The symmetric difference metric

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Let (Ω, Σ, μ) be a probability space. For $A, B \in \Sigma$, define

$$d_\mu(A, B) = \mu(A \triangle B).$$

This is a pseudometric on Σ:

$$\begin{aligned} d_\mu(A, C) &= \mu(A \triangle C) \\ &= \mu((A \triangle B) \triangle (B \triangle C)) \\ &\leq \mu((A \triangle B) \cup (B \triangle C)) \\ &\leq \mu(A \triangle B) + \mu(B \triangle C) \\ &= d_\mu(A, B) + d_\mu(B, C). \end{aligned}$$

The relation $A \sim B$ if and only if $d_\mu(A, B) = 0$ is an equivalence relation on Σ, and $d_\mu([A], [B]) = d_\mu(A, B)$ is a metric on the collection Σ_μ of equivalence classes. We call d_μ the **symmetric difference metric**.

The following theorem shows that (Σ_μ, d_μ) is a complete metric space.[1]

Theorem 1. *If (Ω, Σ, μ) is a probability space, then (Σ_μ, d_μ) is a complete metric space.*

Proof. Suppose that $[B_n]$ is a Cauchy sequence in (Σ_μ, d_μ). As for any Cauchy sequence in a metric space, there is a subsequence $[A_n]$ of $[B_n]$ such that $d_\mu([A_k], [A_n]) < 2^{-n}$ for $k \geq n$. Define

$$E_n = \bigcup_{k \geq n} A_k.$$

[1] V. I. Bogachev, *Measure Theory*, volume I, p. 54, Theorem 1.12.16.

We have

$$\begin{aligned}
E_n \setminus A_n &= \bigcup_{k=n+1}^{\infty} (A_k \setminus A_n) \\
&= \bigcup_{k=n+1} \left(A_k \setminus \bigcup_{j=n}^{k-1} A_j \right) \\
&\subset \bigcup_{k=n+1} (A_k \setminus A_{k-1}) \\
&= \bigcup_{k=n}^{\infty} (A_{k+1} \setminus A_k),
\end{aligned}$$

hence

$$\mu(E_n \triangle A_n) = \mu(E_n \setminus A_n) \le \sum_{k=n}^{\infty} \mu(A_{k+1} \setminus A_k) < \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}. \qquad (1)$$

Now, define

$$A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} E_n,$$

for which

$$\begin{aligned}
\mu(A_n \triangle A) &= \mu(A_n \setminus A) \\
&= \mu\left(A_n \cap \left(\bigcap_{k=1}^{\infty} E_k \right)^c \right) \\
&= \mu\left(A_n \cap \bigcup_{k=1}^{\infty} E_k^c \right) \\
&= \mu\left(\bigcup_{k=1}^{\infty} (A_n \cap E_k^c) \right) \\
&= \lim_{k\to\infty} \mu(A_n \cap E_k^c) \\
&= \lim_{k\to\infty} \mu\left(\bigcap_{j\ge k} (A_n \setminus A_j) \right) \\
&\le \lim_{k\to\infty} \mu(A_n \setminus A_k) \\
&< 2^{-n}.
\end{aligned}$$

Using (1),

$$d_\mu(A_n, A) \le \mu(E_n \triangle A_n) + \mu(A_n \triangle A) < 2^{-n+1} + 2^{-n} = 3 \cdot 2^{-n},$$

showing that $[A_n]$ converges to $[A]$ as $n \to \infty$, and because $[A_n]$ is a subsequence of the Cauchy sequence $[B_n]$, it follows that $[B_n]$ converges to $[A]$ and therefore that (Σ_μ, d_μ) is a complete metric space. □

Lemma 2. *For* $A, B \in \Sigma$,

$$|\mu(A) - \mu(B)| \le \mu(A \triangle B).$$

Proof.

$$\begin{aligned} |\mu(A) - \mu(B)| &= |(\mu(A \setminus B) + \mu(A \cap B)) - (\mu(B \setminus A) + \mu(B \cap B))| \\ &= |\mu(A \setminus B) - \mu(B \setminus A)| \\ &\le \mu(A \setminus B) + \mu(B \setminus A) \\ &= \mu((A \setminus B) \cup (B \setminus A)) \\ &\le \mu(A \triangle B). \end{aligned}$$

□

The following theorem connects the metric space (Σ_μ, d_μ) with the Banach space $L^1(\mu)$.[2]

Theorem 3. *If* (Σ_μ, d_μ) *is separable then* $L^1(\mu)$ *is separable.*

[2] John B. Conway, *A Course in Abstract Analysis*, p. 90, Proposition 2.7.13.