Singular integral operators and the Riesz transform

Jordan Bell

November 17, 2017

1 Calderón-Zygmund kernels

Let ω_{n-1} be the measure of S^{n-1}. It is

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Let v_n be the measure of the unit ball in $\mathbb{R}^n$. It is

$$v_n = \frac{\omega_{n-1}}{n} = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

For $k, N \geqslant 0$ and $\phi \in C^\infty(\mathbb{R}^n)$ let

$$p_{k,N}(\phi) = \max_{|\alpha| \leqslant k} \sup_{x \in \mathbb{R}^n} (1+|x|)^N |(\partial^\alpha \phi)(x)|.$$

A Borel measurable function $K : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ is called a **Calderón-Zygmund kernel** if there is some B such that

1. $|K(x)| \leqslant B|x|^{-n}$, $x \neq 0$
2. $\int_{|x| \geqslant 2|y|} |K(x-y) - K(x)| dx \leqslant B$, $y \neq 0$
3. $\int_{R_1 < |x| < R_2} K(x) dx = 0$, $0 < R_1 < R_2 < \infty$.

The following lemma gives a tractable condition under which Condition 2 is satisfied.[1]

Lemma 1. *If* $|(\nabla K)(x)| \leqslant C|x|^{-n-1}$ *for all* $x \neq 0$ *then for* $y \neq 0$,

$$\int_{|x| \geqslant 2|y|} |K(x-y) - K(x)| dx \leqslant v_n 2^n C.$$

[1] Camil Muscalu and Wilhelm Schlag, *Classical and Multilinear Harmonic Analysis*, volume I, p. 167, Lemma 7.2.

Proof. For $|x| > 2|y| > 0$, if $0 \leqslant t \leqslant 1$ then

$$|x - ty| \geqslant |x| - t|y| \geqslant |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}.$$

Write $f(t) = K(x-ty)$, for which $f'(t) = -(\nabla K)(x-ty)\cdot y$. By the fundamental theorem of calculus,

$$K(x-y) - K(x) = f(1) - f(0) = \int_0^1 f'(t)dt = -\int_0^1 (\nabla K)(x-ty)\cdot y dt,$$

thus

$$|K(x-y) - K(x)| \leqslant \int_0^1 |(\nabla K)(x-ty)||y|dt \leqslant C|y| \int_0^1 |x-ty|^{-n-1}dt.$$

Then using $|x - ty| > \frac{|x|}{2}$,

$$|K(x-y) - K(x)| \leqslant C|y| \left(\frac{|x|}{2}\right)^{-n-1} = 2^{n+1}C|y||x|^{-n-1}.$$

For $|y| > 0$, using spherical coordinates,[2]

$$\begin{aligned}\int_{|x|\geqslant 2|y|} |K(x-y) - K(x)|dx &\leqslant \int_{|x|\geqslant 2|y|} 2^{n+1}C|y||x|^{-n-1}dx \\ &= 2^{n+1}C|y| \int_{2|y|}^{\infty} \left(\int_{S^{n-1}} |r\gamma|^{-n-1}d\sigma(\gamma)\right) r^{n-1}dr \\ &= v_n 2^{n+1}C|y| \int_{2|y|}^{\infty} r^{-2}dr \\ &= v_n 2^{n+1}C|y| \cdot \frac{1}{2|y|} \\ &= v_n 2^n C.\end{aligned}$$

□

For a Calderón-Zygmund kernel K, for $f \in \mathscr{S}(\mathbb{R}^n)$, for $x \in \mathbb{R}^n$, and for $\epsilon > 0$, using Condition 3 with $R_1 = \epsilon$ and $R_2 = 1$,[3]

$$\begin{aligned}&\int_{|x-y|\geqslant \epsilon} K(x-y)f(y)dy \\ &= \int_{\epsilon\leqslant|x-y|\leqslant 1} K(x-y)(f(y)-f(x))dy + \int_{|x-y|\geqslant 1} K(x-y)f(y)dy.\end{aligned}$$

