The Poincaré-Dulac normal form theorem for formal vector fields

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April 3, 2014

1 Introduction

In this note we present proofs of the Poincaré normal form theorem and the Poincaré-Dulac normal form theorem for formal vector fields. Other accounts in the literature do not explicitly work out the proofs by induction of these theorems. Our presentation is a more precise and detailed version of the presentation in [5, §§3–5]. These topics are also covered in [1, §I.3], [2, Chapter 5], and [3, §A.5]. The history of the problem of normalization of vector fields is presented by Yakovenko in review 96a:34021 in Mathematical Reviews. The computation of normal forms is discussed in [6] and [7, Chapter 19].

The Poincaré-Dulac normal form has recently been used in [4], which proves the unconditional uniqueness of solutions of the periodic one-dimensional cubic nonlinear Schrödinger equation.

In §6 we give detailed examples where we explicitly compute the leading terms of the formal maps which conjugate formal vector fields to their Poincaré normal form and Poincaré-Dulac normal form.

2 Formal vector fields

Let $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_n]]$ be the algebra of formal power series in the variables $x_1, \ldots, x_n$:

$$\mathbb{C}[[x]] = \left\{ \sum_{|\alpha| \geq 0} c_\alpha x^\alpha : c_\alpha \in \mathbb{C} \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

A *formal vector field* is an element of $\mathfrak{g} = \mathbb{C}[[x]]^n$, that is, an n-tuple of formal power series. $\mathfrak{g}$ is a Lie algebra with the vector field commutator as its Lie bracket, defined for $F, G \in \mathfrak{g}$ by

$$[F, G](x) = \frac{\partial G}{\partial x}(x) F(x) - \frac{\partial F}{\partial x}(x) G(x).$$

For $F \in \mathfrak{g}$, we define $\mathrm{ad}_F : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_F(G) = [F, G]$ for $G \in \mathfrak{g}$.

Let $\mathfrak{m} \subset \mathbb{C}[[x]]$ be the set of formal power series with constant term 0. An element of $\mathfrak{m}^n$ (an n-tuple of elements of $\mathfrak{m}$) is said to be a *formal map*. If $H = (h_1, \ldots, h_n)$ is a formal map and $f(x) = \sum_{|\alpha| \geq 0} c_\alpha x^\alpha$ is a formal power series, then

$$f(H(x)) = \sum_{|\alpha| \geq 0} c_\alpha h_1(x)^{\alpha_1} \cdots h_n(x)^{\alpha_n}$$

is a formal power series. We call elements of $\mathfrak{m}^n$ formal maps because we can compose formal power series with them. On the other hand, $f(x) = \sum_{k=0}^\infty x^k$ is a formal power series, but for $H(x) = 1 + x$ (which has nonzero constant coefficient),

$$f(H(x)) = \sum_{k=0}^\infty (1+x)^k = \sum_{k=0}^\infty \sum_{j=0}^k \binom{k}{j} x^j$$

is not a formal power series because, for instance, the constant coefficient is infinite (indeed, each coefficient is infinite).

Two formal vector fields F, F' are said to be *equivalent* if there is a formal map H such that

$$\frac{\partial H}{\partial x}(x) F(x) = F'(H(x)).$$

It is clear that if $F(0) = 0$ and F is equivalent to F', then $F'(0) = 0$.

Let $\mathscr{H}_m \subset \mathbb{C}[[x]]$ be the vector space whose elements are homogeneous polynomials of degree m in the variables $x_1, \ldots, x_n$, and 0, and let $\mathscr{D}_m = \mathscr{H}_m^n \subset \mathfrak{g}$.

For a formal vector field F, the *linearization* of F is the $n \times n$ matrix A defined by $A_{i,j} = \frac{\partial F_i}{\partial x_j}(0)$, i.e., $A = \frac{\partial F}{\partial x}(0)$. A formal vector field F with $F(0) = 0$ and with linearization A can be written as

$$F(x) = Ax + \sum_{j=2}^\infty V^j(x)$$

for some $V^j \in \mathscr{D}_j$.

The following theorem is the inverse function theorem for formal maps [5, pp. 32–33].

Theorem 1. If H is a formal map and $\frac{\partial H}{\partial x}(0)$ is invertible, then there is a formal map H^{-1} such that $H(H^{-1}(x)) = x$ and $H^{-1}(H(x)) = x$.

The following theorem shows that any formal vector field is equivalent to a formal vector field whose linearization is in Jordan normal form.

Theorem 2. If a formal vector field F has linearization A and $A = QBQ^{-1}$, then F is equivalent to a formal vector field with linearization B.

