Harmonic analysis on the p-adic numbers

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1 p-adic numbers

Let p be prime and let $N_p = \{0, \ldots, p-1\}$. $\mathbb{Q}_p \subset \prod_{\mathbb{Z}} N_p$. For $x \in \mathbb{Q}_p$,

$$x = \lim_{m\to\infty} \sum_{k\leq m} x(k)p^k = \sum_{k\in\mathbb{Z}} x(k)p^k = \sum_{k\geq v_p(x)} x(k)p^k$$

for

$$v_p(x) = \inf\{k \in \mathbb{Z} : x(k) \neq 0\}.$$

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \geq 0\}.$$

For $x, y \in \mathbb{Q}_p$,

$$v_p(xy) = v_p(x) + v_p(y), \qquad v_p(x+y) \geq \min(v_p(x), v_p(y)),$$

and $v_p(x) = \infty$ if and only if $x = 0$. The p-integers $\mathbb{Z}_p$ with the valuation v_p are a Euclidean domain: for $f, g \in \mathbb{Z}_p$ with $v_p(f) \geq v_p(g)$ we have $f \cdot g^{-1} \in \mathbb{Z}_p$. $\mathbb{Z}_p^*$ is the set of those $x \in \mathbb{Z}_p$ for which there is some $y \in \mathbb{Z}_p$ satisfying $xy = 1$.

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : v_p(x) = 0\}.$$

The ideals of the ring $\mathbb{Z}_p$ are $\{0\}$ and $p^n\mathbb{Z}_p$, $n \geq 0$. From this it follows that $\mathbb{Z}_p$ is a **discrete valuation ring**, a principal ideal domain with exactly one maximal ideal, namely $p\mathbb{Z}_p$; $\mathbb{Z}_p$ is the **valuation ring** of $\mathbb{Q}_p$ with the valuation v_p. For $n \geq 1$, $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is isomorphic as a ring with $\mathbb{Z}/p^n\mathbb{Z}$.

$$|x|_p = p^{-v_p(x)}, \qquad d_p(x,y) = |x-y|_p.$$

With the topology induced by the metric d_p, $\mathbb{Q}_p$ is a locally compact abelian group, and $(\mathbb{Q}_p, d_p)$ is a complete metric space. $(\mathbb{Q}_p, |\cdot|_p)$ is a complete nonarchimedean valued field. For $x \in \mathbb{Q}_p$,

$$\{x + p^n\mathbb{Z}_p : n \in \mathbb{Z}\}$$

is a local base at x for the topology of $\mathbb{Q}_p$.

$$[x]_p = \sum_{k\geq 0} x(k)p^k \in \mathbb{Z}_p, \quad \{x\}_p = \sum_{k<0} x(k)p^k \in [0,1) \cap \mathbb{Z}[1/p].$$

$$\psi_p(x) = e^{2\pi i\{x\}_p}$$

is a continuous group homomorphism $\mathbb{Q}_p \to S^1$. Its image is the discrete abelian group

$$\mathbb{Z}[p^\infty] = \{e^{2\pi i m p^{-n}} : m, n \geq 0\},$$

the Prüfer p-group, and its kernel is $\mathbb{Z}_p$. $\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{Z}[p^\infty]$ are isomorphic as discrete abelian groups. There is a complete algebraically closed nonarchimedean valued field $\mathbb{C}_p$, unique up to unique isomorphism, that is an extension of $(\mathbb{Q}_p, |\cdot|_p)$.

2 Pontryagin dual

Denote by $\widehat{\mathbb{Q}}_p$ the **Pontryagin dual** of the locally compact abelian group $(\mathbb{Q}_p, +)$. For $\xi \in \widehat{\mathbb{Q}}_p$ and $x \in \mathbb{Q}_p$,

$$x = \sum_{k \in \mathbb{Z}} x(k) p^k$$

and

$$\langle x, \xi \rangle = \xi(x) = \prod_{k \in \mathbb{Z}} \xi(x(k)p^k) = \prod_{k \in \mathbb{Z}} \xi(p^k)^{x(k)}. \tag{1}$$