By Condition 1 there is some B such that $|K(x)| \leqslant B|x|^{-n}$, and combining this with $|f(y) - f(x)| \leqslant \|\nabla f\|_\infty |y-x|$,

$$|K(x-y)(f(y)-f(x))| \leqslant B\|\nabla f\|_\infty |y-x|^{-n+1},$$

[2] See http://individual.utoronto.ca/jordanbell/notes/sphericalmeasure.pdf

[3] https://math.aalto.fi/~parissi1/notes/harmonic.pdf, p. 115, Lemma 6.15.

which is integrable on $\{|x-y| \leqslant 1\}$. Then by the dominated convergence theorem,

$$\lim_{\epsilon \to 0} \int_{\epsilon \leqslant |x-y| \leqslant 1} K(x-y)(f(y)-f(x))dy = \int_{|x-y| \leqslant 1} K(x-y)(f(y)-f(x))dy.$$

Lemma 2. *For a Calderón-Zygmund kernel K, for $f \in \mathscr{S}(\mathbb{R}^n)$, and for $x \in \mathbb{R}^n$, the limit*

$$\lim_{\epsilon \to 0} \int_{|x-y| \geqslant \epsilon} K(x-y)f(y)dy$$

exists.

2 Singular integral operators

For a Calderón-Zygmund kernel K on $\mathbb{R}^n$, for $f \in \mathscr{S}(\mathbb{R}^n)$, and for $x \in \mathbb{R}^n$, let

$$(Tf)(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geqslant \epsilon} K(x-y)f(y)dy.$$

We call T a **singular integral operator**. By Lemma 2 this makes sense.

We prove that singular integral operators are $L^2 \to L^2$ bounded.[4]

Theorem 3. *There is some C_n such that for any Calderón-Zygmund kernel K and any $f \in \mathscr{S}(\mathbb{R}^n)$,*

$$\|Tf\|_2 \leqslant C_n B \|f\|_2 .$$

Proof. For $0 < r < s < \infty$ and for $\xi \in \mathbb{R}^n$ define

$$m_{r,s}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} 1_{r<|x|<s}(x) K(x) dx.$$

Take $r < |\xi|^{-1} < s$, for which

$$m_{r,s}(\xi) = \int_{r<|x|<|\xi|^{-1}} e^{-2\pi i x \cdot \xi} K(x) dx + \int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x \cdot \xi} K(x) dx.$$

For the first integral, using Condition 3 with $R_1 = r$ and $R_2 = |\xi|^{-1}$ and then

[4] Camil Muscalu and Wilhelm Schlag, *Classical and Multilinear Harmonic Analysis*, volume I, p. 168, Proposition 7.3; Elias M. Stein, *Singular Integrals and Differentiability Properties of Functions*, p. 35, §3.2, Theorem 2; `http://math.uchicago.edu/~may/REU2013/REUPapers/Talbut.pdf`

using Condition 1,

$$\begin{aligned}\left|\int_{r<|x|<|\xi|^{-1}} e^{-2\pi i x\cdot\xi} K(x)dx\right| &= \left|\int_{r<|x|<|\xi|^{-1}} (e^{-2\pi i x\cdot\xi}-1)K(x)dx\right| \\ &\leqslant \int_{|x|<|\xi|^{-1}} |e^{-2\pi i x\cdot\xi}-1||K(x)|dx \\ &\leqslant \int_{|x|<|\xi|^{-1}} 2\pi|x||\xi||K(x)|dx \\ &\leqslant 2\pi|\xi|\int_{|x|<|\xi|^{-1}} B|x|^{-n+1}dx \\ &= 2\pi|\xi|\cdot v_n|\xi|^{-1}.\end{aligned}$$

For the second integral, let $z = \frac{\xi}{2|\xi|^2}$, and

$$\begin{aligned}\int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x\cdot\xi} K(x)dx &= -\int_{|\xi|^{-1}<|x|<s} e^{-2\pi i (x+z)\cdot\xi} K(x)dx \\ &= -\int_{|\xi|^{-1}<|x-z|<s} e^{-2\pi i x\cdot\xi} K(x-z)dx.\end{aligned}$$

