Abstract Fourier series and Parseval's identity

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1 Orthonormal basis

Let H be a separable complex Hilbert space.[1] If $e_i \in H$, $i \geq 1$, and $\langle e_i, e_j \rangle = \delta_{i,j}$, we say that the set $\{e_i\}$ is *orthonormal*. If $\text{span}\{e_i : i \geq 1\}$ is a dense subspace of H, we say that $\{e_i : i \geq 1\}$ is an *orthonormal basis* for H. We can write this in another way. If $S_\alpha, \alpha \in I$ are subsets of H, let $\bigvee_{\alpha \in I} S_\alpha$ be the closure of the span of $\bigcup_{\alpha \in I} S_\alpha$. To say that $\{e_i\}$ is an orthonormal basis for H is to say that $\{e_i\}$ is orthonormal and that $H = \bigvee_{i \geq 1} \{e_i\}$.

2 Abstract Fourier series

If $a_k \in \mathbb{C}$ and the sequence $\sum_{k=1}^n a_k e_k$ converges in H, we denote its limit by

$$\sum_{k=1}^\infty a_k e_k.$$

This is a *definition* of an infinite sum in H. Since H is complete, one usually shows that a sequence converges by showing that the sequence is Cauchy, and hence to show that $\sum_{k=1}^n a_k e_k$ converges it is equivalent to show that

$$\sum_{k=m+1}^n a_k e_k \to 0$$

as $m, n \to \infty$. And showing this is equivalent to showing that

$$\left\langle \sum_{k=m+1}^n a_k e_k, \sum_{k=m+1}^n a_k e_k \right\rangle \to 0$$

as $m, n \to \infty$. This is equivalent to

$$\sum_{k=m+1}^n |a_k|^2 \to 0$$

[1]One talk do everything we are doing and obtain the same results for nonseparable Hilbert spaces, but one has to define what uncountable sums mean. This is done in John B. Conway, *A Course in Functional Analysis*, second ed., chapter I.

as $m, n \to \infty$, and this is equivalent to the series

$$\sum_{k=1}^{\infty} |a_k|^2$$

converging. Thus, the series $\sum_{k=1}^{\infty} a_k e_k$ converges if and only if the series $\sum_{k=1}^{\infty} |a_k|^2$ converges.[2]

Let $\{e_i : i \geq 1\}$ be an orthonormal basis for H; it is a fact that one exists. Let $v \in H$ and define

$$s_n = \sum_{k=1}^{n} \langle v, e_k \rangle e_k.$$

If $1 \leq i \leq n$ then

$$\langle v - s_n, e_i \rangle = \langle v, e_i \rangle - \sum_{k=1}^{n} \langle v, e_k \rangle \langle e_k, e_i \rangle = \langle v, e_i \rangle - \langle v, e_i \rangle = 0,$$

hence

$$\langle v - s_n, s_n \rangle = 0.$$

It follows that

$$\begin{aligned} \sum_{k=1}^{n} | \langle v, e_k \rangle |^2 &= \langle s_n, s_n \rangle \\ &\leq \langle s_n, s_n \rangle + \langle v - s_n, v - s_n \rangle \\ &= \langle v, v \rangle , \end{aligned}$$

where we used $\langle v - s_n, s_n \rangle = 0$ in the third line. Therefore the series $\sum_{k=1}^{\infty} | \langle v, e_k \rangle |^2$ converges, and so the sequence s_n converges to some $v' = \sum_{k=1}^{\infty} \langle v, e_k \rangle e_k \in H$. Since s_n converges to v', in particular it converges weakly to v', i.e., for any $w \in H$,

$$\lim_{n\to\infty} \langle s_n, w \rangle = \langle v', w \rangle .$$

Therefore for any j,

$$\langle v - v', e_j \rangle = \langle v, e_j \rangle - \langle v', e_j \rangle = \langle v, v_j \rangle - \lim_{n\to\infty} \langle s_n, e_j \rangle = \langle v, v_j \rangle - \langle v, v_j \rangle = 0;$$

this is because for $n \geq j$ we have $\langle v - s_n, e_j \rangle = 0$ and hence $\langle v, e_j \rangle = \langle s_n, e_j \rangle$. As $\langle v - v', e_j \rangle = 0$ for all j, it follows that $v - v' = 0$, i.e. $v = v'$. Hence,

$$v = \sum_{k=1}^{\infty} \langle v, e_k \rangle e_k.$$

[2] Furthermore, using the triangle inequality rather than the orthonormality of the e_k, one can check that if the series $\sum_{k=1}^{\infty} |a_k|$ converges then the series $\sum_{k=1}^{\infty} a_k e_k$ converges.

We call this an *abstract Fourier series* for v.[3] It can be written as

$$v = \sum_{k=1}^{\infty} (e_k \otimes e_k) v,$$

and thus can be written without v as

$$\mathrm{id}_H = \sum_{k=1}^{\infty} e_k \otimes e_k;$$

$e_k \otimes e_k \in B(H)$ is a projection with rank 1, and the above series conveges in the *strong operator topology* on $B(H)$. Writing the identity map in this way is called a *resolution of the identity*.

3 Parseval's identity

On the one hand

$$\lim_{n\to\infty} \|s_n\|^2 = \sum_{k=1}^{\infty} |\langle v, e_k \rangle|^2.$$

On the other hand,

$$\lim_{n\to\infty} \|s_n\|^2 = \|v\|^2.$$

Hence

$$\|v\|^2 = \sum_{k=1}^{\infty} |\langle v, e_k \rangle|^2,$$

which is *Parseval's identity*.

[3] If $H = L^2(\mathbb{T})$, one checks that $e_k = e^{ik}, k \in \mathbb{Z}$, is an orthonormal basis for H. Then,

$$\langle f, e_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ik} dt$$

and f is the limit in H of $\sum_{k=0}^{n} \langle f, e_k \rangle e^{ik}$. Thus in H,

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k \rangle e^{ik}.$$