For $y \in \mathbb{Q}_p$, define $m_y : \mathbb{Q}_p \to \mathbb{Q}_p$ by $m_y(x) = y \cdot x$, which is a continuous group homomorphism. Then $\xi_y = \psi_p \circ m_y$ is a continuous group homomorphism $\mathbb{Q}_p \to S^1$, namely $\xi_y \in \widehat{\mathbb{Q}}_p$. The kernel of ξ_y is $\{x \in \mathbb{Q}_p : yx \in \mathbb{Z}_p\}$, in other words

$$\ker \xi_y = \{x \in \mathbb{Q}_p : |x|_p \leq |y|_p^{-1}\}$$

where $|0|_p^{-1} = \infty$. If $y \neq 0$ then

$$\ker \xi_y = \{x \in \mathbb{Q}_p : |x|_p \leq |y|_p^{-1}\} = p^{-v_p(y)}\mathbb{Z}_p.$$

We shall prove that $y \mapsto \xi_y$ is an isomorphism of topological groups $\mathbb{Q}_p \to \widehat{\mathbb{Q}}_p$. We will use the following lemma.[1]

Lemma 1. *If $\xi \in \widehat{\mathbb{Q}}_p$ then there is some $n \in \mathbb{Z}$ such that $\langle x, \xi \rangle = 1$ for $x \in p^n\mathbb{Z}_p$.*

Proof. Let $U = \{e^{2\pi i\theta} : |\theta| < \frac{1}{4}\}$, which is an open set in S^1. As $\xi(0) \in U$ and $\{p^n\mathbb{Z}_p : n \in \mathbb{Z}\}$ is a local base at 0, there is some $n \in \mathbb{Z}$ such that $p^n\mathbb{Z}_p \subset \xi^{-1}(U)$. This means that $\xi(p^n\mathbb{Z}_p) \subset U$, and because $\xi : \mathbb{Q}_p \to S^1$ is a group homomorphism, $\xi(p^n\mathbb{Z}_p)$ is therefore a subgroup of S^1 contained in U. But the only subgroup of S^1 contained in U is $\{1\}$, and therefore $\xi(p^n\mathbb{Z}_p) = \{1\}$. □

[1] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 92, Lemma 4.9.

Suppose $\xi \in \widehat{\mathbb{Q}}_p$, $\xi \neq 1$. By (1) there is then some k such that $\xi(p^k) \neq 1$. Now, $|p^j|_p = p^{-j} \to 0$ as $j \to \infty$, so $p^j \to 0$ in $\mathbb{Q}_p$ and therefore $\xi(p^j) \to 1$ as $j \to \infty$. Let

$$j_\xi - 1 = \max\{k \in \mathbb{Z} : \langle p^k, \xi \rangle \neq 1\}.$$

Then $\langle p^{j_\xi - 1}, \xi, \neq \rangle 1$ and $\langle p^j, \xi \rangle = 1$ for $j \geq j_\xi$. In particular, $j_\xi = 0$ is equivalent with $\langle 1, \xi \rangle = 1$ and $\langle p, \xi \rangle \neq 1$.[2]

Lemma 2. *Suppose that $\xi \in \widehat{\mathbb{Q}}_p$ with $\langle 1, \xi \rangle = 1$ and $\langle p^{-1}, \xi \rangle \neq 1$. Then there are $c_j \in N_p$, $j \geq 0$, with $c_0 \neq 0$, such that*

$$\langle p^{-k}, \xi \rangle = \exp\left(2\pi i \sum_{j=1}^{k} c_{k-j} p^{-j}\right), \qquad k \geq 1.$$

Proof. Let $\omega_0 = \langle 1, \xi \rangle = 1$ and for $k \geq 1$ let $\omega_k = \langle p^{-k}, \xi \rangle \in S^1$, which satisfy

$$\omega_{k+1}^p = \langle p^{-k}, \xi \rangle = \omega_k.$$

Because $\omega_1^p = 1$ this means that there is some $c_0 \in N_p$ such that $\omega_1 = e^{2\pi i c_0 p^{-1}}$, and by hypothesis $\omega_1 \neq 1$, which means $c_0 \neq 0$. By induction, suppose for some $k \geq 1$ and $c_0, \ldots, c_{k-1} \in N_p$, $c_0 \neq 0$, such that