Let

$$\begin{aligned}R &= \int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x\cdot\xi} K(x-z)dx - \int_{|\xi|^{-1}<|x-z|<s} e^{-2\pi i x\cdot\xi} K(x-z)dx \\ &= -\int_{|\xi|^{-1}<|x+z|<s} e^{-2\pi i x\cdot\xi} K(x)dx + \int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x\cdot\xi} K(x)dx,\end{aligned}$$

with which

$$\int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x\cdot\xi} K(x)dx = \frac{1}{2}\int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x\cdot\xi}(K(x)-K(x-z))dx + \frac{R}{2}.$$

On the one hand, applying Condition 2, as $|z| = \frac{1}{2|\xi|}$,

$$\begin{aligned}\left|\int_{|\xi|^{-1}<|x|<s} e^{-2\pi i x\cdot\xi}(K(x)-K(x-z))dx\right| &\leqslant \int_{|x|>|\xi|^{-1}} |K(x)-K(x-z)|dx \\ &= \int_{|x|>2|z|} |K(x)-K(x-z)|dx \\ &\leqslant B.\end{aligned}$$

On the other hand, let

$$D = D_1 \triangle D_2 = \{x : |\xi|^{-1} < |x+z| < s\} \triangle \{x : |\xi|^{-1} < |x| < s\}.$$

For $x \in D_1$ we have

$$|x| \geqslant |x+z| - |z| > \frac{1}{|\xi|} - \frac{1}{2|\xi|} = \frac{1}{2|\xi|},$$

and for $x \in D_2$ we have $|x| > \frac{1}{|\xi|}$, so for $x \in D$,

$$|x| > \frac{1}{2|\xi|}.$$

Applying Condition 1,

$$|K(x)| \leqslant B|x|^{-n} < 2^n B|\xi|^n.$$

Furthermore, for $x \in D_1 \backslash D_2$ we have $|x| \leqslant |\xi|^{-1}$, and for $x \in D_2 \backslash D_1$ we have

$$|x| \leqslant |x+z| + |z| = |x+z| + \frac{1}{2|\xi|} \leqslant \frac{1}{|\xi|} + \frac{1}{2|\xi|} = \frac{3}{2|\xi|}.$$

Hence

$$D \subset \left\{ x : \frac{1}{2|\xi|} < |x| \leqslant \frac{3}{2|\xi|} \right\},$$

so

$$\lambda(D) \leqslant \left(\frac{3}{2|\xi|} \right)^n v_n = \left(\frac{3}{2} \right)^n |\xi|^{-n} v_n.$$

Therefore

$$|R| \leqslant 2^n B |\xi|^n \cdot \left(\frac{3}{2} \right)^n |\xi|^{-n} v_n = 3^n v_n B$$

and then

$$\left| \int_{|\xi|^{-1} < |x| < s} e^{-2\pi i x \cdot \xi} K(x) dx \right| \leqslant \frac{1}{2} B + \frac{1}{2} \cdot 3^n v_n B,$$

and finally[5]

$$|m_{r,s}(\xi)| \leqslant 2\pi v_n + \frac{1}{2} B + \frac{1}{2} \cdot 3^n v_n B = C_n B.$$

Define

$$(T_{r,s} f)(x) = \int_{\mathbb{R}^n} 1_{r<|y|<s}(y) K(y) f(x-y) dy, \qquad x \in \mathbb{R}^n.$$

Then

$$\begin{aligned}
\widehat{T_{r,s} f}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \left(\int_{\mathbb{R}^n} 1_{r<|y|<s}(y) K(y) f(x-y) dy \right) dx \\
&= \int_{\mathbb{R}^n} 1_{r<|y|<s}(y) K(y) \left(\int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x-y) dx \right) dy \\
&= \int_{\mathbb{R}^n} 1_{r<|y|<s}(y) K(y) e^{-2\pi i y \cdot \xi} \widehat{f}(\xi) dy \\
&= m_{r,s}(\xi) \widehat{f}(\xi),
\end{aligned}$$

