Orthonormal bases for product measures

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October 22, 2015

1 Measure and integration theory

Let $\mathscr{B}$ be the Borel σ-algebra of $\mathbb{R}$, and let $\overline{\mathscr{B}}$ be the Borel σ-algebra of $[-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}$: the elements of $\overline{\mathscr{B}}$ are those subsets of $\overline{\mathbb{R}}$ of the form $B, B \cup \{-\infty\}, B \cup \{\infty\}, B \cup \{-\infty,\infty\}$, with $B \in \mathscr{B}$.

Let $(X,\mathscr{A},\mu)$ be a measure space. It is a fact that if f_n is a sequence of $\mathscr{A} \to \overline{\mathscr{B}}$ measurable functions then $\sup_n f_n$ and $\inf_n f_n$ are $\mathscr{A} \to \overline{\mathscr{B}}$ measurable, and thus if f_n is a sequence of $\mathscr{A} \to \overline{\mathscr{B}}$ measurable functions that converge pointwise to a function $f : X \to \overline{\mathbb{R}}$, then f is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable.[1] If $f_1, \ldots, f_n$ are $\mathscr{A} \to \overline{\mathscr{B}}$ measurable, then so are $f_1 \vee \cdots \vee f_n$ and $f_1 \wedge \cdots \wedge f_n$, and a function $f : X \to \overline{\mathbb{R}}$ is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable if and only if both $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$ are $\mathscr{A} \to \overline{\mathscr{B}}$ measurable. In particular, if f is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable then so is $|f| = f^+ + f^-$.

A **simple function** is a function $f : X \to \mathbb{R}$ that is $\mathscr{A} \to \mathscr{B}$ measurable and whose range is finite. Let $E = E(\mathscr{A})$ be the collection of nonnegative simple functions. It is straightforward to prove that

$$u, v \in E,\ \alpha \geq 0 \quad \Rightarrow \quad \alpha u,\ u+v,\ u \cdot v,\ u \vee v,\ u \wedge v \in E.$$

Define $I_\mu : E \to [0,\infty]$ by

$$I_\mu u = \sum_{i=1}^n a_i \mu(A_i),$$

where u has range $\{a_1, \ldots, a_n\}$ and $A_i = u^{-1}(a_i)$. One proves that $I_\mu : E \to [0,\infty]$ is positive homogeneous, additive, and order preserving.[2]

It is a fact[3] that if u_n is a nondecreasing sequence in E and $u \in E$ then

$$u \leq \sup_n u_n \quad \Rightarrow \quad I_\mu u \leq \sup_n I_\mu u_n.$$

It follows that if u_n and v_n are sequences in E then

$$\sup_n u_n = \sup_n v_n \quad \Rightarrow \quad \sup_n I_\mu u_n = \sup_n I_\mu v_n. \tag{1}$$

[1] Heinz Bauer, *Measure and Integration Theory*, p. 52, Corollary 9.7.
[2] Heinz Bauer, *Measure and Integration Theory*, pp. 55–56, §10.
[3] Heinz Bauer, *Measure and Integration Theory*, p. 57, Theorem 11.1.

Define $E^* = E^*(\mathscr{A})$ to be the set of all functions $f : X \to [0,\infty]$ for which there is a nondecreasing sequence u_n in E satisfying $\sup_n u_n = f$, in other words, there is a sequence u_n in E satisfying $u_n \uparrow f$. From (1), for $f \in E^*$ and sequences $u_n, v_n \in E$ with $\sup_n u_n = f$ and $\sup_n v_n = f$, it holds that $\sup_n I_\mu u_n = \sup_n I_\mu v_n$. Also, if $u \in E$ then $u_n = u$ is a nondecreasing sequence in E with $u = \sup_n u_n$, so $u \in E^*$. Then it makes sense to extend I_μ from $E \to [0,\infty]$ to $E^* \to [0,\infty]$ by defining $I_\mu f = \sup_n I_\mu u_n$. One proves[4] that

$$f, g \in E^*,\ \alpha \geq 0 \quad \Rightarrow \quad \alpha f,\ f+g,\ f \cdot g,\ f \vee g,\ f \wedge g \in E^*$$

and that $I_\mu : E^* \to [0,\infty]$ is positive homogeneous, additive, and order preserving.