$$\omega_k = \exp\left(2\pi i \sum_{j=1}^{k} c_{k-j} p^{-j}\right).$$

Generally, if $z^p = e^{i\theta}$ then there is some $c \in N_p$ such that $z = e^{\frac{1}{p} i\theta} e^{2\pi i c p^{-1}}$. Thus, the fact that $\omega_{k+1}^p = \omega_k$ means that there is some $c_k \in N_p$ such that

$$\omega_{k+1} = \exp\left(\frac{1}{p} \cdot 2\pi i \sum_{j=1}^{k} c_{k-j} p^{-j}\right) \cdot e^{2\pi i c_k p^{-1}} = \exp\left(2\pi i \sum_{j=1}^{k+1} c_{k+1-j} p^{-j}\right).$$

□

We prove a final lemma.[3]

Lemma 3. *Suppose that $\xi \in \widehat{\mathbb{Q}}_p$ with $\langle 1, \xi \rangle = 1$ and $\langle p^{-1}, \xi \rangle \neq 1$. Then there is some $y \in \mathbb{Q}_p$ with $|y|_p = 1$ and $\xi = \xi_y$.*

Proof. By Lemma 2 there are $c_j \in N_p$, $j \geq 0$, $c_0 \neq 0$, such that

$$\langle p^{-k}, \xi \rangle = \exp\left(2\pi i \sum_{j=1}^{k} c_{k-j} p^{-j}\right), \qquad k \geq 1.$$

[2] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 92, Lemma 4.10.
[3] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 92, Lemma 4.11.

Define $y \in \mathbb{Q}_p$ by $y(j) = c_j$ for $j \geq 0$ and $y(j) = 0$ for $j < 0$. As $y(0) = c_0 \neq 0$, $|y|_p = 1$. For $k \geq 1$ and $-k \leq j \leq -1$ we have $(p^{-k}y)(j) = y(j+k) = c_{j+k}$, and for $j < -k$ we have $(p^{-k}y)(j) = y(j+k) = 0$, so

$$\{p^{-k}y\}_p = \sum_{j<0}(p^{-k}y)(j)p^j = \sum_{-k\leq j\leq -1}(p^{-k}y)(j)p^j = \sum_{-k\leq j\leq -1} c_{j+k}p^j,$$

yielding

$$\left\langle p^{-k}, \xi \right\rangle = \exp\left(2\pi i \sum_{-k\leq j\leq -1} c_{k+j}p^j\right) = \exp(2\pi i\{p^{-k}y\}_p),$$

i.e. $\left\langle p^{-k}, \xi \right\rangle = \psi_p(p^{-k}y) = \left\langle p^{-k}, \xi_y \right\rangle$. But $\langle 1, \xi \rangle = 1$ implies that $\left\langle p^k, \xi \right\rangle = 1$ for $k \geq 0$, and because $y(k) = 0$ for $k < 0$,

$$\langle 1, \xi_y \rangle = e^{2\pi i\{y\}_p} = 1,$$

which implies that $\left\langle p^k, \xi \right\rangle = 1$ for $k \geq 0$. Therefore $\left\langle p^k, \xi \right\rangle = \left\langle p^k, \xi_y \right\rangle$ for all $k \in \mathbb{Z}$, which implies that $\xi = \xi_y$. □

We now have worked out enough to prove that $y \mapsto \xi_y$ is an isomorphism.[4]

Theorem 4. *$y \mapsto \xi_y$ is an isomorphism of topological groups $\mathbb{Q}_p \to \widehat{\mathbb{Q}}_p$.*

Proof. For $x \in \mathbb{Q}_p$,

$$\langle x, \xi_y\xi_z \rangle = \langle x, \xi_y \rangle \langle x, \xi_z \rangle = \psi_p(yx)\psi_p(zx) = \psi_p(yx + zx) = \langle x, \xi_{y+z} \rangle,$$