[5]The way I organize the argument, I want to use $\|m_{r,s}\|_\infty \leqslant C_n B$, while we have only obtained this bound for $r < |\xi|^{-1} < s$. To make the argument correct I may need to do things in a different order, e.g. apply Fatou's lemma and then use an inequality instead of using an inequality and then apply Fatou's lemma.

and so

$$\left\|\widehat{T_{r,s}f}\right\|_2^2 = \int_{\mathbb{R}^n} |\widehat{T_{r,s}f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |m_{r,s}(\xi)\widehat{f}(\xi)|^2 d\xi \leqslant \|m_{r,s}\|_\infty^2 \left\|\widehat{f}\right\|_2^2,$$

by Plancherel's theorem and the inequality we got for $|m_{r,s}(\xi)|$,

$$\|T_{r,s}f\|_2^2 \leqslant \|m_{r,s}\|_\infty^2 \|f\|_2^2 \leqslant (C_n B)^2 \|f\|_2^2.$$

For each $x \in \mathbb{R}^n$, $(T_{r,s}f)(x) \to (Tf)(x)$ as $r \to 0$ and $s \to \infty$, and thus using Fatou's lemma,

$$\int_{\mathbb{R}^n} |(Tf)(x)|^2 dx \leqslant \liminf_{r\to 0, s\to\infty} \int_{\mathbb{R}^n} |(T_{r,s}f)(x)|^2 dx = (C_n B)^2 \|f\|_2^2.$$

That is,

$$\|Tf\|_2 \leqslant C_n B \|f\|_2.$$

□

3 The Riesz transform

Let

$$c_n = \frac{1}{\pi v_{n-1}} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

For $1 \leqslant j \leqslant n$, let

$$K_j(x) = c_n \frac{x_j}{|x|^{n+1}}.$$

This is a Calderón-Zygmund kernel. For $\phi \in \mathscr{S}(\mathbb{R}^n)$ define

$$(R_j\phi)(x) = \lim_{\epsilon\to 0} \int_{|y-x|\geqslant\epsilon} K_j(x-y)\phi(y)dy = \lim_{\epsilon\to 0}\int_{|y|\geqslant\epsilon} K_j(y)\phi(x-y)dy.$$

We call each R_j, $1 \leqslant j \leqslant n$, a **Riesz transform**.

For $1 \leqslant j \leqslant n$ define $W_j : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle W_j, \phi\rangle = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \lim_{\epsilon\to 0} \int_{|y|\geqslant\epsilon} K_j(y)\phi(y)dy. \tag{1}$$

For $\epsilon > 0$,

$$\begin{aligned}\left|\int_{\epsilon\leqslant|y|\leqslant 1} K_j(y)\phi(y)dy\right| &= \left|\int_{\epsilon\leqslant|y|\leqslant 1} K_j(y)(\phi(y)-\phi(0))dy\right| \\ &\leqslant \int_{\epsilon\leqslant|y|\leqslant 1} c_n|y|^{-n}\cdot\|\nabla\phi\|_\infty |y|dy \\ &= c_n\|\nabla\phi\|_\infty \omega_{n-1}\int_\epsilon^1 r^{-n+1}\cdot r^{n-1}dr \\ &= c_n\|\nabla\phi\|_\infty\omega_{n-1}(1-\epsilon).\end{aligned}$$

For $|y| \geqslant 1$,

$$\int_{|y|\geqslant 1} |K_j(y)\phi(y)|dy \leqslant c_n \int_{|y|\geqslant 1} |y|^{-n}(1+|y|^2)^{-1/2} p_{0,1}(\phi)dy$$
$$= c_n\omega_n \int_1^\infty r^{-n}(1+r)^{-1}\cdot r^{n-1}dr$$
$$= c_n\omega_n \log 2.$$

It then follows from the dominated convergence theorem that the limit

$$\lim_{\epsilon\to 0}\int_{|y|\geqslant\epsilon} K_j(y)\phi(y)dy$$

exists, which shows that the definition (1) makes sense. It is apparent that W_j is linear. Then prove that if $\phi_k \to \phi$ in $\mathscr{S}(\mathbb{R}^n)$ then $\langle W_j, \phi_k\rangle \to \langle W_j, \phi\rangle$. This being true means that $W_j \in \mathscr{S}'(\mathbb{R}^n)$, namely that each W_j is a tempered distribution.

