Test functions, distributions, and Sobolev's lemma

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1 Introduction

If X is a topological vector space, we denote by X^* the set of continuous linear functionals on X. With the weak-* topology, X^* is a locally convex space, whether or not X is a locally convex space. (But in this note, we only talk about locally convex spaces.)

The purpose of this note is to collect the material given in Walter Rudin, *Functional Analysis*, second ed., chapters 6 and 7, involved in stating and proving Sobolev's lemma.

2 Test functions

Suppose that Ω is an open subset of $\mathbb{R}^n$. We denote by $\mathscr{D}(\Omega)$ the set of all $\phi \in C^\infty(\Omega)$ such that $\operatorname{supp} \phi$ is a compact subset of Ω. Elements of $\mathscr{D}(\Omega)$ are called *test functions*. For $N = 0, 1, \ldots$ and $\phi \in \mathscr{D}(\Omega)$, write

$$\|\phi\|_N = \sup\{|(D^\alpha \phi)(x)| : x \in \Omega, |\alpha| \leq N\},$$

where

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

For each compact subset K of Ω, we define

$$\mathscr{D}_K = \{\phi \in \mathscr{D}(\Omega) : \operatorname{supp} \phi \subseteq K\},$$

and define τ_K to be the locally convex topology on $\mathscr{D}_K$ determined by the family of seminorms $\{\|\cdot\|_N : N \geq 0\}$. One proves that $\mathscr{D}_K$ with the topology τ_K is a Fréchet space. As sets,

$$\mathscr{D}(\Omega) = \bigcup_K \mathscr{D}_K.$$

Define β to be the collection of all convex balanced subsets W of $\mathscr{D}(\Omega)$ such that for every compact subset K of Ω we have $W \cap \mathscr{D}_K \in \tau_K$; to say that W is *balanced* means that if c is a complex number with $|c| \leq 1$ then $cW \subseteq W$. One proves that $\{\phi + W : \phi \in \mathscr{D}(\Omega), W \in \beta\}$ is a basis for a topology τ on $\mathscr{D}(\Omega)$, that β is a local basis at 0 for this topology, and that with the topology τ, $\mathscr{D}(\Omega)$ is a locally convex space.[1] For each compact subset K of Ω, one proves that the topology τ_K is equal to the subspace topology on $\mathscr{D}_K$ inherited from $\mathscr{D}(\Omega)$.[2]

We write $\mathscr{D}'(\Omega) = (\mathscr{D}(\Omega))^*$, and elements of $\mathscr{D}'(\Omega)$ are called *distributions*. With the weak-* topology, $\mathscr{D}'(\Omega)$ is a locally convex space.

It is a fact that a linear functional Λ on $\mathscr{D}(\Omega)$ is continuous if and only if for every compact subset K of Ω there is a nonnegative integer N and a constant C such that $|\Lambda\phi| \leq C \|\phi\|_N$ for all $\phi \in \mathscr{D}_K$.[3]

For $\Lambda \in \mathscr{D}'(\Omega)$ and α a multi-index, we define

$$(D^\alpha \Lambda)(\phi) = (-1)^{|\alpha|}\Lambda(D^\alpha \phi), \qquad \phi \in \mathscr{D}(\Omega).$$

Let K be a compact subset of Ω. As Λ is continuous, there is a nonnegative integer N and a constant C such that $|\Lambda\phi| \leq C \|\phi\|_N$ for all $\phi \in \mathscr{D}_K$. Then

$$|(D^\alpha \Lambda)(\phi)| = |\Lambda(D^\alpha \phi)| \leq C \|D^\alpha \phi\|_N \leq C \|\phi\|_{N+|\alpha|},$$

which shows that $D^\alpha \Lambda \in \mathscr{D}'(\Omega)$.

The *Leibniz formula* is the statement that for all $f, g \in C^\infty(\mathbb{R}^n)$,

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)(D^\beta g),$$

where $\binom{\alpha}{\beta}$ are multinomial coefficients.