showing that $y \mapsto \xi_y$ is a group homomorphism. Suppose that $\xi_y = 1$. Then for all $x \in \mathbb{Q}_p$ we have $\langle x, \xi_y \rangle = 1$, i.e. $e^{2\pi i\{yx\}_p} = 1$, i.e. $\{yx\}_p = 0$, i.e. $yx \in \mathbb{Z}_p$. This implies $y = 0$, showing that $y \mapsto \xi_y$ is injective. It remains to show that $y \mapsto \xi_y$ is surjective, that it is continuous, and that it is an open map. But in fact, the open mapping theorem for locally compact groups[5] tells us that if $f : G \to H$ is a continuous group homomorphism of locally compact groups that is surjective and G is σ-compact then f is open. $\mathbb{Q}_p$ is σ-compact: $\mathbb{Q}_p = \bigcup_{n\in\mathbb{Z}} p^n\mathbb{Z}_p$. So to prove the claim it suffices to prove that $y \mapsto \xi_y$ is surjective and continuous.

Let $\xi \in \widehat{\mathbb{Q}}_p$, $\xi \neq 1$. By Lemma 1, let

$$j - 1 = \max\{k \in \mathbb{Z} : \left\langle p^k, \xi \right\rangle \neq 1\},$$

for which $\left\langle p^{j-1}, \xi \right\rangle \neq 1$ and $\left\langle p^j, \xi \right\rangle = 1$. Define $\eta \in \widehat{\mathbb{Q}}_p$ by

$$\langle x, \eta \rangle = \left\langle p^j x, \xi \right\rangle,$$

[4] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 92, Theorem 4.12.

[5] Karl H. Hofmann and Sidney A. Morris, *The Structure of Compact Groups*, 2nd revised and augmented edition, p. 669, Appendix 1.

which satisfies $\langle 1, \eta\rangle = \left\langle p^j x, \xi\right\rangle = 1$ and $\left\langle p^{-1}, \eta\right\rangle = \left\langle p^{j-1}, \xi\right\rangle \neq 1$. Thus we can apply Lemma 3: there is some $z \in \mathbb{Q}_p$, $|z|_p = 1$, such that $\eta = \xi_z$. Now let $y = p^{-j} z \in \mathbb{Q}_p$, which satisfies

$$\langle x, \xi_y\rangle = e^{2\pi i\{yx\}_p} = e^{2\pi i\{z\cdot p^{-j}x\}_p} = \left\langle p^{-j}x, \xi_z\right\rangle = \left\langle p^{-j}x, \eta\right\rangle = \langle x, \xi\rangle,$$

from which it follows that $\xi = \xi_y$. Therefore $y \mapsto \xi_y$ is surjective.

For $j \geq 1$ and $k \geq 1$ define

$$N(j,k) = \{\xi \in \widehat{\mathbb{Q}}_p : |\langle x, \xi\rangle - 1| < j^{-1} \text{ for } |x|_p \leq p^{-k}\}.$$

It is a fact that $\{N(j,k) : j \geq 1, k \geq 1\}$ is a local base at 1 for the topology of $\widehat{\mathbb{Q}}_p$. Suppose $y \in \mathbb{Z}_p$. For $j \geq 1$, $k \geq 1$ and $|x|_p \leq p^{-k}$, we have $xy \in \mathbb{Z}_p$ and hence $\langle x, \xi_y\rangle = 1$, hence $y \in N(j,k)$. This shows that $\xi(\mathbb{Z}_p) \subset N(j,k)$, and therefore $y \mapsto \xi_y$ is continuous at 0. □

3 Haar measure

For a locally compact abelian group G, a **Haar measure on** G is a Borel measure m on G such that (i) $m(x + E) = m(E)$ for each Borel set E and $x \in G$, (ii) if K is a compact set then $m(K) < \infty$, (iii) if E is a Borel set then

$$m(E) = \inf\{m(U) : E \subset U,\, U \text{ open}\},$$

and (iv) if U is an open set then

$$m(E) = \sup\{m(K) : K \subset U,\, K \text{ compact}\},$$

It is a fact that for any locally compact abelian group G there is a Haar measure m that is not identically 0. One proves that if U is an open set then $m(U) > 0$ and that if m_1, m_2 are Haar measures that are not identically 0 then for some positive real c, $m_1 = cm_2$.[6]

