The Gottschalk-Hedlund theorem, cocycles, and small divisors

Jordan Bell

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1 Introduction

This note consists of my working through details in the paper *Resonances and small divisors* by Étienne Ghys.[1] Aside from containing mathematics, Ghys makes thoughtful remarks about the history of physics, unlike the typically thoughtless statements people make about the Ptolemaic system. He insightfully states “Kepler’s zeroth law”: “If the orbit of a planet is bounded, then it is periodic.” I can certainly draw a three dimensional bounded curve that is not closed, but that curve is not the orbit of a planet. It is also intellectually lazy to scorn Kepler’s correspondence between orbits and the Platonic solids (“Kepler’s fourth law”).

2 Almost periodic functions

Suppose that $f : \mathbb{R} \to \mathbb{C}$ is continuous. For $\epsilon > 0$, we call $T \in \mathbb{R}$ an **ϵ-period of f** if

$$|f(t+T) - f(t)| < \epsilon, \qquad t \in \mathbb{R}.$$

T is a period of f if and only if it is an ϵ-period for all $\epsilon > 0$.

We say that f is **almost periodic** if for every $\epsilon > 0$ there is some $M_\epsilon > 0$ such that if I is an interval of length $> M_\epsilon$ then there is an ϵ-period in I.

If f is periodic, then there is some $M > 0$ such that if I an interval of length $> M$ then at least one multiple T of M lies in I, and hence for any $t \in \mathbb{R}$ we have $f(t+T) - f(t) = f(t) - f(t) = 0$. Thus, for every $\epsilon > 0$, if I is an interval of length $> M$ then there is an ϵ-period in I. Therefore, with a periodic function, the length of the intervals I need not depend on ϵ, while for an almost periodic function they may.

[1] `http://perso.ens-lyon.fr/ghys/articles/resonancesmall.pdf`

3 The Gottschalk-Hedlund theorem

The **Gottschalk-Hedlund theorem** is stated and proved in Katok and Hasselblatt.[2] The following case of the Gottschalk-Hedlund theorem is from Ghys. We denote by

$$\pi_1 : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z}, \qquad \pi_2 : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$$

the projection maps.

Theorem 1 (Gottschalk-Hedlund theorem)**.** *Suppose that* $u : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ *is continuous, that*

$$\int_0^1 u(x)dx = 0,$$

that $x_0 \in \mathbb{R}/\mathbb{Z}$*, and that* α *is irrational. If there is some* C *such that*

$$\left|\sum_{k=0}^{n} u(x_0 + k\alpha)\right| \leq C, \qquad n \geq 0, \tag{1}$$

then there is a continuous function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ *such that*

$$u(x) = v(x + \alpha) - v(x), \qquad x \in \mathbb{R}/\mathbb{Z}.$$

Proof. Say there is some $C > 0$ satisfying (1). Define $g : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ by

$$g(x, y) = (x + \alpha, y + u(x)), \qquad x \in \mathbb{R}/\mathbb{Z}.$$

For $n \geq 0$,

$$g^n(x_0, 0) = \left(x_0 + n\alpha, \sum_{k=0}^{n} u(x_0 + k\alpha)\right)$$

The set $\{g^n(x_0, 0) : n \geq 0\}$, namely the orbit of $(x_0, 0)$ under g, is contained in $\mathbb{R}/\mathbb{Z} \times [-C, C]$. Let K be the closure of this orbit. Because K is a metrizable topological space, for $(x, y) \in K$ there is a sequence $a(n)$ such that $g^{a(n)}(x_0, 0) \to (x, y)$. As g is continuous we get $g^{a(n)+1}(x_0, 0) \to g(x, y)$, which implies that $g(x, y) \in K$. This shows that K is invariant under g. Let $\mathscr{K}$ be the collection of nonempty compact sets contained in K and invariant under g. Thus $K \in \mathscr{K}$, so $\mathscr{K}$ is nonempty. We order $\mathscr{K}$ by $A \prec B$ when $A \subset B$. If $\mathscr{C} \subset \mathscr{K}$ is a chain, let $C_0 = \bigcap_{C \in \mathscr{C}} C$. It follows from K being compact that C_0 is nonempty, hence $C_0 \in \mathscr{K}$ and is a lower bound for the chain $\mathscr{C}$. Since every chain in $\mathscr{K}$ has a lower bound in $\mathscr{K}$, by Zorn's lemma there exists a minimal element M in $\mathscr{K}$: for every $A \in \mathscr{K}$ we have $M \prec A$, i.e. $M \subset A$. To say that M is invariant under g means that $g(M) \subset M$, and M being a nonempty compact set contained in K implies that $g(M)$ is a nonempty compact set contained in K, hence by the minimality of M we obtain $g(M) = M$.