For $\Lambda \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, we define

$$(f\Lambda)(\phi) = \Lambda(f\phi), \qquad \phi \in \mathscr{D}(\Omega);$$

this makes sense because $f\phi \in \mathscr{D}(\Omega)$ when $\phi \in \mathscr{D}(\Omega)$. It is apparent that $f\Lambda$ is linear, and in the following lemma we prove that $f\Lambda$ is continuous.[4]

Lemma 1. *If $\Lambda \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, then $f\Lambda \in \mathscr{D}'(\Omega)$.*

Proof. Suppose that K is a compact subset of Ω. Because Λ is continuous, there is some nonnegative integer N and some constant C such that

$$|\Lambda\phi| \leq C \|\phi\|_N, \qquad \phi \in \mathscr{D}_K.$$

For $|\alpha| \leq N$, by the Leibniz formula, for all $\phi \in \mathscr{D}_K$,

$$D^\alpha(f\phi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)(D^\beta \phi).$$

[1] Walter Rudin, *Functional Analysis*, second ed., p. 152, Theorem 6.4; cf. Helmut H. Schaefer, *Topological Vector Spaces*, p. 57.

[2] Walter Rudin, *Functional Analysis*, second ed., p. 153, Theorem 6.5.

[3] Walter Rudin, *Functional Analysis*, second ed., p. 156, Theorem 6.8.

[4] Walter Rudin, *Functional Analysis*, second ed., p. 159, §6.15.

Because $f \in C^\infty(\Omega)$, there is some C_α such that $|(D^{\alpha-\beta} f)(x)| \leq C_\alpha$ for $\beta \leq \alpha$ and for $x \in K$. Using $\phi(x) = 0$ for $x \notin K$, the above statement of the Leibniz formula, and the inequality just obtained, it follows that there is some C'_α such that $|(D^\alpha(f\phi))(x)| \leq C'_\alpha \|\phi\|_N$ for all $x \in \Omega$. This gives

$$\|f\phi\|_N = \sup_{|\alpha| \leq N} \sup_{x \in \Omega} |(D^\alpha(f\phi))(x)| \leq \sup_{|\alpha| \leq N} C'_\alpha \|\phi\|_N = C' \|\phi\|_N ;$$

the last equality is how we define C', which is a maximum of finitely many C'_α and so finite. Then,

$$|(f\Lambda)(\phi)| = |\Lambda(f\phi)| \leq C \|f\phi\|_N \leq CC' \|\phi\|_N , \qquad \phi \in \mathscr{D}_K.$$

This bound shows that $f\Lambda$ is continuous. $\square$

The above lemma shows that $f\Lambda \in \mathscr{D}'(\Omega)$ when $f \in C^\infty(\Omega)$ and $\Lambda \in \mathscr{D}'(\Omega)$. Therefore $D^\alpha(f\Lambda) \in \mathscr{D}(\Omega)$, and the following lemma, proved in Rudin, states that the Leibniz formula can be used with $f\Lambda$.[5]

Lemma 2. *If $f \in C^\infty(\Omega)$ and $\Lambda \in \mathscr{D}'(\Omega)$, then*

$$D^\alpha(f\Lambda) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)(D^\beta \Lambda).$$

If $f : \Omega \to \mathbb{C}$ is locally integrable, define

$$\Lambda\phi = \int_\Omega \phi(x) f(x) dx, \qquad \phi \in \mathscr{D}(\Omega).$$

For $\phi \in \mathscr{D}_K$,

$$|\Lambda\phi| \leq \|\phi\|_0 \int_K |f| dx,$$

from which it follows that Λ is continuous. If μ is a complex Borel measure on $\mathbb{R}^n$ or a positive Borel measure on $\mathbb{R}^n$ that assigns finite measure to compact sets, define

$$\Lambda\phi = \int_\Omega \phi d\mu, \qquad \phi \in \mathscr{D}(\Omega).$$

For $\phi \in \mathscr{D}_K$,

$$|\Lambda\phi| \leq \|\phi\|_0 |\mu|(K),$$

from which it follows that Λ is continuous. Thus, we can encode certain functions and measures as distributions. I will dare to say that we can encode most functions and measures that we care about as distributions.

If $\Lambda_1, \Lambda_2 \in \mathscr{D}'(\Omega)$ and ω is an open subset of Ω, we say that $\Lambda_1 = \Lambda_2$ *in* ω if $\Lambda_1\phi = \Lambda_2\phi$ for all $\phi \in \mathscr{D}(\omega)$.

Let $\Lambda \in \mathscr{D}'(\Omega)$ and let ω be an open subset of Ω. We say that Λ *vanishes on* ω if $\Lambda\phi = 0$ for all $\phi \in \mathscr{D}(\omega)$. Taking W to be the union of all open subsets ω of Ω on which Λ vanishes, we define the *support of* Λ to be the set $\Omega \setminus W$.