$\mathbb{Q}_p$ is a locally compact abelian group, so there is a Haar measure m on $\mathbb{Q}_p$ that is not identically 0. Because $\mathbb{Z}_p$ is compact, $m(\mathbb{Z}_p) < \infty$, and because $\mathbb{Z}_p$ is open, $m(\mathbb{Z}_p) > 0$. Then let $\mu = \frac{1}{m(\mathbb{Z}_p)} m$, which is the unique Haar measure on $\mathbb{Q}_p$ satisfying

$$\mu(\mathbb{Z}_p) = 1.$$

Lemma 5. *For* $k \in \mathbb{Z}$,

$$\mu(p^k \mathbb{Z}_p) = p^{-k}.$$

Proof. If $k > 0$, then $p^k\mathbb{Z}_p$ is an ideal in $\mathbb{Z}_p$ and $\mathbb{Z}_p/p^k\mathbb{Z}_p$ is isomorphic as a ring with $\mathbb{Z}/p^k\mathbb{Z}$. So there are $x_j \in \mathbb{Z}_p$, $1 \leq j \leq p^k$, such that $\mathbb{Z}_p = \bigcup_{1\leq j\leq p^k}(x_j + p^k\mathbb{Z}_p)$, and the sets $x_j + p^k\mathbb{Z}_p$ are pairwise disjoint. Therefore

$$1 = \mu(\mathbb{Z}_p) = \sum_{j=1}^{p^k} \mu(x_j + p^k\mathbb{Z}_p) = \sum_{j=1}^{p_k} \mu(p^k\mathbb{Z}_p) = p^k \mu(p^k\mathbb{Z}_p),$$

[6] Walter Rudin, *Fourier Analysis on Groups*, pp. 1–2.

yielding $\mu(p^k\mathbb{Z}_p) = p^{-k}$.

If $k < 0$, then $p^k\mathbb{Z}_p$ is a ring and $\mathbb{Z}_p$ is an ideal in this ring. $\square$

We calculate $\mu(x \cdot E)$.[7]

Lemma 6. *For A a Borel set in $\mathbb{Q}_p$ and $x \in \mathbb{Q}_p$,*

$$\mu(x \cdot A) = |x|_p\mu(A).$$

Proof. If $x = 0$ then $x \cdot A = \{0\}$ and $\mu(x \cdot A) = 0$ and $|x|_p\mu(A) = 0 \cdot \mu(A) = 0$. (The set $\mathbb{Q}_p$ is infinite and μ is translation invariant, so finite sets have measure 0.) For $x \neq 0$, write $M_x(y) = x^{-1} \cdot y$, which is an isomorphism of locally compact groups $(\mathbb{Q}_p, +) \to (\mathbb{Q}_p, +)$. Let μ_x be the pushforward of μ by M_x:

$$\mu_x(E) = \mu(M_x^{-1}E) = \mu(\{y \in \mathbb{Q}_p : x^{-1}y \in E\}) = \mu(x \cdot E).$$

Because M_x is an isomorphism, it follows that μ_x is a Haar measure on $\mathbb{Q}_p$. And because $\mu_x(\mathbb{Q}_p) = \mu(\mathbb{Q}_p) = \infty$, showing μ_x is not identically 0, there is some $c_x > 0$ such that $\mu_x = c_x\mu$.

Now, as $x \neq 0$, $v_p(x) \in \mathbb{Z}$ and $|x|_p = p^{-v_p(x)}$. Then $p^{-v_p(x)}x \in \mathbb{Z}_p^*$, so there is some $y \in \mathbb{Z}_p^*$ such that $x = p^{v_p(x)}y$. As $y \in \mathbb{Z}_p^*$, $y \cdot \mathbb{Z}_p = \mathbb{Z}_p$ and hence $x \cdot \mathbb{Z}_p = p^{v_p(x)} \cdot \mathbb{Z}_p$. By Lemma 5, $\mu(p^{v_p(x)}\mathbb{Z}) = p^{-v_p(x)}$, so

$$\mu_x(\mathbb{Z}_p) = \mu(x \cdot \mathbb{Z}_p) = \mu(p^{v_p(x)}\mathbb{Z}) = p^{-v_p(x)}$$

and therefore

$$p^{-v_p(x)} = c_x\mu(\mathbb{Z}_p) = c_x,$$

and $|x|_p = p^{-v_p(x)}$ so $c_x = |x|_p$. Therefore $\mu_x = |x|_p\mu$. $\square$

