{1}{q}} \|f\|_p.$$

Proof. Let $K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt}$, the Fejér kernel. From this expression we get $|K_n(t)| \leq K_n(0) = n+1$. It's straightforward to show that $K_n(t) = \frac{1}{n+1}\left(\frac{\sin \frac{n+1}{2} t}{\sin \frac{1}{2} t}\right)^2$. Since $\sin \frac{t}{2} > \frac{t}{\pi}$ for $0 < t < \pi$, we get $|K_n(t)| \leq \frac{\pi^2}{(n+1)t^2}$, and thus we obtain

$$|K_n(t)| \leq \min\left(n+1, \frac{\pi^2}{(n+1)t^2}\right).$$

Then, for any $r \geq 1$,

$$\begin{aligned}
\|K_n\|_r^r &= \frac{1}{2\pi} \int_0^{2\pi} |K_n(t)|^r dt \\
&\leq \frac{1}{2\pi} \int_0^{\frac{\pi}{n+1}} (n+1)^r dt + \frac{1}{2\pi} \int_{\frac{\pi}{n+1}}^{2\pi} \left(\frac{\pi^2}{(n+1)t^2}\right)^r dt \\
&= \frac{(n+1)^{r-1}}{2} + \frac{1}{2} \frac{1}{(n+1)^r} \frac{1}{2r-1} \left((n+1)^{2r-1} - \frac{1}{2^{2r-1}}\right) \\
&\leq \frac{(n+1)^{r-1}}{2} + \frac{1}{2} \frac{1}{(n+1)^r} \frac{1}{2r-1} (n+1)^{2r-1} \\
&\leq (n+1)^{r-1}.
\end{aligned}$$

Hence $\|K_n\|_r \leq (n+1)^{1-\frac{1}{r}}$.

Let $V_n(t) = 2K_{2n+1}(t) - K_n(t)$, the de la Vallée Poussin kernel [1, p. 16]. Then

$$\|V_n\|_r \leq 2\|K_{2n+1}\|_r + \|K_n\|_r \leq 2(2n+2)^{1-\frac{1}{r}} + (n+1)^{1-\frac{1}{r}} \leq 5(n+1)^{1-\frac{1}{r}}.$$

For $|j| \leq n+1$ we have $\widehat{V_n}(j) = 1$, and one thus checks that $V_n * f = f$. Take $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$. By Young's inequality we have

$$\|f\|_q = \|V_n * f\|_q \leq \|V_n\|_r \|f\|_p \leq 5(n+1)^{\frac{1}{p}-\frac{1}{q}} \|f\|_p.$$

□

References

[1] Yitzhak Katznelson, *An introduction to harmonic analysis*, third ed., Cambridge Mathematical Library, Cambridge University Press, 2004.