[5]Walter Rudin, *Functional Analysis*, second ed., p. 160, §6.15.

3 The Fourier transform

Let $C_0(\mathbb{R}^n)$ be the set of those continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for every $\epsilon > 0$, there is some compact set K such that $|f(x)| < \epsilon$ for $x \notin K$. With the supremum norm $\|\cdot\|_\infty$, $C_0(\mathbb{R}^n)$ is a Banach space.

Let m_n be *normalized Lebesgue measure on* $\mathbb{R}^n$:

$$dm_n(x) = (2\pi)^{-n/2} dx.$$

Using m_n, we define

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p dm_n \right)^{1/p}, \qquad 1 \le p < \infty$$

and

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dm_n(y).$$

For $t \in \mathbb{R}^n$, define $e_t : \mathbb{R}^n \to \mathbb{C}$ by

$$e_t(x) = \exp(it \cdot x), \qquad x \in \mathbb{R}^n.$$

The *Fourier transform* of $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$(\mathscr{F} f)(t) = \hat{f}(t) = \int_{\mathbb{R}^n} f e_{-t} dm_n, \qquad t \in \mathbb{R}^n.$$

Using the dominated convergence theorem, one shows that $\hat{f}$ is continuous.

For $f \in C^\infty(\mathbb{R}^n)$ and N a nonnegative integer, write

$$p_N(f) = \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |(D^\alpha f)(x)|,$$

and let $\mathscr{S}_n$ be the set of those $f \in C^\infty(\mathbb{R}^n)$ such that for every nonnegative integer N, $p_N(f) < \infty$. $\mathscr{S}_n$ is a vector space, and with the locally convex topology determined by the family of seminorms $\{p_N : N \ge 0\}$ it is a Fréchet space.[6] Further, one proves that $\mathscr{F} : \mathscr{S}_n \to \mathscr{S}_n$ is a continuous linear map.[7]

The *Riemann-Lebesgue lemma* is the statement that if $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.[8]

The *inversion theorem*[9] is the statement that if $g \in \mathscr{S}_n$ then

$$g(x) = \int_{\mathbb{R}^n} \hat{g} e_x dm_n, \qquad x \in \mathbb{R}^n,$$

and that if $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, and we define $f_0 \in C_0(\mathbb{R}^n)$ by

$$f_0(x) = \int_{\mathbb{R}^n} \hat{f} e_x dm_n, \qquad x \in \mathbb{R}^n,$$

[6] Walter Rudin, *Functional Analysis*, second ed., p. 184, Theorem 7.4.
[7] Walter Rudin, *Functional Analysis*, second ed., p. 184, Theorem 7.4.
[8] Walter Rudin, *Functional Analysis*, second ed., p. 185, Theorem 7.5.
[9] Walter Rudin, *Functional Analysis*, second ed., p. 186, Theorem 7.7.

then $f(x) = f_0(x)$ for almost all $x \in \mathbb{R}^n$. For $g \in \mathscr{S}_n$, as $\hat{g} \in \mathscr{S}_n$, the function $f(t) = \hat{g}(-t)$ belongs to $\mathscr{S}_n$. The inversion theorem tells us that for all $x \in \mathbb{R}^n$,

$$g(x) = \int_{\mathbb{R}^n} \hat{g}(t)e_x(t)dm_n(t) = \int_{\mathbb{R}^n} \hat{g}(-t)e_x(-t)dm_n(t) = \int_{\mathbb{R}^n} f(t)e_{-x}(t)dm_n(t),$$

and hence that $g = \hat{f}$. This shows that $\mathscr{F} : \mathscr{S}_n \to \mathscr{S}_n$ is onto. Using the inversion theorem, one checks that

$$\int_{\mathbb{R}^n} f\overline{g}dm_n = \int_{\mathbb{R}^n} \hat{f}\overline{\hat{g}}dm_n, \qquad f, g \in \mathscr{S}_n,$$

and so $\|f\|_{L^2} = \|\mathscr{F}f\|_{L^2}$ for $f \in \mathscr{S}_n$. It is a fact that $\mathscr{S}_n$ is a dense subset of the Hilbert space $L^2(\mathbb{R}^n)$, and it follows that there is a unique bounded linear operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, that is equal to $\mathscr{F}$ on $\mathscr{S}_n$, and that is unitary. We denote this $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