[2] Anatole Katok and Boris Hasselblat, *Introduction to the Modern Theory of Dynamical Systems*, p. 102, Theorem 2.9.4.

The set M is nonempty, so take $(x,y) \in M$. Because M is invariant under g, $\{g^n(x,y) : n \geq 0\} \subset M$. The set

$$\pi_1\{g^n(x,y) : n \geq 0\} = \{x + n\alpha : n \geq 0\}$$

is dense in $\mathbb{R}/\mathbb{Z}$, hence $\pi_1(M)$ is dense in $\mathbb{R}/\mathbb{Z}$. Moreover, M being compact implies that $\pi_1(M)$ is closed, so $\pi_1(M) = \mathbb{R}/\mathbb{Z}$.

For $t \in \mathbb{R}$, define $\tau_t : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ by $\tau_t(x,y) = (x, y+t)$. For any t,

$$\tau_t \circ g(x,y) = \tau_t(x+\alpha, y+u(x)) = (x+\alpha, y+u(x)+t) = g(x, y+t) = g \circ \tau_t(x,y),$$

so $\tau_t \circ g = g \circ \tau_t$. Hence, if $A \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ and $g(A) \subset A$, then $g(\tau_t(A)) = \tau_t \circ g(A) \subset \tau_t(A)$, namely, if A is invariant under g then $\tau_t(A)$ is invariant under g. Therefore $\tau_t(M)$ is invariant under g, and so $M \cap \tau_t(M)$ is invariant under g. This intersection is compact and is contained in K, so either $M \cap \tau_t(M) = \emptyset$ or by the minimality of M, $M \cap \tau_t(M) = M$. Suppose by contradiction that for some nonzero t, $M \cap \tau_t(M) = M$. Then using $g(M) = M$ we get $\tau_t(M) = M$, and hence for any positive integer k we have $\tau_{kt}(M) = \tau_t^k(M) = M$. But because M is compact, $\pi_2(M)$ is contained in some compact interval I, and then there is some positive integer k such that $\pi_2(\tau_{kt}(M))$ is not contained in I, a contradiction. Therefore, when $t \neq 0$ we have $M \cap \tau_t(M) = \emptyset$. Let $x \in \mathbb{R}/\mathbb{Z}$. If there were distinct $y_1, y_2 \in \mathbb{R}$ such that $(x,y_1), (x,y_2) \in M$, then with $t = y_2 - y_1 \neq 0$ we get $\tau_t(x,y_1) = (x,y_2) \in M$, contradicting $M \cap \tau_t(M) = \emptyset$. This shows that for each $x \in \mathbb{R}/\mathbb{Z}$ there is a unique $y \in \mathbb{R}$ such that $(x,y) \in M$, and we denote this y by $v(x)$, thus defining a function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. Then M is the graph of v, and because M is compact, it follows that the function v is continuous. Let $(x, v(x)) \in M$. As M is invariant under g,

$$(x+\alpha, v(x) + u(x)) = g(x, v(x)) \in M,$$

and as M is the graph of v we get $v(x) + u(x) = v(x+\alpha)$ and hence $v(x+\alpha) - v(x) = u(x)$, completing the proof. □

4 Cohomology

In this section I am following Tao.[3] Suppose that a group $(G, \cdot)$ acts on a set X and that $(A, +)$ is an abelian group. A **cocycle** is a function $\rho : G \times X \to A$ such that

$$\rho(gh, x) = \rho(h, x) + \rho(g, hx), \qquad g, h \in G, \quad x \in X. \tag{2}$$

If $F : X \to A$ is a function, we call the function $\rho(g,x) = F(gx) - F(x)$ a **coboundary**. This satisfies

$$\rho(gh,x) - \rho(g,hx) = F((gh)x) - F(x) - F(g(hx)) + F(hx) = F(hx) - F(x) = \rho(h,x),$$

[3] Terence Tao, *Cohomology for dynamical systems*, http://terrytao.wordpress.com/2008/12/21/cohomology-for-dynamical-systems/

showing that a coboundary is a cocycle. We now show how to fit the notions of cocycle and coboundary into a general sitting of cohomology. We show that they correspond respectively to a 1-cocycle and a 1-coboundary.

For $n \geq 0$, an n-**simplex** is an element of $G^n \times X$, i.e., a thing of the form $(g_1, \ldots, g_n, x)$, for $g_1, \ldots, g_n \in G$ and $x \in X$. We denote by $C_n(G, X)$ the free abelian group generated by the collection of all n-simplices, and an element of $C_n(G, X)$ is called an n-**chain**. In particular, the elements of $C_0(G, X)$ are formal $\mathbb{Z}$-linear combinations of elements of X. For $n < 0$, we define $C_n(G, X)$ to be the trivial group.