The **monotone convergence theorem**[5] states that if f_n is a sequence in E^* then $\sup_n f_n \in E^*$ and

$$I_\mu \left(\sup_n f_n \right) = \sup_n I_\mu f_n.$$

We now prove a characterization of E^*.[6]

Theorem 1. *E^* is equal to the set of functions $X \to [0,\infty]$ that are $\mathscr{A} \to \overline{\mathscr{B}}$ measurable.*

Proof. If $f \in E^*$, then there is a sequence u_n in E with $u_n \uparrow f$. Because each u_n is measurable $\mathscr{A} \to \overline{\mathscr{B}}$, so is f.

Now suppose that $f : X \to [0,\infty]$ is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable. For $n \geq 1$ and $0 \leq i \leq n2^n - 1$ let

$$A_{i,n} = \{f \geq i2^{-n}\} \cap \{f < (i+1)2^{-n}\} = \{i2^{-n} \leq f < (i+1)2^{-n}\},$$

and for $i = n2^n$ let

$$A_{i,n} = \{f \geq n\}.$$

Because f is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable, the sets $A_{i,n}$ belong to $\mathscr{A}$. For each n, the sets $A_{0,n}, \ldots A_{n2^n-1,n}, A_{n2^n,n}$ are pairwise disjoint and their union is equal to X. It is apparent that

$$A_{i,n} = A_{2i,n+1} \cup A_{2i+1,n+1}, \qquad 0 \leq i \leq n2^n - 1. \tag{2}$$

Define

$$u_n = \sum_{i=0}^{n2^n} i2^{-n} 1_{A_{i,n}},$$

which belongs to E. For $x \in X$, either $f(x) = \infty$ or $0 \leq f(x) < \infty$. In the first case, $u_n(x) = n$ for all $n \geq 1$. In the second case, $u_n(x) \leq f(x) < u_n(x) + 2^{-n}$ for all $n > f(x)$. Therefore $u_n(x) \uparrow f(x)$ as $n \to \infty$, and because this is true for each $x \in X$, this means $u_n \uparrow f$ and so $f \in E^*$. □

[4]Heinz Bauer, *Measure and Integration Theory*, pp. 58–59, §11.
[5]Heinz Bauer, *Measure and Integration Theory*, p. 59, Theorem 11.4.
[6]Heinz Bauer, *Measure and Integration Theory*, p. 61, Theorem 11.6.

So far we have defined $I_\mu : E^* \to [0,\infty]$. Suppose that $f : X \to \overline{\mathbb{R}}$ is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable. Then $f^+, f^- : X \to [0,\infty]$ are $\mathscr{A} \to \overline{\mathscr{B}}$ measurable so by Theorem 1, $f^+, f^- \in E^*$. Then $I_\mu f^+, I_\mu f^- \in [0,\infty]$. We say that a function $f : X \to \overline{\mathbb{R}}$ is **μ-integrable** if it is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable and $I_\mu f^+ < \infty$ and $I_\mu f^- < \infty$. One checks that a function $f : X \to \overline{\mathbb{R}}$ is μ-integrable if and only if it is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable and $I_\mu |f| < \infty$. If $f : X \to \overline{\mathbb{R}}$ is μ-integrable, we now define $I_\mu f \in \mathbb{R}$ by

$$I_\mu f = I_\mu f^+ - I_\mu f^-.$$

For example, if $\mu(X) < \infty$ and S is a subset of X that does not belong to $\mathscr{A}$, define $f : X \to \mathbb{R}$ by $f = 1_S - 1_{X\setminus S}$. Then $f^+ = 1_S$ and $f^- = 1_{X\setminus S}$, and thus f is not $\mathscr{A} \to \overline{\mathscr{B}}$ measurable, so it is not μ-integrable. But $|f| = 1$ belongs to E, and $I_\mu |f| = \mu(X) < \infty$ by hypothesis, showing that $|f|$ is μ-integrable while f is not.

One proves that if $f, g : X \to \overline{\mathbb{R}}$ are μ-integrable and $\alpha \in \mathbb{R}$ then αf is μ-integrable and

$$I_\mu(\alpha f) = \alpha I_\mu f,$$

if $f + g$ is defined on all X then $f + g$ is μ-integrable and

$$I_\mu(f + g) = I_\mu f + I_\mu g,$$

and $f \vee g, f \wedge g$ are μ-integrable.[7] Furthermore, I_μ is order preserving.