For a function $f : \mathbb{R}^n \to \mathbb{C}$, write

$$\widetilde{f}(x) = f(-x), \qquad (\tau_y f)(x) = f(x-y).$$

For $u \in \mathscr{S}'(\mathbb{R}^n)$ and $h \in \mathscr{S}(\mathbb{R}^n)$, define

$$\langle h * u, \phi\rangle = \left\langle u, \widetilde{h} * \phi\right\rangle, \qquad \phi \in \mathscr{S}(\mathbb{R}^n).$$

It is a fact that $h*u \in \mathscr{S}'(\mathbb{R}^n)$, and this tempered distribution is induced by the C^∞ function $x \mapsto \left\langle u, \tau_x\widetilde{h}\right\rangle$.[6] The Fourier transform of a tempered distribution u is defined by

$$\langle \widehat{u}, \phi\rangle = \left\langle u, \widehat{\phi}\right\rangle, \qquad \phi \in \mathscr{S}(\mathbb{R}^n),$$

where

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\phi(x)dx, \qquad \xi \in \mathbb{R}^n.$$

It is a fact that $\widehat{u}$ is itself a tempered distribution. Finally, for a tempered distribution u and a Schwartz function h, we define

$$\langle hu, \phi\rangle = \langle u, h\phi\rangle, \qquad \phi \in \mathscr{S}(\mathbb{R}^n).$$

It is a fact that hu is itself a tempered distribution. It is proved that[7]

$$\widehat{\phi * u} = \widehat{\phi}\widehat{u}.$$

The left-hand side is the Fourier transform of the tempered distribution $\phi*u$, and the right-hand side is the product of the Schwartz function $\widehat{\phi}$ and the tempered distribution $\widehat{u}$.

[6] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 116, Theorem 2.3.20.
[7] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 120, Proposition 2.3.22.

Lemma 4. *For $1 \leqslant j \leqslant n$, for $\phi \in \mathscr{S}(\mathbb{R}^n)$, and for $x \in \mathbb{R}^n$,*

$$(R_j\phi)(x) = (\phi * W_j)(x).$$

We will use the following identity for integrals over S^{n-1}.[8]

Lemma 5. *For $\xi \neq 0$ and for $1 \leqslant j \leqslant n$,*

$$\int_{S^{n-1}} \operatorname{sgn}(\xi \cdot \theta)\theta_j d\sigma(\theta) = \frac{2\omega_{n-2}}{n-1}\frac{\xi_j}{|\xi|}.$$

Proof. It is a fact that

$$\int_{S^{n-1}} \operatorname{sgn}(\theta_k)\theta_j d\sigma(\theta) = \begin{cases} 0 & k \neq j \\ \int_{S^{n-1}} |\theta_j| d\sigma(\theta) & k = j. \end{cases} \tag{2}$$

It suffices to prove the claim when $\xi \in S^{n-1}$. For $1 \leqslant j \leqslant n$ there is $A_j = (a_{i,k})_{i,k} \in SO_n(\mathbb{R})$ such that[9]

$$A_j e_j = \xi,$$

for which $a_{i,j} = \xi_i$. Using that $A_j^T = A_j^{-1}$ and that σ is invariant under $O(n)$ we calculate

$$\begin{aligned} \int_{S^{n-1}} \operatorname{sgn}(\xi \cdot \theta)\theta_j d\sigma(\theta) &= \int_{S^{n-1}} \operatorname{sgn}(A_j e_j \cdot \theta)(AA^{-1}\theta)_j d\sigma(\theta) \\ &= \int_{S^{n-1}} \operatorname{sgn}(e_j \cdot A_j^{-1}\theta)(AA^{-1}\theta)_j d\sigma(\theta) \\ &= \int_{S^{n-1}} \operatorname{sgn}(e_j \cdot \theta)(A\theta)_j d(A_j^{-1}{}_*\sigma)(\theta) \\ &= \int_{S^{n-1}} \operatorname{sgn}(e_j \cdot \theta)(A\theta)_j d\sigma(\theta) \\ &= \int_{S^{n-1}} \operatorname{sgn}(\theta_j) \sum_{k=1}^n a_{j,k}\theta_k d\theta. \end{aligned}$$