It is a fact that $\mathscr{D}(\mathbb{R}^n)$ is a dense subset of $\mathscr{S}_n$ and that the identity map $i : \mathscr{D}(\mathbb{R}^n) \to \mathscr{S}_n$ is continuous.[10] If $L_1, L_2 \in (\mathscr{S}_n)^*$ are distinct, then there is some $f \in \mathscr{S}_n$ such that $L_1 f \neq L_2 f$, and as $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}_n$, there is a sequence $f_j \in \mathscr{D}(\mathbb{R}^n)$ with $f_j \to f$ in $\mathscr{S}_n$. As

$$(L_1 \circ i)(f_j) - (L_2 \circ i)(f_j) = L_1 f_j - L_2 f_j \to L_1 f_j - L_2 f_j \neq 0,$$

there is some f_j with $(L_1 \circ i)(f_j) \neq (L_2 \circ i)(f_j)$, and hence $L_1 \circ i \neq L_2 \circ i$. This shows that $L \mapsto L \circ i$ is a one-to-one linear map $(\mathscr{S}_n)^* \to \mathscr{D}'(\mathbb{R}^n)$. Elements of $\mathscr{D}'(\mathbb{R}^n)$ of the form $L \circ i$ for $L \in (\mathscr{S}_n)^*$ are called *tempered distributions*, and we denote the set of tempered distributions by $\mathscr{S}'_n$. It is a fact that every distribution with compact support is tempered.[11]

4 Sobolev's lemma

Suppose that Ω is an open subset of $\mathbb{R}^n$. We say that a measurable function $f : \Omega \to \mathbb{C}$ is *locally* L^2 if $\int_K |f|^2 dm_n < \infty$ for every compact subset K of Ω. We say that $\Lambda \in \mathscr{D}'(\Omega)$ is *locally* L^2 if there is a function g that is locally L^2 in Ω such that $\Lambda\phi = \int_\Omega \phi g dm_n$ for every $\phi \in \mathscr{D}(\Omega)$.

The following proof of Sobolev's lemma follows Rudin.[12]

Theorem 3 (Sobolev's lemma)**.** *Suppose that n, p, r are integers, $n > 0$, $p \geq 0$, and*

$$r > p + \frac{n}{2}.$$

Suppose that Ω is an open subset of $\mathbb{R}^n$, that $f : \Omega \to \mathbb{C}$ is locally L^2, and that the distribution derivatives $D_j^k f$ are locally L^2 for $1 \leq j \leq n$, $1 \leq k \leq r$. Then there is some $f_0 \in C^p(\Omega)$ such that $f_0(x) = f(x)$ for almost all $x \in \Omega$.

[10] Walter Rudin, *Functional Analysis*, second ed., p. 189, Theorem 7.10.

[11] Walter Rudin, *Functional Analysis*, second ed., p. 190, Example 7.12 (a).

[12] Walter Rudin, *Functional Analysis*, second ed., p. 202, Theorem 7.25.

Proof. To say that the distribution derivative $D_j^k f$ is locally L^2 means that there is some $g_{j,k} : \Omega \to \mathbb{C}$ that is locally L^2 such that

$$D_j^k \Lambda_f = \Lambda_{g_{j,k}}.$$

Suppose that ω is an open subset of Ω whose closure K is a compact subset of Ω. There is some $\psi \in \mathscr{D}(\Omega)$ with $\psi(x) = 1$ for $x \in K$, and we define $F : \mathbb{R}^n \to \mathbb{C}$ by

$$F(x) = \begin{cases} \psi(x)f(x) & x \in \Omega, \\ 0 & x \notin \Omega; \end{cases}$$

in particular, for $x \in K$ we have $F(x) = f(x)$, and for $x \notin \operatorname{supp}\psi$ we have $F(x) = 0$. Because $\operatorname{supp}\psi \subset \Omega$ is compact and f is locally L^2,

$$\|F\|_{L^2} = \left(\int_{\operatorname{supp}\psi} |\psi f|^2 dm_n\right)^{1/2} \leq \|\psi\|_0 \left(\int_{\operatorname{supp}\psi} |f|^2 dm_n\right)^{1/2} < \infty,$$

and using the Cauchy-Schwarz inequality, $\|F\|_{L^1} \leq \|F\|_{L^2} m_n(\operatorname{supp}\psi)^{1/2} < \infty$, so

$$F \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).$$

Then,

$$\int_{\mathbb{R}^n} |\widehat{F}|^2 dm_n < \infty. \tag{1}$$

Because $\Lambda_F = \psi\Lambda_f$ in Ω, the Leibniz formula tells us that in Ω,