Proof. Let $H(x) = Q^{-1}x$, and define F' by $F'(x) = Q^{-1}F(Qx)$. F' has linearization

$$\frac{\partial F'}{\partial x}(0) = Q^{-1} \frac{\partial F}{\partial x}(0) Q = Q^{-1}AQ = B,$$

and

$$\frac{\partial H}{\partial x}(x)F(x) = Q^{-1}F(x) = F'(H(x)),$$

so F is equivalent to F'. $\square$

A vector $\lambda \in \mathbb{C}^n$ is said to be *resonant* if there is some $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \geq 2$ and some $1 \leq k \leq n$ such that $\lambda_k = \langle \alpha, \lambda \rangle$. We define $\langle \alpha, \lambda \rangle = \sum_{j=1}^n \alpha_j \lambda_j$. An $n \times n$ matrix A is said to be resonant if the vector of its eigenvalues is resonant, and a formal vector field is said to be resonant if its linearization is resonant. $|\alpha|$ is the *order* of the resonance.

3 Poincaré normal form theorem for formal vector fields

The following theorem is the *Poincaré normal form theorem*, which states that a nonresonant formal vector field with constant term 0 whose linearization is in Jordan normal form is equivalent to its linearization. By Theorem 2 any formal vector field is equivalent to a formal vector field whose linearization is in Jordan normal form, so it follows that any nonresonant formal vector field with constant term 0 can be linearized.

Theorem 3. If F is a nonresonant formal vector field with constant term 0 and the linearization A of F is in Jordan normal form, then F is equivalent to the formal vector field F' defined by $F'x = Ax$.

Proof. We prove the claim by induction. Let $F_2 = F$. We can write

$$F_2(x) = Ax + \sum_{j=2}^{\infty} V_2^j(x)$$

where $V_2^j \in \mathscr{D}_j$. Let $H_1(x) = x$. Then $\frac{\partial H_1}{\partial x}F(x) = F_2(H_1(x))$, and thus F is equivalent to the formal vector field F_2. Assume that for some m there are $V_m^j \in \mathscr{D}_j$, $j = m, \ldots$, such that F is equivalent to

$$F_m(x) = Ax + \sum_{j=m}^{\infty} V_m^j(x).$$

We want to show that there are $V_{m+1}^j \in \mathscr{D}_j$, $j = m+1, \ldots$, such that F_m is equivalent to

$$F_{m+1}(x) = Ax + \sum_{j=m+1}^{\infty} V_{m+1}^j(x). \tag{1}$$

That is, we want to show that there exists a formal map H_m and $V_{m+1}^j \in \mathscr{D}_j$ so that if F_{m+1} is defined by (1) then

$$\frac{\partial H_m}{\partial x}(x)F_m(x) = F_{m+1}(H_m(x)). \tag{2}$$

If there exists a formal map H_m and $V^j_{m+1} \in \mathscr{D}_j$ that satisfy (2) and H_m is of the form $H_m(x) = x + P_m(x)$ for some $P_m \in \mathscr{D}_m$, then

$$(I + \frac{\partial P_m}{\partial x}(x))(Ax + \sum_{j=m}^{\infty} V^j_m(x)) = Ax + AP_m(x) + \sum_{j=m+1}^{\infty} V^j_{m+1}(H_m(x)). \quad (3)$$

Comparing terms of degree m we get

$$V^m_m(x) + \frac{\partial P_m}{\partial x}(x)Ax = AP_m(x)$$

or

$$-V^m_m = \mathrm{ad}_A(P_m).$$

This equation is called the *homological equation*.

By Corollary 5, $\mathrm{ad}_A|_{\mathscr{D}_m} : \mathscr{D}_m \to \mathscr{D}_m$ is a linear isomorphism, and hence we can define P_m by $P_m = (\mathrm{ad}_A)^{-1}(-V^m_m)$. Then the terms V^j_{m+1}, $j = m+1, \ldots$ are determined by setting

$$\sum_{j=m+1}^{\infty} V^j_m(x) + \frac{\partial P_m}{\partial x}(x) \sum_{j=m}^{\infty} V^j_m(x) = \sum_{j=m+1}^{\infty} V^j_{m+1}(H_m(x)).$$

Therefore if we define F_{m+1} by (1), the formal vector fields F_m, F_{m+1} are equivalent.

Then $H^{(m)}(x) = H_m \circ \cdots \circ H_1(x)$ is a formal map such that $\frac{\partial H^{(m)}}{\partial x}(x)F(x) = F_{m+1}(H^{(m)}(x))$. Since $H^{(m+1)} = H_{m+1} \circ H^{(m)}$ and $H^{(m)}$ have the same terms of degree $\leq m$, $\lim_{m\to\infty} H^{(m)}(x)$ exists in $\mathfrak{m}^n$; let H be this limit. Then we can check that H is a formal map such that $\frac{\partial H}{\partial x}(x)F(x) = F'(H(x))$, and so F is equivalent to F'. $\square$

For any $n \times n$ matrix A (resonant or nonresonant) and for $P \in \mathscr{D}_m$, we have $\mathrm{ad}_A(P)(x) = \frac{\partial P}{\partial x}(x)Ax - AP(x) \in \mathscr{D}_m$, hence $\mathscr{D}_m$ is an invariant subspace of ad_A.