Applying Lemma 2 and $a_{j,j} = \xi_j$, this becomes

$$\int_{S^{n-1}} \operatorname{sgn}(\xi \cdot \theta)\theta_j d\sigma(\theta) = \int_{S^{n-1}} \xi_j |\theta_j| d\sigma(\theta) = \frac{\xi_j}{|\xi|}\int_{S^{n-1}} |\theta_j| d\sigma(\theta).$$

Hence for each $1 \leqslant j \leqslant n$,

$$\int_{S^{n-1}} \operatorname{sgn}(\xi \cdot \theta)\theta_j d\sigma(\theta) = \frac{\xi_j}{|\xi|}\int_{S^{n-1}} |\theta_1| d\sigma(\theta).$$

[8] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 261, Lemma 4.1.15.

[9] `http://www.math.umn.edu/~garrett/m/mfms/notes/08_homogeneous.pdf`

It is a fact that[10]

$$\int_{RS^{n-1}} f(\theta)d\sigma(\theta) = \int_{-R}^{R}\int_{\sqrt{R^2-s^2}S^{n-2}} f(s,\phi)d\phi\frac{Rds}{\sqrt{R^2-s^2}}.$$

Using this with $f(\theta) = f(\theta_1,\dots,\theta_n) = |\theta_1|$ and using that the measure of RS^{n-2} is $R^{n-2}\omega_{n-1}$, we calculate

$$\begin{aligned}\int_{S^{n-1}} |\theta_1|d\sigma(\theta) &= \int_{-1}^{1}\int_{\sqrt{1-s^2}S^{n-2}} |s|d\phi\frac{ds}{\sqrt{1-s^2}}\\ &= \int_{-1}^{1}(1-s^2)^{\frac{n-2}{2}-\frac{1}{2}}\omega_{n-2}|s|d\phi\\ &= 2\omega_{n-2}\int_0^1(1-s^2)^{\frac{n-3}{2}}sds\\ &= \omega_{n-2}\int_0^1 u^{\frac{n-3}{2}}du\\ &= \frac{2\omega_{n-2}}{n-1}.\end{aligned}$$

□

We now calculate the Fourier transform of the W_j. We show that the Fourier transform of the tempered distribution W_j is induced by the function $\xi \mapsto -i\frac{\xi_j}{|\xi|}$.[11]

Theorem 6. *For $1 \leqslant j \leqslant n$ and for $\phi \in \mathscr{S}(\mathbb{R}^n)$,*

$$\left\langle \widehat{W}_j, \phi\right\rangle = \int_{\mathbb{R}^n} -i\phi(x)\frac{x_j}{|x|}dx.$$

Proof. We calculate

$$\begin{aligned}\left\langle W_j, \widehat{\phi}\right\rangle &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\lim_{\epsilon\to 0}\int_{|\xi|\geqslant\epsilon} K_j(\xi)\widehat{\phi}(\xi)d\xi\\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\lim_{\epsilon\to 0}\int_{\epsilon\leqslant|\xi|\leqslant 1/\epsilon} K_j(\xi)\widehat{\phi}(\xi)d\xi\\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\lim_{\epsilon\to 0}\int_{\epsilon\leqslant|\xi|\leqslant 1/\epsilon}\left(\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\phi(x)dx\right)\frac{\xi_j}{|\xi|^{n+1}}d\xi\\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\lim_{\epsilon\to 0}\int_{\mathbb{R}^n}\phi(x)\left(\int_{\epsilon\leqslant|\xi|\leqslant 1/\epsilon} e^{-2\pi i x\cdot\xi}\frac{\xi_j}{|\xi|^{n+1}}d\xi\right)dx.\end{aligned}$$

[10] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 441, Appendix D.2.
[11] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 260, Proposition 4.1.14.