$$D_j^r \Lambda_F = D_j^r(\psi\Lambda_f) = \sum_{s=0}^{r} \binom{r}{s} (D_j^{r-s}\psi)(D_j^s \Lambda_f) = \sum_{s=0}^{r} \binom{r}{s} (D_j^{r-s}\psi)(\Lambda_{g_{j,s}}),$$

hence, defining $H_j : \mathbb{R}^n \to \mathbb{C}$ by

$$H_j(x) = \begin{cases} \sum_{s=0}^{r} \binom{r}{s} (D_j^{r-s}\psi)(x) g_{j,s}(x) & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

we have $D_j^r \Lambda_F = \Lambda_{H_j}$ in Ω. It is apparent that $H_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Let $\phi \in \mathscr{D}(\mathbb{R}^n)$. There are $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{R}^n)$ with $\phi = \phi_1 + \phi_2$ and $\operatorname{supp}\phi_1 \subset \Omega$, $\operatorname{supp}\phi_2 \subset \mathbb{R}^n \setminus \operatorname{supp}\psi$.[13] We have just established that $(D_j^r \Lambda_F)\phi_1 = \Lambda_{H_j}\phi_1$. For ϕ_2, it is apparent that

$$(D_j^r \Lambda_F)\phi_2 = \Lambda_F(D_j^r \phi_2) = \int_{\mathbb{R}^n} (D_j^r \phi_2)(x) F(x) dm_n(x) = 0$$

and

$$\Lambda_{H_j}\phi_2 = \int_{\mathbb{R}^n} \phi_2(x) H_j(x) dm_n(x) = 0.$$

[13] ϕ_1 and ϕ_2 are constructed using a partition of unity. See Walter Rudin, *Functional Analysis*, second ed., p. 162, Theorem 6.20.

Hence $(D_j^r \Lambda_F)(\phi) = \Lambda_{H_j}\phi$. It is apparent that Λ_{H_j} has compact support, so $D_j^r \Lambda_F = \Lambda_{H_j}$ are tempered distributions. Let $\xi \in \mathscr{S}_n$, and take $\phi \in \mathscr{S}_n$ with $\xi = \hat{\phi}$. Then,

$$\begin{aligned}
(D_j^r \Lambda_F)\phi &= \Lambda_F D_j^r \phi \\
&= \int_{\mathbb{R}^n} (D_j^r \phi)(x) F(x) dm_n(x) \\
&= \int_{\mathbb{R}^n} \mathscr{F}(D_j^r \phi)(y) \widehat{F}(y) dm_n(y) \\
&= \int_{\mathbb{R}^n} (iy_j)^r \xi(y) \widehat{F}(y) dm_n(y),
\end{aligned}$$

and

$$\Lambda_{H_j}\phi = \int_{\mathbb{R}^n} \phi(x) H_j(x) dm_n(x) = \int_{\mathbb{R}^n} \xi(y) \widehat{H_j}(y) dm_n(y).$$

It follows that $(iy_j)^r \widehat{F}(y) = \widehat{H_j}(y)$ for all $y \in \mathbb{R}^n$. But $\widehat{H_j} \in L^2(\mathbb{R}^n)$, so

$$\int_{\mathbb{R}^n} y_i^{2r} |\widehat{F}(y)|^2 dm_n(y) < \infty, \qquad 1 \le i \le n. \tag{2}$$

Using (1), (2), and the inequality

$$(1+|y|)^{2r} < (2n+2)^r (1 + y_1^{2r} + \cdots + y_n^{2r}), \qquad y \in \mathbb{R}^n,$$

we get

$$J = \int_{\mathbb{R}^n} (1+|y|)^{2r} |\widehat{F}(y)|^2 dm_n(y) < \infty.$$

Let σ_{n-1} be surface measure on S^{n-1}, with $\sigma_{n-1}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Using the Cauchy-Schwarz inequality and the change of variable $y = tu$, $u \in S^{n-1}$, $t \ge 0$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} (1+|y|)^p |\widehat{F}(y)| dm_n(y) \right)^2 &= \left(\int_{\mathbb{R}^n} (1+|y|)^r |\widehat{F}(y)| (1+|y|)^{p-r} dm_n(y) \right)^2 \\
&\le J \int_{\mathbb{R}^n} (1+|y|)^{2p-2r} dm_n(y) \\
&= J(2\pi)^{-n/2} \int_0^\infty \int_{S^{n-1}} (1+t)^{2p-2r} t^{n-1} d\sigma_{n-1}(u) dt \\
&= \frac{2J}{\Gamma(n/2)} \int_0^\infty (1+t)^{2p-2r} t^{n-1} dt.
\end{aligned}$$