A basis for $\mathscr{D}_m$ consists of $F_{k,\alpha}(x) = x^\alpha e_k$, $k = 1, \ldots, n$, $|\alpha| = m$. Let $w_j = \sqrt{p_{n-j+1}}$, where p_j is the jth prime; these are real numbers $w_1 > \cdots > w_n > 0$ that are independent over $\mathbb{Q}$. Assign the weight w_k to x_k and the weight $-w_k$ to e_k. Each element in the basis thus has a weight, and we can check that the only distinct elements with the same weights are $x^\alpha x_j e_j$ and $x^\alpha x_k e_k$ for $j \neq k$. If we order the basis decreasing in weight and decree that $x^\alpha x_j e_j$ is before $x^\alpha x_{j+1} e_{j+1}$, then the basis is well-ordered. In the second example in §6, we write out the ordered bases for $\mathscr{D}_2$ and $\mathscr{D}_3$.

Lemma 4. If A is in Jordan normal form, then in the ordered basis $F_{k,\alpha}$ of $\mathscr{D}_m$, $\mathrm{ad}_A|_{\mathscr{D}_m}$ is a lower triangular matrix with diagonal entries $\langle\lambda, \alpha\rangle - \lambda_k$, and if A if diagonal then $\mathrm{ad}_A|_{\mathscr{D}_m}$ is diagonal.

Proof. Let A have eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct), and let $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Let J_j be the $n \times n$ matrix whose $(j, j+1)$ entry is 1 and all whose other entries are 0. For some index set $J \subseteq \{1, \dots, n-1\}$,

$$A = \Lambda + \sum_{j \in J} J_j.$$

The ith row of $F_{k,\alpha}(x)$ is $\delta_{i,k} x^\alpha$, hence $\Lambda F_{k,\alpha} = \lambda_k F_{k,\alpha}$. The entry in row i and column j of the matrix $\frac{\partial F_{k,\alpha}}{\partial x}(x)$ is $\delta_{i,k} x^\alpha \frac{\alpha_j}{x_j}$, hence

$$\frac{\partial F_{k,\alpha}}{\partial x}(x)\Lambda x = x^\alpha \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \frac{\lambda_1 \alpha_1}{x_1} & \cdots & \frac{\lambda_n \alpha_n}{x_n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} x = x^\alpha \begin{bmatrix} 0 \\ \vdots \\ \langle \lambda, \alpha \rangle \\ \vdots \\ 0 \end{bmatrix} = \langle \lambda, \alpha \rangle F_{k,\alpha}(x).$$

Then $\mathrm{ad}_\Lambda F_{k,\alpha}(x) = \frac{\partial F_{k,\alpha}}{\partial x}(x) A x - A F_{k,\alpha}(x) = (\langle \lambda, \alpha \rangle - \lambda_k) F_{k,\alpha}(x)$. Thus the basis vectors $F_{k,\alpha}$ are eigenvectors of ad_Λ with eigenvalues $\langle \lambda, \alpha \rangle - \lambda_k$.

We shall now show that $\mathrm{ad}_A |_{\mathscr{D}_m}$ is a lower-triangular matrix whose diagonal is $\mathrm{ad}_\Lambda |_{\mathscr{D}_m}$. Note that

$$\mathrm{ad}_{J_j}(F_{k,\alpha})(x) = [J_j, F_{k,\alpha}](x) = \frac{\partial F_{k,\alpha}}{\partial x}(x) J_j x - \frac{\partial J_j}{\partial x} F_{k,\alpha} = x^\alpha \frac{\alpha_j x_j}{x_{j+1}} e_k - \delta_{j+1,k} x^\alpha e_{j+1}.$$

If $\alpha_{j+1} \neq 0$ then the first term has weight $\sum_{i=1}^n \alpha_i w_i + w_{j+1} - w_j - w_k$, which is greater than the weight of $F_{k,\alpha}$. If $j = k+1$, then the second term has weight $\sum_{i=1}^n \alpha_i w_i - w_{j+1}$, which is also greater than the weight of $F_{k,\alpha}$. Therefore written in the ordered basis $F_{k,\alpha}$, the matrix $\mathrm{ad}_{J_j} |_{\mathscr{D}_m}$ is strictly lower triangular.

But $\mathrm{ad}_A = \mathrm{ad}_\Lambda + \sum_{j \in J} \mathrm{ad}_{J_j}$, completing the proof. $\square$

Corollary 5. If A is in Jordan normal form and A is nonresonant, then $\mathrm{ad}_A |_{\mathscr{D}_m} : \mathscr{D}_m \to \mathscr{D}_m$ is a linear isomorphism.