Let $f : X \to \mathbb{C}$ be a function and write $f = u + iv$. One proves that f is Borel measurable (i.e. $\mathscr{A} \to \mathscr{B}_{\mathbb{C}}$ measurable), if and only if u and v are measurable $\mathscr{A} \to \mathscr{B}$. We define f to be μ-integrable if both u and v are μ-integrable, and define

$$I_\mu f = I_\mu u + i I_\mu v.$$

2 $\mathscr{L}^2$

Let $(X, \mathscr{A}, \mu)$ be a measure space and for $1 \leq p < \infty$ let $\mathscr{L}^p(\mu)$ be the collection of Borel measurable functions $f : X \to \mathbb{C}$ such that $|f|^p$ is μ-integrable. For complex a, b, because $x \mapsto x^p$ is convex we have by Jensen's inequality

$$\left|\frac{a+b}{2}\right|^p \leq \left(\frac{1}{2}|a| + \frac{1}{2}|b|\right)^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p = \frac{1}{2}(|a|^p + |b|^p),$$

so $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. Thus if $f, g \in \mathscr{L}^p(\mu)$ then

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p),$$

which implies that $\mathscr{L}^p(\mu)$ is a linear space.

For Borel measurable $f : X \to \mathbb{C}$ define

$$\|f\|_{L^p} = \left(\int_X |f|^p d\mu\right)^{1/p}.$$

[7] Heinz Bauer, *Measure and Integration Theory*, p. 65, Theorem 12.3.

For $f, g \in \mathscr{L}^p(\mu)$, by Hölder's inequality, with $\frac{1}{p} + \frac{1}{p'} = 1$ (for which $p' = \frac{p}{p-1}$),

$$\begin{aligned}\|f+g\|_{L^p}^p &\leq \int_X |f||f+g|^{p-1} d\mu + \int_X |g||f+g|^{p-1} d\mu \\ &\leq \|f\|_{L^p} \left\||f+g|^{p-1}\right\|_{L^{p'}} + \|g\|_{L^p} \left\||f+g|^{p-1}\right\|_{L^{p'}} \\ &= \|f\|_{L^p} \|f+g\|_{L^p}^{p-1} + \|g\|_{L^p} \|f+g\|_{L^p}^{p-1},\end{aligned}$$

which implies that $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$, and hence $\|\cdot\|_{L^p}$ is a seminorm on $\mathscr{L}^p(\mu)$.

Let $\mathscr{N}^p(\mu)$ be the set of those $f \in \mathscr{L}^p(\mu)$ such that $\|f\|_{L^p} = 0$. $\mathscr{N}^p(\mu)$ is a linear subspace of $\mathscr{L}^p(\mu)$, and we define

$$L^p(\mu) = \mathscr{L}^p(\mu)/\mathscr{N}^p(\mu) = \{f + \mathscr{N}^p(\mu) : f \in \mathscr{L}^p(\mu)\}.$$

$L^p(\mu)$ is a normed linear space with the norm $\|\cdot\|_{L^p}$.

It is a fact that if V is a normed linear space then V is complete if and only if each absolutely convergent series in V converges in V. Suppose that f_k is a sequence in $\mathscr{L}^p(\mu)$ with $\sum_{k=1}^\infty \|f\|_{L^p} < \infty$. For $n \geq 1$ let $g_n(x) = \left(\sum_{k=1}^n |f_k(x)|\right)^p$ and define $g : X \to [0, \infty]$ by

$$g(x) = \left(\sum_{k=1}^\infty |f_k(x)|\right)^p = \lim_{n\to\infty} g_n(x),$$

which is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable, being the pointwise limit of a sequence of functions each of which is $\mathscr{A} \to \overline{\mathscr{B}}$ measurable. Because $g_1 \leq g_2 \leq \cdots$, by the monotone convergence theorem,

$$\int_X g d\mu = \lim_{n\to\infty} \int_X g_n d\mu.$$

But

$$\left(\int_X g_n d\mu\right)^{1/p} = \left\|\sum_{k=1}^n |f_k|\right\|_{L^p} \leq \sum_{k=1}^n \|f_k\|_{L^p} \leq \sum_{k=1}^\infty \|f_k\|_{L^p},$$

which implies that $\int_X g d\mu < \infty$, meaning that $g : X \to [0, \infty]$ is integrable. The fact that g is integrable implies $\mu(E) = 0$, where $E = \{x \in X : g(x) = \infty\} \in \mathscr{A}$. For $x \in X \setminus E$, $\sum_{k=1}^\infty |f_k(x)| < \infty$ and because $\mathbb{C}$ is complete this implies that $\sum_{k=1}^\infty f_k(x) \in \mathbb{C}$, and so it makes sense to define $f : X \to \mathbb{C}$ by