For the inside integral, because $\theta \mapsto \cos(-2\pi r x_j \theta_j)\theta_j$ is an odd function,

$$\begin{aligned}
\int_{\epsilon \leqslant |\xi| \leqslant 1/\epsilon} e^{-2\pi i x\cdot\xi} \frac{\xi_j}{|\xi|^{n+1}} d\xi &= \int_{\epsilon \leqslant r \leqslant 1/\epsilon} \left(\int_{S^{n-1}} e^{-2\pi i x\cdot(r\theta)} \frac{r\theta_j}{r^{n+1}} d\sigma(\theta) \right) r^{n-1} dr \\
&= \int_{\epsilon \leqslant r \leqslant 1/\epsilon} \left(\int_{S^{n-1}} e^{-2\pi i r x\cdot\theta} \theta_j d\sigma(\theta) \right) r^{-1} dr \\
&= \int_{\epsilon \leqslant r \leqslant 1/\epsilon} \left(\int_{S^{n-1}} i \sin(-2\pi r x\cdot\theta) \theta_j d\sigma(\theta) \right) r^{-1} dr \\
&= -i \int_{\epsilon \leqslant r \leqslant 1/\epsilon} \left(\int_{S^{n-1}} \sin(2\pi r x\cdot\theta) \theta_j d\sigma(\theta) \right) r^{-1} dr \\
&= -i \int_{S^{n-1}} \left(\int_{\epsilon \leqslant r \leqslant 1/\epsilon} \sin(2\pi r x\cdot\theta) r^{-1} dr \right) \theta_j d\sigma(\theta).
\end{aligned}$$

Call the whole last expression $f_\epsilon(x)$. It is a fact that for $0 < a < b < \infty$,

$$\left| \int_a^b \frac{\sin t}{t} dt \right| \leqslant 4,$$

thus for $x \neq 0$,

$$|f_\epsilon(x)| \leqslant 4\omega_{n-1}.$$

As[12]

$$\lim_{\epsilon \to 0} f_\epsilon(x) = -i \int_{S^{n-1}} \operatorname{sgn}(x\cdot\theta) \frac{\pi}{2} \theta_j d\sigma(\theta),$$

applying the dominated convergence theorem yields

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \phi(x) \left(\int_{\epsilon \leqslant |\xi| \leqslant 1/\epsilon} e^{-2\pi i x\cdot\xi} \frac{\xi_j}{|\xi|^{n+1}} d\xi \right) dx \\
&= \int_{\mathbb{R}^n} \phi(x) \left(-i \int_{S^{n-1}} \operatorname{sgn}(x\cdot\theta) \frac{\pi}{2} \theta_j d\sigma(\theta) \right) dx \\
&= -i \frac{\pi}{2} \int_{\mathbb{R}^n} \phi(x) \left(\int_{S^{n-1}} \operatorname{sgn}(x\cdot\theta) \theta_j d\sigma(\theta) \right) dx.
\end{aligned}$$

Then using Lemma 5 and putting the above together we get

$$\begin{aligned}
\left\langle W_j, \widehat{\phi} \right\rangle &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot -i \frac{\pi}{2} \int_{\mathbb{R}^n} \phi(x) \left(\int_{S^{n-1}} \operatorname{sgn}(x\cdot\theta) \theta_j d\sigma(\theta) \right) dx \\
&= -i \frac{\pi}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \phi(x) \frac{2\omega_{n-2}}{n-1} \frac{x_j}{|x|} dx.
\end{aligned}$$

We work out that

$$\frac{\pi}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{2\omega_{n-2}}{n-1} = 1,$$

[12] Loukas Grafakos, *Classical Fourier Analysis*, second ed., p. 263, Exercise 4.1.1.

and therefore

$$\left\langle W_j, \widehat{\phi} \right\rangle = -i \int_{\mathbb{R}^n} \phi(x) \frac{x_j}{|x|} dx,$$

completing the proof. □

Because $R_j h = h * W_j$,

$$\left\langle \widehat{R_j h}, \phi \right\rangle = \left\langle \widehat{h} \widehat{W}_j, \phi \right\rangle = \left\langle \widehat{W}_j, \widehat{h} \phi \right\rangle = \int_{\mathbb{R}^n} -i \widehat{h}(\xi) \phi(\xi) \frac{\xi_j}{|\xi|} d\xi.$$