4 Poincaré-Dulac normal form theorem for formal vector fields

Say that A is in Jordan normal form and that A has a resonance of order m. Then in the basis $F_{k,\alpha}$ for $\mathscr{D}_m$, the matrix $\mathrm{ad}_A |_{\mathscr{D}_m}$ will be lower triangular with a zero on the diagonal, and hence will not be invertible. For each m, let $\mathscr{N}_m$ be a subspace of $\mathscr{D}_m$ such that

$$\mathscr{D}_m = \mathscr{N}_m + \mathrm{ad}_A(\mathscr{D}_m);$$

we do not suppose here that $\mathscr{N}_m \cap \mathrm{ad}_A(\mathscr{D}_m) = \{0\}$.

Lemma 6. Let F be a formal vector field with constant term 0 whose linearization A is in Jordan normal form and let $\mathscr{N}_m$ satisfy $\mathscr{D}_m = \mathscr{N}_m + \mathrm{ad}_A(\mathscr{D}_m)$. Then F is equivalent to a formal vector field with constant term 0 and linearization A whose nonlinear terms of degree m belong to $\mathscr{N}_m$.

Proof. Let $F_2 = F$, and write

$$F_2(x) = Ax + \sum_{j=2}^{\infty} V_2^j(x)$$

for $V_2^j \in \mathscr{D}_j$. For $H_1(x) = x$, we have $\frac{\partial H_1}{\partial x} F(x) = F_2(H_1(x))$, and hence F is equivalent to the formal vector field F_2. Assume that for some m there are $V_m^j \in \mathscr{N}_j$, $j = 2, \ldots, m-1$ and $V_m^j \in \mathscr{D}_j$, $j = m, \ldots$, such that F is equivalent to

$$F_m(x) = Ax + \sum_{j=2}^{\infty} V_m^j(x).$$

Since $V_m^m \in \mathscr{D}_m$, there are $P_m \in \mathscr{D}_m$ and $V_{m+1}^m \in \mathscr{N}_m$ such that $\mathrm{ad}_A(P_m) = V_{m+1}^m - V_m^m$. Let $H_m(x) = x + P_m(x)$, and let $V_{m+1}^j = V_j^m$ for $j = 2, \ldots, m-1$.

Let $U_{m+1}^j \in \mathscr{D}_j$, $j = m+1, \ldots$ be determined by

$$\sum_{j=m+1}^{\infty} V_m^j(x) + \frac{\partial P_m}{\partial x}(x) \sum_{j=2}^{\infty} V_m^j(x) = \sum_{j=m+1}^{\infty} U_{m+1}^j(x),$$

and then let $V_{m+1}^j \in \mathscr{D}_j$, $j = m+1, \ldots$ be determined by

$$\sum_{j=2}^{\infty} V_{m+1}^j(x + P_m(x)) = \sum_{j=2}^{m} V_{m+1}^j(x) + \sum_{j=m+1}^{\infty} U_{m+1}^j(x);$$

we can check that indeed this determines V_{m+1}^j.

Let $F_{m+1}(x) = Ax + \sum_{j=2}^{\infty} V_{m+1}^j(x)$. Then $\frac{\partial H_m}{\partial x}(x) F_m(x) = F_{m+1}(H_m(x))$, and hence F_m is equivalent to the formal vector field F_{m+1}, where $V_{m+1}^j \in \mathscr{N}_j$ for $j = 2, \ldots, m$, and $V_{m+1}^j \in \mathscr{D}_j$ for $j = m+1, \ldots$.

Then $H^{(m)}(x) = H_m \circ \cdots \circ H_1(x)$ is a formal map such that $\frac{\partial H^{(m)}}{\partial x}(x) F(x) = F_{m+1}(H^{(m)}(x))$. Since $H^{(m+1)} = H_{m+1} \circ H^{(m)}$ and $H^{(m)}$ have the same terms of degree $\leq m$, $\lim_{m\to\infty} H^{(m)}(x)$ exists in $\mathfrak{m}^n$; let H be this limit. Then we can check that H is a formal map such that $\frac{\partial H}{\partial x}(x) F(x) = F'(H(x))$, and so F is equivalent to F'. $\square$

If $\lambda_k = \langle \lambda, \alpha \rangle$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then $F_{k,\alpha} = x^\alpha e_k$ is said to be a *resonant monomial vector* (with respect to A). For $m = |\alpha|$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, the resonant monomial vectors are a basis for $\ker \mathrm{ad}_\Lambda |_{\mathscr{D}_m}$.

The following theorem is the *Poincaré-Dulac normal form theorem*, which states that a resonant formal vector field with constant term 0 whose linearization is in Jordan normal form is equivalent to a formal vector field with constant term 0 and the same linear term whose nonlinear terms are the resonant monomial vectors. We say that a formal vector field with constant term 0 and linearization A is in *Poincaré-Dulac normal form* if its nonlinear terms are resonant monomial vectors with respect to A.