Lemma 7. *For $f \in L^1(\mathbb{Q}_p)$ and $x \neq 0$,*

$$\int_{\mathbb{Q}_p} f(x^{-1}y)d\mu(y) = |x|_p \int_{\mathbb{Q}_p} f(y)d\mu(y).$$

Proof. μ_x is the pushforward of μ by $M_x(y) = x^{-1} \cdot y$, and by the change of variables formula,

$$\int_{\mathbb{Q}_p} f(x^{-1}y)d\mu(y) = \int_{\mathbb{Q}_p} (f \circ M_x)(y)d\mu(y) = \int_{\mathbb{Q}_p} f(y)d\mu_x(y) = |x|_p \int_{\mathbb{Q}_p} f(y)d\mu(y).$$

$\square$

The restriction of μ to the Borel σ-algebra of $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ is a Borel measure on $\mathbb{Q}_p^*$. We prove that the Borel measure on $\mathbb{Q}_p^*$ whose density with respect to μ is $x \mapsto \frac{1}{|x|_p}$ is a Haar measure.[8]

[7] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 254, Lemma 13.2.1.

[8] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 255, Proposition 13.2.2.

Theorem 8. *$\frac{1}{|x|_p} d\mu(x)$ is a Haar measure on the multiplicative group $\mathbb{Q}_p^*$.*

Proof. For $f \in C_c(\mathbb{Q}_p^*)$ and $y \in \mathbb{Q}_p^*$, writing $g_y(x) = \frac{f(x)}{|yx|_p}$, by Lemma 7 we have

$$\begin{aligned}
\int_{\mathbb{Q}_p^*} f(y^{-1}x) \frac{1}{|x|_p} d\mu(x) &= \int_{\mathbb{Q}_p^*} (g_y \circ M_y)(x) d\mu(x) \\
&= \int_{\mathbb{Q}_p^*} g_y(x) d\mu_y(x) \\
&= |y|_p \int_{\mathbb{Q}_p^*} g_y(x) d\mu(x) \\
&= |y|_p \int_{\mathbb{Q}_p^*} \frac{f(x)}{|yx|_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p^*} f(x) \frac{1}{|x|_p} d\mu(x).
\end{aligned}$$

□

Write $d\nu_0(x) = \frac{1}{|x|_p} d\mu(x)$. For $x \in \mathbb{Q}_p^*$, $p^{-v_p(x)} x \in \mathbb{Z}_p^*$, i.e. $x \in p^{v_p(x)} \mathbb{Z}_p^*$, and $\mathbb{Z}_p^*$ is the kernel of the group homomorphism $x \mapsto v_p(x)$, $\mathbb{Q}_p^* \to \mathbb{Z}$. It follows that the sets $p^k \mathbb{Z}_p^*$, $k \in \mathbb{Z}$, are pairwise disjoint and $\mathbb{Q}_p^* = \bigcup_{k \in \mathbb{Z}} p^k \mathbb{Z}_p^*$. For $k \in \mathbb{Z}$, because $p^k \mathbb{Z}_p^*$ is a compact open set in $\mathbb{Q}_p$ it is the case that $1_{p^k \mathbb{Z}_p^*} \in C_c(\mathbb{Q}_p)$ so by Lemma 7,

$$\begin{aligned}
\nu_0(p^k \mathbb{Z}_p^*) &= \int_{\mathbb{Q}_p^*} 1_{p^k \mathbb{Z}_p^*}(x) \frac{1}{|x|_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p^*}(p^{-k} x) \frac{1}{|p^{-k} \cdot p^k x|_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p^*}(x) \frac{1}{|p^k x|_p} d\mu_{p^k}(x) \\
&= |p^k|_p \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p^*}(x) \frac{1}{|p^k x|_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p^*} \frac{1}{|x|_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p^*} d\mu(x) \\
&= \mu(\mathbb{Z}_p^*).
\end{aligned}$$

Check that $1 + p\mathbb{Z}_p$ is a subgroup of $\mathbb{Z}_p^*$ with index $p-1$: the sets $a + p\mathbb{Z}_p$, $a \in N_p$, $a \neq 0$, are contained in $\mathbb{Z}_p^*$ and are pairwise disjoint. This implies

$$\mu(\mathbb{Z}_p^*) = (p-1)\mu(p\mathbb{Z}_p) = \frac{p-1}{p}.$$

Then

$$d\nu(x) = \frac{p}{p-1}\frac{1}{|x|_p}d\mu(x)$$

is a Haar measure on $\mathbb{Q}_p^*$ with $\nu(\mathbb{Z}_p^*) = 1$.