$$f(x) = 1_{X\setminus E}(x) \sum_{k=1}^\infty f_k(x),$$

which is Borel measurable. Furthermore, $|f|^p \leq g$, and because g is integrable this implies that $f \in \mathscr{L}^p(\mu)$. For $x \in X \setminus E$,

$$\lim_{n\to\infty} \left|\sum_{k=1}^n f_k(x) - f(x)\right|^p = 0$$

and

$$\left| \sum_{k=1}^{n} f_k(x) - f(x) \right|^p \le g(x),$$

so by the dominated convergence theorem,[8]

$$\lim_{n\to\infty} \int_X \left| \sum_{k=1}^{n} f_k(x) - f(x) \right|^p d\mu = 0.$$

Because $x \mapsto x^{1/p}$ is continuous this implies

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} f_k - f \right\|_{L^p} = 0.$$

Hence, if f_k is a sequence in $L^p(\mu)$ such that $\sum_{k=1}^{\infty} \|f_k\|_{L^p} < \infty$ then there is some $f \in L^p(\mu)$ such that $\sum_{k=1}^{n} f_k \to f$ in the norm $\|\cdot\|_{L^p}$. This implies that $L^p(\mu)$ is a Banach space.

We say that the σ-algebra $\mathscr{A}$ is **countably generated** if there is a countable subset $\mathscr{C}$ of $\mathscr{A}$ such that $\mathscr{A} = \sigma(\mathscr{C})$ and we say that a topological space is **separable** if there exists a countable dense subset of it. It can be proved that if $\mathscr{A}$ is countably generated and μ is σ-finite, then for $1 \le p < \infty$ there is a countable collection of simple functions that is dense in $L^p(\mu)$, showing that $L^p(\mu)$ is separable.[9]

Theorem 2. *Let $(X, \mathscr{A}, \mu)$ be a measure space and let $1 \le p < \infty$. $L^p(\mu)$ with the norm $\|\cdot\|_{L^p}$ is a Banach space, and if $\mathscr{A}$ is countably generated and μ is σ-finite then $L^p(\mu)$ is separable.*

For $f, g \in \mathscr{L}^2(\mu)$, let

$$\langle f, g\rangle_{L^2(\mu)} = \int_X f \cdot \overline{g} d\mu.$$

This is an inner product on $L^2(\mu)$, and thus $L^2(\mu)$ is a Hilbert space.

3 Product measures

Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_1, \mathscr{A}_1, \mu_1)$ be measure spaces and let $\mathscr{A}_1 \otimes \mathscr{A}_2$ be the product σ-algebra. For $Q \subset X_1 \times X_2$, write

$$Q_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in Q\}, \qquad Q_{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in Q\}.$$

One proves that if μ_1 and μ_2 are σ-finite, then for each $Q \in \mathscr{A}_1 \otimes \mathscr{A}_2$ the function $x_1 \mapsto \mu_2(Q_{x_1})$ is $\mathscr{A}_1 \to \overline{\mathscr{B}}$ measurable and the function $x_2 \mapsto \mu_1(Q_{x_2})$

[8] Heinz Bauer, *Measure and Integration Theory*, p. 83, Theorem 15.6.
[9] Donald L. Cohn, *Measure Theory*, second ed., p. 102, Proposition 3.4.5.

is $\mathscr{A}_2 \to \overline{\mathscr{B}}$ measurable.[10] If μ_1 and μ_2 are σ-finite, one proves[11] that there is a unique measure $\mu : \mathscr{A}_1 \otimes \mathscr{A}_2 \to [0, \infty]$ that satisfies

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \qquad A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2.$$

The measure μ satisfies

$$\mu(Q) = \int_{X_1} \mu_2(Q_{x_1}) d\mu_1(x_1) = \int_{X_2} \mu_1(Q_{x_2}) d\mu_2(x_2)$$

for $Q \in \mathscr{A}_1 \otimes \mathscr{A}_2$, and is itself σ-finite. We write $\mu = \mu_1 \otimes \mu_2$, and call μ the **product measure of μ_1 and μ_2.**