For $n > 0$, we define the **boundary map** $\partial : C_n(G, X) \to C_{n-1}(G, X)$ by

$$\begin{aligned} \partial(g_1, \ldots, g_n, x) &= (g_1, \ldots, g_{n-1}, g_n x) \\ &\quad + \sum_{k=1}^{n-1} (-1)^{n-k} (g_1, \ldots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \ldots, g_n, x) \\ &\quad + (-1)^n (g_2, \ldots, g_n, x). \end{aligned}$$

For $n \leq 0$ we define $\partial : C_n(G, X) \to C_{n-1}(G, X)$ to be the trivial map. If $n \leq 1$ then of course $\partial^2 = 0$. If $n \geq 2$, one writes out $\partial^2(g_1, \ldots, g_n, x)$ and checks that it is equal to 0, and hence that $\partial^2 = 0$. Thus the sequence of abelian groups $C_n(G, X)$ and the boundary maps $\partial : C_n(G, X) \to C_{n-1}(G, X)$ are a **chain complex**.

We denote the kernel of $\partial : C_n(G, X) \to C_{n-1}(G, X)$ by $Z_n(G, X)$, and elements of $Z_n(G, X)$ are called n-**cycles**. We denote the image of $\partial : C_{n+1}(G, X) \to C_n(G, X)$ by $B_n(G, X)$, and elements of $B_n(G, X)$ are called n-**boundaries**. Because $\partial^2 = 0$, an n-boundary is an n-cycle. $Z_n(G, X)$ and $B_n(G, X)$ are abelian groups and $B_n(G, X)$ is contained in $Z_n(G, X)$, and we write

$$H_n(G, X) = Z_n(G, X)/B_n(G, X),$$

and call $H_n(G, X)$ the n**th homology group**.

We define $C^n(G, X, A) = \mathrm{Hom}(C_n(G, X), A)$, which is an abelian group. Elements of $C^n(G, X, A)$ are called n-**cochains**. That is, an n-cochain is a group homomorphism $C_n(G, X) \to A$. Because $C_n(G, X)$ is a free abelian group generated by the collection of all n-simplices, an n-cochain is determined by the values it assigns to n-simplices. We thus identity n-cochains with functions $G^n \times X \to A$.

We define the **coboundary map** $\delta : C^{n-1}(G, X, A) \to C^n(G, X, A)$ by

$$(\delta F)(c) = F(\partial c), \qquad F \in C^{n-1}(G, X, A), c \in C_n(G, X).$$

Explicitly, for $F \in C^{n-1}(G, X, A)$ and for an n-simplex $(g_1, \ldots, g_n, x)$,

$$\begin{aligned} (\delta F)(g_1, \ldots, g_n, x) &= F(\partial(g_1, \ldots, g_n, x)) \\ &= F(g_1, \ldots, g_{n-1}, g_n x) \\ &\quad + \sum_{k=1}^{n-1} (-1)^{n-k} F(g_1, \ldots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \ldots, g_n, x) \\ &\quad + (-1)^n F(g_2, \ldots, g_n, x). \end{aligned}$$

For $F \in C^{n-2}(G, X, A)$, write $G = \delta F$ and take and $c \in C_n(G, X)$. Then,

$$(\delta^2 F)(c) = (\delta G)(c) = G(\partial c) = (\delta F)(\partial c) = F(\partial^2 c) = F(0) = 0,$$

showing that $\delta^2 = 0$. Thus the sequence of abelian groups $C^n(G, X, A)$ and the coboundary maps $\delta : C^{n-1}(G, X, A) \to C^n(G, X, A)$ are a **cochain complex**.

We denote the kernel of $\delta : C^n(G, X, A) \to C^{n+1}(G, X, A)$ by $Z^n(G, X, A)$, and elements of $Z^n(G, X, A)$ are called n**-cocycles**. We denote the image of $\delta : C^{n-1}(G, X, A) \to C^n(G, X, A)$ by $B^n(G, X, A)$, and elements of $B^n(G, X, A)$ are called n**-coboundaries**. Because $\delta^2 = 0$, an n-coboundary is an n-cocycle. $Z^n(G, X, A)$ and $B^n(G, X, A)$ are abelian groups and $B^n(G, X, A)$ is contained in $Z^n(G, X, A)$, and we write

$$H^n(G, X, A) = Z^n(G, X, A)/B^n(G, X, A),$$

which we call the n**th cohomology group**.