4 Integration

As $\mathbb{Z}_p \setminus \{0\} = \bigcup_{n\geq 0} p^n\mathbb{Z}_p^*$, for $\operatorname{Re} s > -1$,

$$\begin{aligned}
\int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^s d\mu(x) &= \sum_{n\geq 0}\int_{p^n\mathbb{Z}_p^*} |x|_p^s d\mu(x)\\
&= \sum_{n\geq 0} p^{-ns}\mu(p^n\mathbb{Z}_p^*)\\
&= \sum_{n\geq 0} p^{-ns}p^{-n}\cdot\mu(\mathbb{Z}_p^*)\\
&= \sum_{n\geq 0} p^{-ns}p^{-n}\cdot\frac{p-1}{p}\\
&= \frac{p-1}{p(1-p^{-1-s})}.
\end{aligned}$$

For $\operatorname{Re} s > 0$,

$$\begin{aligned}
\int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^s d\nu(x) &= \sum_{n\geq 0}\int_{p^n\mathbb{Z}_p^*} |x|_p^s \frac{p}{p-1}\frac{1}{|x|_p}d\mu(x)\\
&= \frac{p}{p-1}\sum_{n\geq 0}\int_{p^n\mathbb{Z}_p^*}(p^{-n})^{s-1}d\mu(x)\\
&= \frac{p}{p-1}\sum_{n\geq 0} p^{(-s+1)n}p^{-n}\cdot\frac{p-1}{p}\\
&= \sum_{n\geq 0} p^{-ns}\\
&= \frac{1}{1-p^{-s}}.
\end{aligned}$$

It is worth remarking that this is a factor of the Euler product for the Riemann zeta function.

We will use the following when working with the Fourier transform.[9]

Lemma 9. *For $n \in \mathbb{Z}$,*

$$\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x)e^{-2\pi i\{x\}_p}d\mu(x) = \begin{cases} p^{-n} & n\geq 0\\ 0 & \textit{otherwise.}\end{cases}$$

[9] Dorian Goldfeld and Joseph Hundley, *Automorphic Representations and L-Functions for the General Linear Group*, volume I, p. 16, Lemma 1.6.4.

Proof. If $n \geq 0$ and $x \in p^n\mathbb{Z}_p$ then $\{x\}_p = 0$ so

$$\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{x\}_p} d\mu(x) = \mu(p^n\mathbb{Z}_p) = p^{-n}.$$

If $n < 0$, let $y = p^n \in p^n\mathbb{Z}_p$, for which $\{y\}_p = p^n$. Define $T : \mathbb{Q}_p \to \mathbb{Q}_p$ by $T(x) = -y + x$. Then, as μ is translation invariant and as $x + y \in p^n\mathbb{Z}_p$ if and only if $x \in p^n\mathbb{Z}_p$,

$$\begin{aligned}
\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{x\}_p} d\mu(x) &= \int_{\mathbb{Q}_p} (1_{p^n\mathbb{Z}_p} \circ T)(y + x) e^{-2\pi i \{T(y+x)\}_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(y + x) e^{-2\pi i \{y+x\}_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{y+x\}_p} d\mu(x) \\
&= e^{-2\pi i \{y\}_p} \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{x\}_p} d\mu(x).
\end{aligned}$$

Because $e^{-2\pi i \{y\}_p} \neq 1$, for $I = e^{-2\pi i \{y\}_p} I$ we have $I = 0$. □