Let X' be a set and let $f : X_1 \times X_2 \to X'$ be a function. For $x_1 \in X_1$, define $f_{x_1} : X_2 \to X'$ by

$$f_{x_1}(x_2) = f(x_1, x_2), \qquad x_2 \in X_2$$

and for $x_2 \in X_2$, define $f_{x_2} : X_1 \to X'$ by

$$f_{x_2}(x_1) = f(x_1, x_2), \qquad x_1 \in X_1.$$

For $Q \subset X_1 \times X_2$,

$$(1_Q)_{x_1} = 1_{Q_{x_1}}, \qquad (1_Q)_{x_2} = 1_{Q_{x_2}}.$$

It is straightforward to prove that if $(X', \mathscr{A}')$ is a measurable space and $f : (X_1 \times X_2, \mathscr{A}_1 \otimes \mathscr{A}_2) \to (X', \mathscr{A}')$ is measurable, then for each $x_1 \in X_1$ the function $f_{x_1} : X_2 \to X'$ is measurable $\mathscr{A}_2 \to \mathscr{A}'$ and for each $x_2 \in X_2$ the function $f_{x_2} : X_1 \to X'$ is measurable $\mathscr{A}_1 \to \mathscr{A}'$.[12]

Tonelli's theorem[13] states that if $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_1, \mathscr{A}_1, \mu_1)$ are σ-finite measure spaces and $f : X_1 \times X_2 \to [0, \infty]$ is $\mathscr{A}_1 \otimes \mathscr{A}_2 \to \overline{\mathscr{B}}$ measurable, then the functions

$$x_2 \mapsto \int_{X_1} f_{x_2} d\mu_1, \qquad x_1 \mapsto \int_{X_2} f_{x_1} d\mu_2$$

are $\mathscr{A}_2 \to \overline{\mathscr{B}}$ measurable and $\mathscr{A}_1 \to \overline{\mathscr{B}}$ measurable respectively, and

$$\begin{aligned} &\int_{X_1 \times X_2} f d(\mu_1 \otimes \mu_2) \\ &= \int_{X_2} \left(\int_{X_1} f_{x_2} d\mu_1 \right) d\mu_2(x_2) \\ &= \int_{X_1} \left(\int_{X_2} f_{x_1} d\mu_2 \right) d\mu_1(x_1). \end{aligned} \tag{3}$$

Fubini's theorem[14] states that if $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ are σ-finite measure spaces and $f : X_1 \times X_2 \to \overline{\mathbb{R}}$ is $\mu_1 \otimes \mu_2$-integrable then there is some

[10] Heinz Bauer, *Measure and Integration Theory*, p. 135, Lemma 23.2.
[11] Heinz Bauer, *Measure and Integration Theory*, p. 136, Theorem 23.3.
[12] Heinz Bauer, *Measure and Integration Theory*, p. 138, Lemma 23.5.
[13] Heinz Bauer, *Measure and Integration Theory*, p. 138, Theorem 23.6.
[14] Heinz Bauer, *Measure and Integration Theory*, p. 139, Corollary 23.7.

$A_1 \in \mathscr{A}_1$ with $\mu_1(A_1) = 0$ such that for $x_1 \in X_1 \setminus A_1$ the function $f_{x_1} : X_2 \to \overline{\mathbb{R}}$ is μ_2-integrable, and there is some $A_2 \in \mathscr{A}_2$ with $\mu_2(A_2) = 0$ such that for $x_2 \in X_2 \setminus A_2$ the function $f_{x_2} : X_1 \to \overline{\mathbb{R}}$ is μ_1-integrable. Furthermore, define $F_1 : X_1 \to \mathbb{R}$ by $F_1(x_1) = \int_{X_2} f_{x_1} d\mu_2$ for $x_1 \in X_1 \setminus A_1$ and $F_1(x_1) = 0$ for $x_1 \in A_1$, and define $F_2 : X_2 \to \mathbb{R}$ by $F_2(x_2) = \int_{X_1} f_{x_2} d\mu_1$ for $x_2 \in X_2 \setminus A_2$ and $F_2(x_2) = 0$ for $x_2 \in A_2$. The functions F_1 and F_2 are μ_1-integrable and μ_2-integrable respectively, and

$$\int_{X_1 \times X_2} f d(\mu_1 \otimes \mu_2) = \int_{X_1} F_1 d\mu_1 = \int_{X_2} F_2 d\mu_2.$$

