Ramanujan's sum

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1 Definition

Let q and l be positive integers. Define

$$c_q(l) = \sum_{\substack{1 \le j \le q \\ \gcd(h,q)=1}} e^{-2\pi i h l/q} = \sum_{\substack{1 \le h \le q \\ \gcd(h,q)=1}} e^{2\pi i h l/q} = \sum_{\substack{1 \le h \le q \\ \gcd(h,q)=1}} \cos \frac{2\pi h l}{q}.$$

$c_q(l)$ is called *Ramanujan's sum.*

2 Fourier transform on $\mathbb{Z}/q$ and the principal Dirichlet character modulo q

For $F : \mathbb{Z}/q \to \mathbb{C}$, the *Fourier transform* $\widehat{F} : \mathbb{Z}/q \to \mathbb{C}$ of F is defined by

$$\widehat{F}(k) = \frac{1}{q} \sum_{j \in \mathbb{Z}/q} F(j) e^{-2\pi i j k/q}, \qquad k \in \mathbb{Z}/q.$$

Define $\chi : \mathbb{Z}/q \to \mathbb{C}$ by $\chi(j) = 1$ if $\gcd(j,q) = 1$ and $\chi(j) = 0$ if $\gcd(j,q) > 1$. χ is called the *principal Dirichlet character modulo* q. The Fourier transform of χ is

$$\widehat{\chi}(k) = \frac{1}{q} \sum_{j \in \mathbb{Z}/q} \chi(j) e^{-2\pi i j k/q} = \frac{1}{q} \sum_{\substack{1 \le j \le q \\ \gcd(j,q)=1}} e^{-2\pi i j k/q}.$$

Therefore we can write Ramanujan's sum $c_q(l)$ as $c_q(l) = q \cdot \widehat{\chi}(l)$, thus $c_q = q \cdot \widehat{\chi}$.

The above gives us an expression for $c_q(l)$ as a multiple of the Fourier transform of the principal Dirichlet character modulo q. $c_q : \mathbb{Z}/q \to \mathbb{C}$, and we can write the Fourier transform of c_q as

$$\begin{aligned} \widehat{c_q}(k) &= \frac{1}{q} \sum_{j \in \mathbb{Z}/q} c_q(j) e^{-2\pi i j k/q} \\ &= \sum_{j \in \mathbb{Z}/q} \widehat{\chi}(j) e^{-2\pi i j k/q}. \end{aligned}$$

3 Dirichlet series

Here I am following Titchmarsh in §1.5 of his *The theory of the Riemann zeta-function*, second ed. Let μ be the Möbius function. The Möbius inversion formula states that if

$$g(q) = \sum_{d|q} f(d)$$

then

$$f(q) = \sum_{d|q} \mu\left(\frac{q}{d}\right) g(d).$$

($\sum_{d|q}$ is a sum over the positive divisors of q.)

Define

$$\eta_q(k) = \sum_{j \in \mathbb{Z}/q} e^{-2\pi ijk/q}.$$

We have (this is not supposed to be obvious)

$$\eta_q(k) = \sum_{d|q} c_d(k).$$

Therefore by the Möbius inversion formula we have

$$c_q(k) = \sum_{d|q} \mu\left(\frac{q}{d}\right) \eta_d(k).$$

(Hence $|c_q(k)| \leq \sum_{d|k} d = \sigma_1(k)$, where $\sigma_a(k) = \sum_{d|k} d^a$.)

If $q|k$ then $\eta_q(k) = q$, and if $q \not| k$ then $\eta_q(k) = 0$. (To show the second statement: multiply the sum by $e^{-2\pi ik/q}$, and check that this product is equal to the original sum. Since we multplied the sum by a number that is not 1, the sum must be equal to 0.) Thus we can express the Möbius function using Ramanujan's sum as $\mu(q) = c_q(1)$.

Because $\eta_d(k) = d$ if $k|d$ and $\eta_d(k) = 0$ if $k \not| d$, we have

$$c_q(k) = \sum_{d|q, d|k} \mu\left(\frac{q}{d}\right) d = \sum_{dr=q, d|k} \mu(r) d.$$

So

$$\frac{c_q(k)}{q^s} = \sum_{dr=q,d|k} \frac{1}{q^s}\mu(r)d = \sum_{dr=q,d|k} \frac{1}{d^s r^s}\mu(r)d = \sum_{dr=q,d|k} \frac{1}{r^s}\mu(r)d^{1-s}.$$

Therefore

$$\sum_{q=1}^{\infty} \frac{c_q(k)}{q^s} = \sum_{q=1}^{\infty} \sum_{dr=q,d|k} \frac{1}{r^s}\mu(r)d^{1-s} = \sum_{r=1}^{\infty} \sum_{d|k} \frac{1}{r^s}\mu(r)d^{1-s} = \sum_{r=1}^{\infty} \frac{1}{r^s}\mu(r) \sum_{d|k} d^{1-s}.$$

Then

$$\sum_{q=1}^{\infty} \frac{c_q(k)}{q^s} = \sigma_{1-s}(k) \sum_{r=1}^{\infty} \frac{1}{r^s} \mu(r) = \sigma_{1-s}(k) \frac{1}{\zeta(s)};$$

here we used that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

On the other hand, if rather than sum over q we sum over k, then we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{c_q(k)}{k^s} &= \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{d|q, d|k} \mu\left(\frac{q}{d}\right) d \\
&= \sum_{d|q} \sum_{m=1}^{\infty} \frac{1}{(md)^s} \cdot \mu\left(\frac{q}{d}\right) d \\
&= \sum_{d|q} \sum_{m=1}^{\infty} \frac{1}{m^s} \cdot \frac{1}{d^s} \cdot \mu\left(\frac{q}{d}\right) d \\
&= \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{d|q} \frac{1}{d^s} \mu\left(\frac{q}{d}\right) d \\
&= \zeta(s) \cdot \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{1-s}.
\end{aligned}$$