Theorem 7. A formal vector field with constant term 0 whose linearization is in Jordan normal form is equivalent to a formal vector field with constant term 0 and the same linearization whose nonlinear terms are resonant monomial vectors.

Proof. Let F be a formal vector field with constant term 0 and linearization A in Jordan normal form. Say that A has eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct) and let $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

For $m = 2, \dots$, let

$$\mathscr{N}_m = \bigoplus_{\substack{|\alpha|=m \\ \lambda_k = \langle \lambda, \alpha \rangle}} F_{k,\alpha}\mathbb{C}.$$

Then $\mathscr{N}_m = \ker \operatorname{ad}_\Lambda |_{\mathscr{D}_m}$. It follows from Lemma 4 that $\ker \operatorname{ad}_A |_{\mathscr{D}_m} \subseteq \ker \operatorname{ad}_\Lambda |_{\mathscr{D}_m}$. But $\mathscr{D}_m = \ker \operatorname{ad}_A |_{\mathscr{D}_m} + \operatorname{ad}_A(\mathscr{D}_m)$, hence $\mathscr{D}_m = \ker \operatorname{ad}_\Lambda |_{\mathscr{D}_m} + \operatorname{ad}_A(\mathscr{D}_m)$. Therefore $\mathscr{D}_m = \mathscr{N}_m + \operatorname{ad}_A(\mathscr{D}_m)$, and so by Lemma 6, F is equivalent to a formal vector field with constant term 0 and linearization A whose nonlinear terms of degree m belong to $\mathscr{N}_m$, which is the set of resonant monomial vectors of degree m, completing the proof. $\square$

5 Polynomial vector fields

The *Poincaré domain* is the set $\mathfrak{P} \subset \mathbb{C}^n$ of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that the convex hull of the points $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$ does not include the origin. (The complement of the Poincaré domain in $\mathbb{C}^n$ is called the *Siegel domain* $\mathfrak{S}$.)

Theorem 8. If $\lambda \in \mathfrak{P}$, then for all $M > 0$ there are only finitely many $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $1 \leq k \leq n$ such that $|\lambda_k - \langle \alpha, \lambda \rangle| \leq M$.

Proof. Since the convex hull of the points $\lambda_1, \dots, \lambda_n$ does not include the origin, there is a line through the origin that does not intersect the convex hull. It follows that there is an $\mathbb{R}$-linear map $\ell : \mathbb{C} \to \mathbb{R}$ and some $r > 0$ such that $\ell(\lambda_k) \leq -r$ for all k.

Then

$$\ell(\langle \alpha, \lambda \rangle) = \sum_{k=1}^n \alpha_k \ell(\lambda_k) \leq \sum_{k=1}^n \alpha_k(-r) = -r|\alpha|.$$

Let $-R = \min_{1 \leq k \leq n} \ell(\lambda_k)$, and let $\|\ell\| = \max_{|z|=1} |\ell(z)|$. For all $\alpha \in \mathbb{Z}^n_{\geq 0}$ and all k,

$$\|\ell\||\lambda_k - \langle \alpha, \lambda \rangle| \geq |\ell(\lambda_k - \langle \alpha, \lambda \rangle)| \geq \ell(\lambda_k - \langle \alpha, \lambda \rangle) \geq \ell(\lambda_k) + r|\alpha| \geq -R + r|\alpha|.$$

There are only finitely many $\alpha \in \mathbb{Z}^n_{\geq 0}$ such that $\frac{-R+r|\alpha|}{\|\ell\|} \leq M$. Therefore there are only finitely many $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $1 \leq k \leq n$ such that $|\lambda_k - \langle \alpha, \lambda \rangle)| \leq M$. $\square$

In particular, if $\lambda \in \mathfrak{P}$ then there are only finitely many $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $1 \leq k \leq n$ such that $\lambda_k = \langle \alpha, \lambda \rangle$. Thus we have the following corollary to the above theorem.

Corollary 9. Let F be a formal vector field with constant term 0 whose linearization A is in Jordan normal form, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, and let $\lambda = (\lambda_1, \dots, \lambda_n)$. If $\lambda \in \mathfrak{P}$, then there are only finitely many nonlinear terms in the Poincaré-Dulac normal form of F.