Take $n = 1$. We identify $C^1(G, X, A)$, the group of 1-chains, with functions $G \times X \to A$. For $\rho \in C^1(G, X, A)$, to say that ρ is a 1-cocycle is equivalent to saying that for any $(g, h, x) \in G^2 \times X$, $(\delta\rho)(g, h, x) = 0$, i.e. $\rho(g, hx) - \rho(gh, x) + \rho(h, x) = 0$, i.e.

$$\rho(gh, x) = \rho(h, x) + \rho(g, hx).$$

To say that ρ is a 1-coboundary is equivalent to saying that there is a 0-chain F (a function $X \to A$) such that $\rho = \delta F$, i.e., for any $(g, x) \in G \times X$,

$$\rho(g, x) = (\delta F)(g, x) = F(gx) - F(x).$$

5 Small divisors

Suppose that $u : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be C^∞ and satisfies

$$\int_0^1 u(x)dx = 0.$$

For each $n \in \mathbb{Z}$, let

$$\widehat{u}(n) = \int_0^1 e^{-2\pi inx} u(x)dx.$$

We have $\widehat{u}(0) = 0$. For any $x \in \mathbb{R}/\mathbb{Z}$,

$$u(x) = \sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{2\pi inx},$$

and $\sum_{n \in \mathbb{Z}} |\widehat{u}(n)| < \infty$; for these statements to be true it suffices merely that u be C^β for some $\beta > \frac{1}{2}$.

Let α be irrational. We shall find conditions under which there exists a continuous function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that

$$u(x) = v(x + \alpha) - v(x), \qquad x \in \mathbb{R}/\mathbb{Z}. \tag{3}$$

Supposing that for each x, $v(x)$ is equal to its Fourier series evaluated at x and that its Fourier series converges absolutely,

$$v(x) = \sum_{n \in \mathbb{Z}} \widehat{v}(n) e^{2\pi i n x},$$

then for each $x \in \mathbb{R}/\mathbb{Z}$,

$$v(x+\alpha) - v(x) = \sum_{n \in \mathbb{Z}} \widehat{v}(n) \left(e^{2\pi i n (x+\alpha)} - e^{2\pi i n x} \right) = \sum_{n \in \mathbb{Z}} \widehat{v}(n)(e^{2\pi i n \alpha} - 1) e^{2\pi i n x}.$$

Then using $u(x) = v(x+\alpha) - v(x)$ we obtain

$$\widehat{u}(n) = \widehat{v}(n)(e^{2\pi i n \alpha} - 1), \qquad n \in \mathbb{Z},$$

or,

$$\widehat{v}(n) = \frac{\widehat{u}(n)}{e^{2\pi i n \alpha} - 1}, \qquad n \neq 0; \tag{4}$$

because α is irrational, the denominator of the right-hand side is indeed nonzero for $n \neq 0$. The value of $\widehat{v}(0)$ is not determined so far. We shall find conditions under which the continuous function v we desire can be defined using (4).

A real number β is said to be **Diophantine** if there is some $r \geq 2$ and some $C > 0$ such that for all $q > 0$ and $p \in \mathbb{Z}$,

$$\left| \beta - \frac{p}{q} \right| > C q^{-r}. \tag{5}$$

It is immediate that a Diophantine number is irrational. Suppose that α satisfies (5). Let $n \neq 0$ and let p_n be the integer nearest $n\alpha$. Then

$$\begin{aligned} |e^{2\pi i n \alpha} - 1| &= |e^{2\pi i (n\alpha - p_n)} - 1| \\ &\geq \frac{2}{\pi} |2\pi (n\alpha - p_n)| \\ &= 4|n\alpha - p_n| \\ &= 4|n| \left| \alpha - \frac{p_n}{n} \right| \\ &> 4|n| \cdot C|n|^{-r} \\ &= 4C|n|^{-r+1}. \end{aligned}$$

Because $u \in C^\infty$, it is straightforward to prove that for each nonnegative integer k there is some $C_k > 0$ such that

$$|\widehat{u}(n)| \leq C_k |n|^{-k}, \qquad n \neq 0.$$

Therefore, for each nonnegative integer k, using (4) we have

$$|\widehat{v}(n)| = \frac{|\widehat{u}(n)|}{|e^{2\pi i n \alpha} - 1|} < C_k |n|^{-k} \cdot \frac{1}{4C|n|^{-r+1}} = \frac{C_k}{4C} |n|^{r-k-1}, \qquad n \neq 0. \tag{6}$$

One can prove that if h_n are complex numbers satisfying (6) then the function defined by

$$h(x) = \sum_{n \in \mathbb{Z}} h_n e^{2\pi i n x}, \qquad x \in \mathbb{R}/\mathbb{Z}$$

is C^∞. Therefore, we have established that if α is Diophantine then there is some $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ that is C^∞ and that satisfies (3).

On the other hand, for $\alpha = \sum_{n=1}^\infty 10^{-n!}$, Ghys constructs a C^∞ function $u : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that there is no continuous function $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfying $u(x) = v(x + \alpha) - v(x)$ for all $x \in \mathbb{R}/\mathbb{Z}$.