Lemma 10. *For $n \in \mathbb{Z}$ and $y \in \mathbb{Q}_p$,*

$$\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{yx\}_p} d\mu(x) = \begin{cases} p^{-n} & y \in p^{-n}\mathbb{Z}_p \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $y \in p^{-n}\mathbb{Z}_p$ then for any $x \in p^n\mathbb{Z}_p$ we have $yx \in \mathbb{Z}_p$ and so $\{yx\}_p = 0$ and $I = \mu(p^n\mathbb{Z}_p) = p^{-n}$. □

Another lemma.[10]

Lemma 11. *For $n \in \mathbb{Z}$,*

$$\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p^*}(x) e^{-2\pi i \{x\}_p} d\mu(x) = \begin{cases} p^{-n}(1 - p^{-1}) & n \geq 0 \\ -1 & n = -1 \\ 0 & n < -1. \end{cases}$$

Proof. $\mathbb{Z}_p^* = \mathbb{Z}_p - p\mathbb{Z}_p$ and $p^n\mathbb{Z}_p^* = p^n\mathbb{Z}_p - p^{n+1}\mathbb{Z}_p$ and then

$$\begin{aligned}
\int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p^*}(x) e^{-2\pi i \{x\}_p} d\mu(x) &= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{x\}_p} d\mu(x) \\
&\quad - \int_{\mathbb{Q}_p} 1_{p^{n+1}\mathbb{Z}_p}(x) e^{-2\pi i \{x\}_p} d\mu(x) \\
&= I_1 - I_2.
\end{aligned}$$

[10] Dorian Goldfeld and Joseph Hundley, *Automorphic Representations and L-Functions for the General Linear Group*, volume I, p. 16, Proposition 1.6.5.

We apply Lemma 9. If $n \geq 0$ then $I_1 = p^{-n}$ and $I_2 = p^{-n-1}$ so $I = p^{-n} - p^{-n-1} = p^{-n}(1 - p^{-1})$. If $n = -1$ then $I_1 = 0$ and $n + 1 \geq 0$ so $I_2 = p^{-n-1} = 1$ hence $I = -1$. Finally if $n < -1$ then $I_1 = 0$ and $I_2 = 0$ so $I = 0$. $\square$

For $f \in L^1(\mathbb{Q}_p)$ and $y \in \mathbb{Q}_p$, define $\widehat{f} \in C_0(\mathbb{Q}_p)$ by

$$\widehat{f}(y) = (\mathscr{F} f)(y) = \int_{\mathbb{Q}_p} f(x) e^{-2\pi i \{yx\}_p} d\mu(x).$$

Let $\mathscr{S}$ be the set of locally constant functions $\mathbb{Q}_p \to \mathbb{C}$ with compact support. We call an element of $\mathscr{S}$ a **p-adic Schwartz function**.[11] We prove that the Fourier transform of a p-adic Schwartz function is itself a p-adic Schwartz function.[12]

Theorem 12. *If $f \in \mathscr{S}$ then $\widehat{f} \in \mathscr{S}$.*

Proof. Let $n \in \mathbb{Z}$, $a \in \mathbb{Q}_p$, and let $N = a + p^n \mathbb{Z}_p$. For $y \in \mathbb{Q}_p$, applying Lemma 10,

$$\begin{aligned}
\widehat{1}_N(y) &= \int_{\mathbb{Q}_p} 1_{a+p^n\mathbb{Z}_p}(x) e^{-2\pi i \{yx\}_p} d\mu(x) \\
&= \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(-a + x) e^{-2\pi i \{y(-a+x)+ay\}_p} d\mu(x) \\
&= e^{-2\pi i \{ay\}_y} \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(-a + x) e^{-2\pi i \{y(-a+x)\}_p} d\mu(x) \\
&= e^{-2\pi i \{ay\}_y} \int_{\mathbb{Q}_p} 1_{p^n\mathbb{Z}_p}(x) e^{-2\pi i \{yx\}_p} d\mu(x) \\
&= e^{-2\pi i \{ay\}_y} p^{-n} 1_{p^{-n}\mathbb{Z}_p}(y).
\end{aligned}$$

$\square$

[11] cf. A. A. Kirillov and A. D. Gvishiani, *Theorems and Problems in Functional Analysis*, p. 210, no. 639.

[12] Dorian Goldfeld and Joseph Hundley, *Automorphic Representations and L-Functions for the General Linear Group*, volume I, p. 17, Theorem 1.6.8.