The Fourier transform of spherical surface measure and radial functions

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1 Notation

For a topological space X, we denote by $\mathscr{B}_X$ the Borel σ-algebra of X. Let ρ_d be the Euclidean metric on $\mathbb{R}^d$ and let m_d be Lebesgue measure on $\mathbb{R}^d$.

2 Polar coordinates

Let $X = (0, \infty)$, which is a metric space with the metric inherited from $\mathbb{R}$. Define $\mu : \mathscr{B}_X \to [0, \infty]$ by

$$d\mu(r) = r^{d-1} dm_1(r).$$

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$. Define $S : \mathscr{P}(S^{d-1}) \to \mathscr{P}(\mathbb{R}^d)$ by

$$S(E) = \left\{ x \in \mathbb{R}^d : \frac{x}{|x|} \in E, 0 < |x| < 1 \right\}.$$

Namely, $S(E)$ is the sector subtended by the set E. S^{d-1} is a metric space with the metric inherited from $\mathbb{R}^d$, and if E is an open set in (S^{d-1}, ρ_d), then $S(E)$ is an open set in $\mathbb{R}^d$. For $E_\alpha \in \mathscr{P}(S^{d-1})$,

$$S\left(\bigcup E_\alpha\right) = \bigcup S(E_\alpha), \qquad S\left(\bigcap E_\alpha\right) = \bigcap S(E_\alpha),$$

and for $E, F \in \mathscr{P}(S^{d-1})$,

$$S(E \setminus F) = S(E) \setminus S(F).$$

Lemma 1.

$$S(\mathscr{B}_{S^{d-1}}) \subset \mathscr{B}_{\mathbb{R}^d}.$$

We define $\sigma_{d-1} : \mathscr{B}_{S^{d-1}} \to [0, \infty)$ by

$$\sigma_{d-1}(E) = d \cdot m_d(S(E)), \qquad E \in \mathscr{B}_{S^{d-1}}.$$

For $f : \mathbb{R}^d \to \mathbb{C}$ and $\gamma \in S^{d-1}$, define $f^\gamma : (0, \infty) \to \mathbb{C}$ by

$$f^\gamma(r) = f(r\gamma), \qquad r \in (0, \infty).$$

The following is proved in Stein and Shakarchi.[1]

Theorem 2. *If $f \in L^1(\mathbb{R}^d, m_d)$, then (i) for σ-almost all $\gamma \in S^{d-1}$ we have $f^\gamma \in L^1((0, \infty), \mu)$, (ii) the function*

$$\gamma \mapsto \int_0^\infty f^\gamma(r) d\mu(r)$$

belongs to $L^1(S^{d-1}, \sigma)$, and (iii)

$$\int_{\mathbb{R}^d} f(x) dm_d(x) = \int_{S^{d-1}} \left(\int_0^\infty f^\gamma(r) d\mu(r) \right) d\sigma(\gamma).$$

For $r \in (0, \infty)$, define $f_r : S^{d-1} \to \mathbb{C}$ by

$$f_r(\gamma) = f(r\gamma), \qquad \gamma \in S^{d-1}.$$

Theorem 3. *If $f \in L^1(\mathbb{R}^d, m_d)$, then (i) for μ-almost all $r \in (0, \infty)$ we have $f_r \in L^1(S^{d-1}, \sigma)$, (ii) the function*

$$r \mapsto \int_{S^{d-1}} f_r(\gamma) d\sigma(\sigma)$$

belongs to $L^1((0, \infty), \mu)$, and (iii)

$$\int_{\mathbb{R}^d} f(x) dm_d(x) = \int_0^\infty \left(\int_{S^{d-1}} f_r(\gamma) d\sigma(\gamma) \right) d\mu(r).$$

3 The Fourier transform of spherical surface measure

For real $\nu > -\frac{1}{2}$,

$$J_\nu(s) = \frac{\left(\frac{s}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^1 e^{isx}(1 - x^2)^{\nu - \frac{1}{2}} dx, \qquad s \in \mathbb{R}.$$

One checks that J_ν satisfies

$$J_\nu(-s) = e^{i\pi\nu} J_\nu(s), \qquad s \in \mathbb{R}.$$

[1]Elias M. Stein and Rami Shakarchi, *Real Analysis*, p. 280, Chapter 6, Theorem 3.4.

We remind ourselves of **spherical coordinates** for S^{d-1}. The Jacobian of the transformation

$$\begin{aligned}
\gamma_1 &= \cos\phi_1 \\
\gamma_2 &= \sin\phi_1 \cos\phi_2 \\
\gamma_3 &= \sin\phi_1 \sin\phi_2 \cos\phi_3 \\
&\cdots \\
\gamma_{d-1} &= \sin\phi_1 \sin\phi_2 \sin\phi_3 \cdots \sin\phi_{d-2} \cos\phi_{d-1} \\
\gamma_d &= \sin\phi_1 \sin\phi_2 \sin\phi_3 \cdots \sin\phi_{d-2} \sin\phi_{d-1},
\end{aligned}$$

with

$$0 \leq \phi_1, \dots, \phi_{d-2} \leq \pi, \qquad 0 \leq \phi_{d-1} \leq 2\pi,$$

is

$$J = \sin^{d-2}\phi_1 \sin^{d-3}\phi_2 \cdots \sin^2\phi_{d-3} \sin\phi_{d-2}.$$