This integral is finite if and only if $2p - 2r + n - 1 < -1$, and we have assumed that $r > p + \frac{n}{2}$. Therefore,

$$\int_{\mathbb{R}^n} (1+|y|)^p |\widehat{F}(y)| dm_n(y) < \infty,$$

from which we get that $y^\alpha \widehat{F}(y)$ is in $L^1(\mathbb{R}^n)$ for $|\alpha| \le p$.

Define

$$F_\omega(x) = \int_{\mathbb{R}^n} \widehat{F} e_x dm_n, \qquad x \in \mathbb{R}^n.$$

(Note that F depends on ω.) $F, \widehat{F} \in L^1(\mathbb{R}^n)$ so by the inversion theorem we have $F(x) = F_\omega(x)$ for almost all $x \in \mathbb{R}^n$. $F_\omega \in C_0(\mathbb{R}^n)$. If $p \geq 1$, then we shall show that $F_\omega \in C^p(\Omega)$. Take e_k to be the standard basis for $\mathbb{R}^n$. For $1 \leq k_1 \leq n$ and $\epsilon \neq 0$,

$$\begin{aligned} \frac{F_\omega(x + \epsilon e_{k_1}) - F_\omega(x)}{\epsilon} &= \frac{1}{\epsilon} \int_{\mathbb{R}^n} \widehat{F}(y) \left(\exp(i\epsilon e_{k_1} \cdot y) - 1\right) \exp(ix \cdot y) dm_n(y) \\ &= \int_{\mathbb{R}^n} i y_{k_1} \widehat{F}(y) \frac{e^{i\epsilon y_{k_1}} - 1}{i\epsilon y_k} e_x(y) dm_n(y). \end{aligned}$$

But $\left| i y_{k_1} \widehat{F}(y) \frac{e^{i\epsilon y_{k_1}} - 1}{i\epsilon y_{k_1}} e_x(y) \right| \leq |y_{k_1} \widehat{F}(y)|$ and $y_{k_1} \widehat{F}(y)$ belongs to $L^1(\mathbb{R}^n)$ (supposing $p \geq 1$) so we can apply the dominated convergence theorem, which gives us

$$(D_{k_1} F_\omega)(x) = \lim_{\epsilon \to 0} \frac{F_\omega(x + \epsilon e_{k_1}) - F_\omega(x)}{\epsilon} = \int_{\mathbb{R}^n} i y_{k_1} \widehat{F}(y) e_x(y) dm_n(y).$$

From the above expression, it is apparent that $D_{k_1} F_\omega$ is continuous. This is true for all $1 \leq k_1 \leq n$, so $F_\omega \in C^1(\mathbb{R}^n)$. If $p \geq 2$, then $y_{k_1} y_{k_2} \widehat{F}(y)$ is in $L^1(\mathbb{R}^n)$ for any $1 \leq k_2 \leq n$, and repeating the above argument we get $F_\omega \in C^2(\mathbb{R}^n)$. In this way, $F_\omega \in C^p(\mathbb{R}^n)$.

For all $x \in \omega$, $f(x) = F(x)$, so $f(x) = F_\omega(x)$ for almost all $x \in \omega$. If ω' is an open subset of Ω whose closure is a compact subset of Ω and $\omega \cap \omega' \neq \emptyset$, then $F_\omega, F_{\omega'} \in C^p(\mathbb{R}^n)$ satisfy $f(x) = F_\omega(x)$ for almost all $x \in \omega$ and $f(x) = F_{\omega'}(x)$ for almost all $x \in \omega'$, so $F_\omega(x) = F_{\omega'}(x)$ for almost all $x \in \omega \cap \omega'$. Since $F_\omega, F_{\omega'}$ are continuous, this implies that $F_\omega(x) = F_{\omega'}(x)$ for all $x \in \omega \cap \omega'$. Thus, it makes sense to define $f_0(x) = F_\omega(x)$ for $x \in \omega$. Because every point in Ω has an open neighborhood of the kind ω and the restriction of f_0 to each ω belongs to $C^p(\omega)$, it follows that $f_0 \in C^p(\Omega)$. □