6 Examples

First example. Let

$$F(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}.$$

This formal vector field has linearization $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is nonresonant. For all $m \geq 2$, $\mathrm{ad}_A|_{\mathscr{D}_m} = \mathrm{id}_{\mathscr{D}_m}$, and hence for all $m \geq 2$, $P_m(x) = -V_m^m(x)$. $H_1(x) = x$. We shall find $H_m(x)$ for $m = 2, \dots, 5$. This will determine the terms in $H(x)$ of degree ≤ 5.

$$\sum_{j=m+1}^{\infty} V_m^j(x) + \frac{\partial P_m}{\partial x}(x) \sum_{j=m}^{\infty} V_m^j(x) = \sum_{j=m+1}^{\infty} V_{m+1}^j(H_m(x)). \tag{4}$$

$m = 2$: $V_2^2(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$, so $P_2(x) = -V_2^2(x) = \begin{bmatrix} -x_1^2 \\ -x_2^2 \end{bmatrix}$ and $H_2(x) = \begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix}$. For $j \geq 3$, $V_2^j(x) = 0$. Then (4) is

$$\frac{\partial P_2}{\partial x}(x) V_2^2(x) = V_3^3(H_2(x)) + V_3^4(H_2(x)) + V_3^5(H_2(x)) + V_3^6(H_2(x)) + \cdots$$

which is

$$\begin{aligned} \begin{bmatrix} -2x_1 & 0 \\ 0 & -2x_2 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} &= V_3^3\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) + V_3^4\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) + V_3^5\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) \\ &\quad + V_3^6\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) + \cdots \end{aligned}$$

It follows that $V_3^3(x) = \begin{bmatrix} -2x_1^3 \\ -2x_2^3 \end{bmatrix}$. So

$$V_3^3\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) = \begin{bmatrix} -2x_1^3 + 6x_1^4 - 6x_1^5 + 2x_1^6 \\ -2x_2^3 + 6x_2^4 - 6x_2^5 + 2x_2^6 \end{bmatrix}.$$

It follows that $V_3^4(x) = \begin{bmatrix} -6x_1^4 \\ -6x_2^4 \end{bmatrix}$. So

$$V_3^4\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) = \begin{bmatrix} -6x_1^4 + 24x_1^5 - 36x_1^6 + 24x_1^7 - 6x_1^8 \\ -6x_2^4 + 24x_2^5 - 36x_2^6 + 24x_2^7 - 6x_2^8 \end{bmatrix}.$$

It follows that $V_3^5(x) = \begin{bmatrix} -18x_1^5 \\ -18x_2^5 \end{bmatrix}$. So

$$V_3^5\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) = \begin{bmatrix} -18x_1^5 + 90x_1^6 - 180x_1^7 + 180x_1^8 - 90x_1^9 + 18x_1^{10} \\ -18x_2^5 + 90x_2^6 - 180x_2^7 + 180x_2^8 - 90x_2^9 + 18x_2^{10} \end{bmatrix}.$$

It follows that $V_3^6(x) = \begin{bmatrix} -56x_1^6 \\ -56x_2^6 \end{bmatrix}$.

$m = 3$: $V_3^3(x) = \begin{bmatrix} -2x_1^3 \\ -2x_2^3 \end{bmatrix}$, so $P_3(x) = \begin{bmatrix} 2x_1^3 \\ 2x_2^3 \end{bmatrix}$ and $H_3(x) = \begin{bmatrix} x_1 + 2x_1^3 \\ x_2 + 2x_2^3 \end{bmatrix}$. Then (4) is

$$\begin{aligned} &V_3^4(x) + V_3^5(x) + V_3^6(x) + \cdots + \begin{bmatrix} 6x_1^2 & 0 \\ 0 & 6x_2^2 \end{bmatrix} \Big(V_3^3(x) + V_3^4(x) + \cdots \Big) \\ =&V_4^4(x + P_3(x)) + V_4^5(x + P_3(x)) + V_4^6(x + P_3(x)) + \cdots \end{aligned}$$

which is

$$\begin{aligned} &\begin{bmatrix} -6x_1^4 \\ -6x_2^4 \end{bmatrix} + \begin{bmatrix} -18x_1^5 \\ -18x_2^5 \end{bmatrix} + \begin{bmatrix} -56x_1^6 \\ -56x_2^6 \end{bmatrix} + \cdots + \begin{bmatrix} -12x_1^5 \\ -12x_2^5 \end{bmatrix} + \begin{bmatrix} -36x_1^6 \\ -36x_2^6 \end{bmatrix} + \cdots \\ =&V_4^4(x + P_3(x)) + V_4^5(x + P_3(x)) + V_4^6(x + P_3(x)) + \cdots \end{aligned}$$

It follows that $V_4^4(x) = \begin{bmatrix} -6x_1^4 \\ -6x_2^4 \end{bmatrix}$. So

$$V_4^4\Big(\begin{bmatrix} x_1 + 2x_1^3 \\ x_2 + 2x_2^3 \end{bmatrix} \Big) = \begin{bmatrix} -6x_1^4 - 48x_1^6 - 144x_1^8 - 192x_1^{10} - 96x_1^{12} \\ -6x_2^4 - 48x_2^6 - 144x_2^8 - 192x_2^{10} - 96x_1^{12} \end{bmatrix}.$$

It follows that $V_4^5(x) = \begin{bmatrix} -30x_1^5 \\ -30x_2^5 \end{bmatrix}$. In $V_4^5\Big(\begin{bmatrix} x_1 + 2x_1^3 \\ x_2 + 2x_2^3 \end{bmatrix} \Big)$ there are no terms of degree 6, so it follows that $V_4^6(x) = \begin{bmatrix} -44x_1^6 \\ -44x_2^6 \end{bmatrix}$.