Suppose that $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ are σ-finite measure spaces. For $e : X_1 \to \mathbb{C}$ and $f : X_2 \to \mathbb{C}$, define $e \otimes f : X_1 \times X_2 \to \mathbb{C}$ by

$$(e \otimes f)(x_1, x_2) = e(x_1)f(x_2),$$

which is Borel measurable $X_1 \times X_2 \to \mathbb{C}$ if e and f are Borel measurable. If $e \in \mathscr{L}^2(\mu_1)$ and $f \in \mathscr{L}^2(\mu_2)$, then by Tonelli's theorem $e \otimes f : X_1 \times X_2 \to \mathbb{C}$ belongs to $\mathscr{L}^2(\mu_1 \otimes \mu_2)$. For $e, e' \in \mathscr{L}^2(\mu_1)$ and $f, f' \in \mathscr{L}^2(\mu_2)$, by Fubini's theorem,

$$\begin{aligned}
&\langle e \otimes f, e' \otimes f' \rangle_{L^2(\mu_1 \otimes \mu_2)} \\
&= \int_{X_1 \times X_2} e(x_1)f(x_2)\overline{e'(x_1)f'(x_2)} d(\mu_1 \otimes \mu_2)(x_1, x_2) \\
&= \int_{X_2} \left(\int_{X_1} e(x_1)\overline{e'(x_1)} d\mu_1(x_1) \right) f(x_2)\overline{f'(x_2)} d\mu_2(x_2) \\
&= \langle e, e' \rangle_{L^2(\mu_1)} \cdot \langle f, f' \rangle_{L^2(\mu_2)}.
\end{aligned}$$

Therefore, if $E \subset \mathscr{L}^2(\mu_1)$ is an orthonormal set in $L^2(\mu_1)$ and $F \subset \mathscr{L}^2(\mu_2)$ is an orthonormal set in $L^2(\mu_2)$, then $\{e \otimes f : e \in E, f \in F\} \subset \mathscr{L}^2(\mu_1 \otimes \mu_2)$ is an orthonormal set in $L^2(\mu_1 \otimes \mu_2)$.

Theorem 3. *Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be σ-finite measure spaces and suppose that $L^2(\mu_1)$ and $L^2(\mu_2)$ are separable. If $E \subset \mathscr{L}^2(\mu_1)$ is an orthonormal basis for $L^2(\mu_1)$ and $F \subset \mathscr{L}^2(\mu_2)$ is an orthonormal basis for $L^2(\mu_2)$, then $\Phi = \{e \otimes f : e \in E, f \in F\} \subset \mathscr{L}^2(\mu_1 \otimes \mu_2)$ is an orthonormal basis for $L^2(\mu_1 \otimes \mu_2)$.*

Proof. To show that Φ is an orthonormal basis for $L^2(\mu_1 \otimes \mu_2)$ it suffices to prove that if $h \in \mathscr{L}^2(\mu_1 \otimes \mu_2)$ belongs to the orthogonal complement of Φ then $h \in \mathscr{N}^2(\mu_1 \otimes \mu_2)$. Thus, suppose that $h \in \mathscr{L}^2(\mu_1 \otimes \mu_2)$ and that $\langle h, e \otimes f \rangle_{L^2(\mu_1 \otimes \mu_2)} = 0$ for all $e \in E, f \in F$. Using Fubini's theorem,

$$\int_{X_1} e(x_1) \left(\int_{X_2} h_{x_1}(x_2) f(x_2) d\mu_2(x_2) \right) d\mu_1(x_1) = 0.$$

Because this is true for all $e \in E$ and E is dense in $L^2(\mu_1)$, it follows that there is some $A_f \in \mathscr{A}_1$ with $\mu_1(A_f) = 0$ such that $\int_{X_2} h_{x_1} f d\mu_2 = 0$ for $x_1 \notin A_f$. Let

$A_1 = \bigcup_{f \in F} A_f$, for which $\mu_1(A_1) = 0$. If $x_1 \notin A_1$ then $\int_{X_2} h_{x_1} f d\mu_2 = 0$ for all $f \in F$, and because F is dense in $L^2(\mu_2)$ this implies that $h_{x_1} = 0$ μ_2-almost everywhere. Then

$$\begin{aligned}\int_{X_1 \times X_2} |h|^2 d(\mu_1 \otimes \mu_2) &= \int_{X_1} \left(\int_{X_2} |h_{x_1}|^2 d\mu_2 \right) d\mu_1(x_1) \\ &= \int_{X_1 \setminus A_1} \left(\int_{X_2} |h_{x_1}|^2 d\mu_2 \right) d\mu_1(x_1) \\ &= 0,\end{aligned}$$

which implies that $h = 0$ $\mu_1 \otimes \mu_2$-almost everywhere. □