Then, for $\xi = (\xi_1, 0, \dots, 0)$, $\xi_1 \neq 0$,

$$\begin{aligned}
\widehat{\sigma}_{d-1}(\xi) &= \int_{S^{d-1}} e^{-2\pi i \gamma \cdot \xi} d\sigma(\gamma) \\
&= \int_{\phi_1=0}^{\pi} \int_{\phi_2=0}^{\pi} \cdots \int_{\phi_{d-2}=0}^{\pi} \int_{\phi_{d-1}=0}^{2\pi} e^{-2\pi i \xi_1 \cos\phi_1} J d\phi_{d-1} d\phi_{d-2} \cdots d\phi_2 d\phi_1 \\
&= 2\pi \cdot \int_{\phi_1=0}^{\pi} e^{-2\pi i \xi_1 \cos\phi_1} \sin^{d-2}\phi_1 d\phi_1 \cdot \prod_{j=2}^{d-2} \int_{\phi_j=0}^{\pi} \sin^{d-j-1}\phi_j d\phi_j.
\end{aligned}$$

We work out that

$$\int_0^{\pi} \sin^k t dt = \frac{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}.$$

This gives

$$\prod_{j=2}^{d-2} \int_{\phi_j=0}^{\pi} \sin^{d-j-1}\phi_j d\phi_j = \prod_{j=2}^{d-2} \frac{\sqrt{\pi}\Gamma\left(\frac{d-j}{2}\right)}{\Gamma\left(\frac{d-j+1}{2}\right)} = \pi^{\frac{d-3}{2}} \frac{\Gamma\left(\frac{2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} = \frac{\pi^{\frac{d-3}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}.$$

With this we have, for $\xi = (\xi_1, 0, \dots, 0)$, $\xi_1 \neq 0$,

$$\widehat{\sigma}_{d-1}(\xi) = 2\pi \frac{\pi^{\frac{d-3}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{\pi} e^{-2\pi i \xi_1 \cos t} \sin^{d-2} t dt.$$

But doing the change of variable $x = \cos t$, for nonzero real s we have

$$\begin{aligned}\int_0^\pi e^{is\cos t}\sin^{d-2} t dt &= \int_0^\pi e^{is\cos t}(1-\cos^2 t)^{\frac{d-2}{2}} dt \\ &= \int_1^{-1} e^{isx}(1-x^2)^{\frac{d-2}{2}} \frac{-dx}{\sqrt{1-x^2}} \\ &= \int_{-1}^1 e^{isx}(1-x^2)^{\frac{d}{2}-1-\frac{1}{2}} dx \\ &= \frac{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)\sqrt{\pi}}{\left(\frac{s}{2}\right)^{\frac{d}{2}-1}} J_{\frac{d}{2}-1}(s).\end{aligned}$$

Thus, taking $s = -2\pi\xi_1$,

$$\begin{aligned}\widehat{\sigma}_{d-1}(\xi) &= 2\pi \frac{\pi^{\frac{d-3}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \frac{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)\sqrt{\pi}}{\left(\frac{-2\pi\xi_1}{2}\right)^{\frac{d}{2}-1}} J_{\frac{d}{2}-1}(-2\pi\xi_1) \\ &= 2\pi\cdot(-\xi_1)^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(-2\pi\xi_1).\end{aligned}$$

For $\xi_1 < 0$ this is

$$\widehat{\sigma}_{d-1}(\xi) = 2\pi|\xi|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(2\pi|\xi|).$$

In general, take nonzero $\xi \in \mathbb{R}^d$. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be the rotation that sends ξ ti $(0, \ldots, 0, -|\xi|)$. Since $\sigma_{d-1} \circ T = \sigma_{d-1}$ (namely, surface measure σ_{d-1} is invariant under rotations),

$$\widehat{\sigma}_{d-1}(\xi) = \widehat{\sigma}_{d-1}((0, \ldots, 0, -|\xi|)) = 2\pi|\xi|^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(2\pi|\xi|).$$

For real $\nu > -\frac{1}{2}$, we use the following asymptotic formula for $J_\nu(s)$:[2]

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O(s^{-3/2}), \qquad s \to +\infty.$$

We get from this that

$$|\widehat{\sigma}_{d-1}(\xi)| = O(|\xi|^{-\frac{d}{2}+\frac{1}{2}}), \qquad |\xi| \to \infty.$$

4 The Fourier transform of radial functions

A function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be **radial** if there is a function $f_0 : [0, \infty) \to \mathbb{C}$ such that

$$f(x) = f_0(|x|), \qquad x \in \mathbb{R}^d.$$

[2]Elias M. Stein and Rami Shakarchi, *Complex Analysis*, p. 319, Appendix A.1.

For $f \in L^1(\mathbb{R}^d)$, Using polar coordinates we determine the Fourier transform of a radial function. For $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx \\
&= \int_0^\infty \left(\int_{S^{d-1}} e^{-2\pi i r\sigma\cdot\xi} f(r\sigma)d\sigma(\gamma) \right) d\mu(r) \\
&= \int_0^\infty \left(\int_{S^{d-1}} e^{-2\pi i r\gamma\cdot\xi} d\sigma(\gamma) \right) f_0(r)d\mu(r) \\
&= \int_0^\infty \widehat{\sigma}_{d-1}(r\xi) f_0(r)d\mu(r) \\
&= \int_0^\infty 2\pi(r|\xi|)^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(2\pi r|\xi|) f_0(r)d\mu(r) \\
&= 2\pi|\xi|^{-\frac{d}{2}+1} \int_0^\infty r^{-\frac{d}{2}+1} J_{\frac{d}{2}-1}(2\pi r|\xi|) f_0(r)d\mu(r).
\end{aligned}$$