$m = 4$: $V_4^4(x) = \begin{bmatrix} -6x_1^4 \\ -6x_2^4 \end{bmatrix}$, so $P_4(x) = \begin{bmatrix} 6x_1^4 \\ 6x_2^4 \end{bmatrix}$ and $H_4(x) = \begin{bmatrix} x_1 + 6x_1^4 \\ x_2 + 6x_2^4 \end{bmatrix}$. Then (4) is

$$V_4^5(x) + \cdots + \begin{bmatrix} 24x_1^3 & 0 \\ 0 & 24x_2^3 \end{bmatrix} \Big(V_4^4(x) + \cdots \Big) = V_5^5(H_4(x)) + \cdots$$

It follows that $V_5^5(x) = V_4^5(x) = \begin{bmatrix} -30x_1^5 \\ -30x_2^5 \end{bmatrix}$.

Because $V_5^5(x) = \begin{bmatrix} -30x_1^5 \\ -30x_2^5 \end{bmatrix}$, we have $P_5(x) = \begin{bmatrix} 30x_1^5 \\ 30x_2^5 \end{bmatrix}$ and $H_5(x) = \begin{bmatrix} x_1 + 30x_1^5 \\ x_2 + 30x_2^5 \end{bmatrix}$.

Let us figure out $H^{(5)}(x) = H_5 \circ H_4 \circ H_3 \circ H_2 \circ H_1(x)$. $H_1(x) = x$, $H_2(x) = \begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix}$, $H_3(x) = \begin{bmatrix} x_1 + 2x_1^3 \\ x_2 + 2x_2^3 \end{bmatrix}$, $H_4(x) = \begin{bmatrix} x_1 + 6x_1^4 \\ x_2 + 6x_2^4 \end{bmatrix}$, and $H_5(x) = \begin{bmatrix} x_1 + 30x_1^5 \\ x_2 + 30x_2^5 \end{bmatrix}$. Then

$$H^{(3)}(x) = H_3 \circ H_2 \circ H_1(x) = H_3\Big(\begin{bmatrix} x_1 - x_1^2 \\ x_2 - x_2^2 \end{bmatrix} \Big) = \begin{bmatrix} x_1 - x_1^2 + 2x_1^3 - 6x_1^4 + 6x_1^5 - 2x_1^6 \\ x_2 - x_2^2 + 2x_2^3 - 6x_2^4 + 6x_2^6 - 2x_2^6 \end{bmatrix}.$$

We can compute $H^{(4)}(x)$ and then $H^{(5)}(x)$. Each component of $H^{(5)}(x)$ is polynomial of degree 120, and

$$H^{(5)}(x) = \begin{bmatrix} x_1 - x_1^2 + 2x_1^3 + 12x_1^5 - 68x_1^6 + 288x_1^7 - 630x_1^8 - 1662x_1^9 \\ x_2 - x_2^2 + 2x_2^3 + 12x_2^5 - 68x_2^6 + 288x_2^7 - 630x_2^8 - 1662x_2^9 \end{bmatrix} + \begin{bmatrix} O(x_1^{10}) \\ O(x_2^{10}) \end{bmatrix},$$

and thus

$$H(x) = \lim_{m \to \infty} H^{(m)}(x) = \begin{bmatrix} x_1 - x_1^2 + 2x_1^3 + 12x_1^5 \\ x_2 - x_2^2 + 2x_2^3 + 12x_2^5 \end{bmatrix} + \begin{bmatrix} O(x_1^6) \\ O(x_2^6) \end{bmatrix}.$$

Second example. We will determine the Poincaré-Dulac normal form for the formal vector field

$$F(x) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix},$$

and find $H_m(x)$ for $m = 2, 3, 4$, which will determine the terms in $H(x)$ of degree ≤ 4.

The formal vector field $F(x)$ has linearization $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. Let $\lambda_1 = 3, \lambda_2 = 1$.