Theorem 7. *For $1 \leqslant j \leqslant n$ and for $h \in \mathscr{S}(\mathbb{R}^n)$,*

$$\widehat{R_j h}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{h}(\xi), \qquad \xi \in \mathbb{R}^n.$$

In other words, the **multiplier** of the Riesz transform R_j is $m_j(\xi) = -i \frac{\xi_j}{|\xi|}$.

4 Properties of the Riesz transform

Theorem 8.

$$-I = \sum_{j=1}^{n} R_j^2,$$

where $I(h) = h$ for $h \in \mathscr{S}(\mathbb{R}^n)$.

Proof. For $h \in \mathscr{S}(\mathbb{R}^n)$,

$$\widehat{R_j^2 h}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{R_j h}(\xi) = -i \frac{\xi_j}{|\xi|} \cdot -i \frac{\xi_j}{|\xi|} \widehat{h}(\xi) = -\frac{\xi_j^2}{|\xi|^2} \widehat{h}(\xi),$$

hence

$$\sum_{j=1}^{n} \widehat{R_j^2 h} = -\widehat{h}.$$

Taking the inverse Fourier transform,

$$\sum_{j=1}^{n} R_j^2 h = -h,$$

i.e.

$$\sum_{j=1}^{n} R_j^2 = -I.$$

□

For a tempered distribution u, for $1 \leqslant j \leqslant n$, we define

$$\langle \partial_j u, \phi \rangle = (-1) \langle u, \partial_j \phi \rangle, \qquad \phi \in \mathscr{S}(\mathbb{R}^n).$$

It is a fact that $\partial_j u$ is itself a tempered distribution. One proves that

$$\widehat{\partial_j u} = (2\pi i \xi_j)\widehat{u}.$$

Each side of the above equation is a tempered distribution. Then

$$\widehat{\Delta u} = \sum_{j=1}^n \widehat{\partial_j^2 u} = \sum_{j=1}^n (2\pi i \xi_j)^2 \widehat{u} = -4\pi^2 \sum_{j=1}^n \xi_j^2 \widehat{u} = -4\pi^2 |\xi|^2 \widehat{u}.$$

Suppose that f is a Schwartz function and that u is a tempered distribution satisfying

$$\Delta u = f,$$

called **Poisson's equation**. Then

$$-4\pi^2 |\xi|^2 \widehat{u} = \widehat{f}.$$

For $1 \leqslant j, k \leqslant n$,

$$\begin{aligned}
\partial_j \partial_k u &= \mathscr{F}^{-1}(\mathscr{F}(\partial_j \partial_k u)) \\
&= \mathscr{F}^{-1}((2\pi i \xi_j)(2\pi i \xi_k)\widehat{u}) \\
&= \mathscr{F}^{-1}\left(-4\pi^2 \xi_j \xi_k \cdot \frac{\widehat{f}}{-4\pi^2 |\xi|^2}\right) \\
&= \mathscr{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}\right).
\end{aligned}$$

Using Theorem 7,

$$\begin{aligned}
R_j R_k f &= \mathscr{F}^{-1}\mathscr{F}(R_j R_k f) \\
&= \mathscr{F}^{-1}\left(-i \frac{\xi_j}{|\xi|} \widehat{R_k f}\right) \\
&= \mathscr{F}^{-1}\left(-i \frac{\xi_j}{|\xi|} \cdot -i \frac{\xi_k}{|\xi|} \widehat{f}\right) \\
&= \mathscr{F}^{-1}\left(-\frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}\right).
\end{aligned}$$

Therefore

$$\partial_j \partial_k u = -R_j R_k f.$$

Theorem 9. *If f is a Schwartz function and u is a tempered distribution satisfying*

$$\Delta u = f,$$

then for $1 \leqslant j, k \leqslant n$,

$$\partial_j \partial_k u = -R_j R_k f.$$