The monomial basis vectors for $\mathscr{D}_2$ are

$$F_{1,(2,0)} = \begin{bmatrix} x_1^2 \\ 0 \end{bmatrix}, F_{1,(1,1)} = \begin{bmatrix} x_1 x_2 \\ 0 \end{bmatrix}, F_{1,(0,2)} = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix},$$
$$F_{2,(2,0)} = \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}, F_{2,(1,1)} = \begin{bmatrix} 0 \\ x_1 x_2 \end{bmatrix}, F_{2,(0,2)} = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}.$$

The weights of these basis vectors are respectively

$$2w_1 - w_1 = w_1 = 1.73\ldots, w_1 + w_2 - w_1 = w_2 = 1.41\ldots, 2w_2 - w_1 = 1.09\ldots,$$
$$2w_1 - w_2 = 2.04\ldots, w_1 + w_2 - w_2 = w_1 = 1.73\ldots, 2w_2 - w_2 = w_2 = 1.41\ldots.$$

The basis vectors are ordered such that $F_{1,(2,0)}$ is before $F_{2,(1,1)}$ and $F_{1,(1,1)}$ is before $F_{2,(0,2)}$. Therefore the ordering of the basis vectors for $\mathscr{D}_2$ is

$$F_{2,(2,0)} > F_{1,(2,0)} > F_{2,(1,1)} > F_{1,(1,1)} > F_{2,(0,2)} > F_{1,(0,2)}. \tag{5}$$

The monomial basis vectors for $\mathscr{D}_3$ are

$$F_{1,(3,0)} = \begin{bmatrix} x_1^3 \\ 0 \end{bmatrix}, F_{1,(2,1)} = \begin{bmatrix} x_1^2 x_2 \\ 0 \end{bmatrix}, F_{1,(1,2)} = \begin{bmatrix} x_1 x_2^2 \\ 0 \end{bmatrix}, F_{1,(0,3)} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix},$$
$$F_{2,(3,0)} = \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix}, F_{2,(2,1)} = \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix}, F_{2,(1,2)} = \begin{bmatrix} 0 \\ x_1 x_2^2 \end{bmatrix}, F_{2,(0,3)} = \begin{bmatrix} 0 \\ x_2^3 \end{bmatrix}.$$

The weights of these basis vectors are respectively

$$2w_1 = 3.46\ldots, w_1 + w_2 = 3.14\ldots, 2w_2 = 2.82\ldots, 3w_2 - w_1 = 2.51\ldots,$$
$$3w_1 - w_2 = 3.78\ldots, 2w_1 = 3.46\ldots, w_1 + w_2 = 3.14\ldots, 2w_2 = 2.82\ldots.$$

The basis vectors are ordered such that $F_{1,(3,0)}$ is before $F_{2,(2,1)}$, $F_{1,(2,1)}$ is before $F_{2,(1,2)}$, and $F_{1,(1,2)}$ is before $F_{2,(0,3)}$. Therefore the ordering of the basis vectors for $\mathscr{D}_3$ is

$$F_{2,(3,0)} > F_{1,(3,0)} > F_{2,(2,1)} > F_{1,(2,1)} > F_{2,(1,2)} > F_{1,(1,2)} > F_{2,(0,3)} > F_{1,(0,3)}. \tag{6}$$

We calculate that $\mathrm{ad}_A |_{\mathscr{D}_2}$ written in the ordered basis (5) is $\mathrm{diag}(3, 1, -1, 5, 3, 1)$, and we calculate that $\mathrm{ad}_A |_{\mathscr{D}_3}$ written in the ordered basis (6) is $\mathrm{diag}(8, 6, 6, 4, 4, 2, 2, 0)$. Thus $\ker \mathrm{ad}_A |_{\mathscr{D}_3} = \mathrm{span}_{\mathbb{C}}\{F_{1,(0,3)}\}$.

7 Conclusion

This paper is useful for people who want fully worked proofs of the Poincaré normal form theorem and the Poincaré-Dulac normal form theorem for formal vector fields, and examples that explicitly follow the constructions in the proofs.

References

[1] D. V. Anosov, S. Kh. Aranson, V. I. Arnold, I. U. Bronshtein, V. Z. Grines, and Yu. S. Ilyashenko. *Ordinary differential equations and smooth dynamical systems*. Springer, 1997. Translated from the Russian by E. R. Dawson and D. O'Shea.

[2] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*, volume 250 of *Grundlehren der mathematischen Wissenschaften*. Springer, second edition, 1988. Translated from the Russian by Joseph Szücs.

[3] Jean-Paul Dufour and Nguyen Tien Zung. *Poisson structures and their normal forms*, volume 242 of *Progress in Mathematics*. Birkhäuser, 2005.

[4] Zihua Guo, Soonsik Kwon, and Tadahiro Oh. Poincaré-Dulac normal form reduction for unconditional well-posedness of the periodic cubic NLS. arXiv:1103.5271.

[5] Yulij Ilyashenko and Sergei Yakovenko. *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2007.

[6] Jürgen Scheurle and Sebastian Walcher. On normal form computations. In Paul Newton, Philip Holmes, and Alan Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 309–325. Springer, 2002.

[7] Stephen Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*, volume 2 of *Texts in Applied Mathematics*. Springer, second edition, 2003.