Orbital stability for NLS

Jordan Bell

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Let $n = 3$, and take $p < \frac{4}{3}$. Some of the material we will present for general n when it doesn't simplify our work to use $n = 3$.

The (defocusing) nonlinear Schrödinger equation is

$$i\phi_t + \Delta\phi + |\phi|^{p-1}\phi = 0.$$

$\phi(x, 0) = \phi_0 \in H^1$.

For a function ψ on $\mathbb{R}^n$, the *orbit* of the function under the symmetries of NLS is

$$\mathscr{G}_\psi = \{\psi(\cdot + x_0)e^{i\gamma} : (x_0, \gamma) \in \mathbb{R}^n \times \mathbb{T}\}.$$

We say that ψ is *orbitally stable* if initial data being near it implies that the solution of NLS is near it always.

We define

$$\rho(\phi(t), \mathscr{G}_\psi) = \inf_{(x_0,\gamma)\in\mathbb{R}^n\times\mathbb{T}} \|\phi(\cdot + x_0, t)e^{i\gamma} - \psi\|_{H^1}.$$

The *ground state equation* is

$$\Delta u - u + |u|^{p-1}u = 0.$$

The ground state equation comes from the solution $\phi(x, t) = e^{it}u(x)$ of NLS. It is a fact that there is a positive bounded solution R of the ground state equation, which we call a ground state.

Theorem 1. *The ground state R is orbitally stable: for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if*

$$\rho(\phi_0, \mathscr{G}_R) < \delta(\epsilon)$$

then for all $t > 0$

$$\rho(\phi(t), \mathscr{G}_R) < \epsilon.$$

We define the energy functional $\mathscr{E}$ by

$$\mathscr{E}[\phi] = \int |\nabla\phi|^2 + |\phi|^2 - \frac{2}{p+1}|\phi|^{p+1}dx,$$

so $\mathscr{E}[\phi]$ is a function of time but not of space.

It is a fact that for each t there are $x_0 = x_0(t)$ and $\gamma = \gamma(t)$ such that

$$\|\phi(\cdot + x_0, t)e^{i\gamma} - R\|_{H^1} = \rho(\phi(t), \mathscr{G}_R).$$

Let $w = \phi(\cdot + x_0, t)e^{i\gamma} - R$; so $\|w(t)\|_{H^1} = \rho(\phi(t), \mathscr{G}_R)$.

Let $\Delta\mathscr{E} = \mathscr{E}[\phi_0] - \mathscr{E}[R]$. We have

$$\begin{aligned}\Delta\mathscr{E} &= \mathscr{E}[\phi(\cdot, t] - \mathscr{E}[R] \\ &= \mathscr{E}[\phi(\cdot + x_0, t)e^{i\gamma}] - \mathscr{E}[R] \\ &= \mathscr{E}[R + w] - \mathscr{E}[R].\end{aligned}$$

We shall express $\mathscr{E}[R+w]$ as a Taylor expansion about R. We compute the first variation as follows:

$$\begin{aligned}d\mathscr{E}[R]w &= \int \nabla w \nabla \overline{R} + \nabla R \nabla \overline{w} + w\overline{R} + R\overline{w} - |R|^{p-1}(w\overline{R} + R\overline{w}) \\ &= 2\Re \int \nabla w \nabla R + wR - w|R|^{p-1}R \\ &= 2\Re \int w(-\Delta R + R - |R|^{p-1}R) \\ &= 0,\end{aligned}$$

where we used the fact that R is real valued, integration by parts, and the fact that R is a solution of the ground state equation. So the first variation of $\mathscr{E}$ at R is 0.

We now compute the second variation of $\mathscr{E}$.

$$\begin{aligned}d^2\mathscr{E}[R][w] &= 2\Re \int -w\Delta\overline{w} + |w|^2 - \frac{p-1}{2}R^{p-1}w^2 - \frac{p-1}{2}R^{p-1}|w|^2 \\ &\quad -R^{p-1}|w|^2 \\ &= 2\Re \int -w\Delta\overline{w} + |w|^2 - \frac{p-1}{2}R^{p-1}w^2 - \frac{p+1}{2}R^{p-1}|w|^2\end{aligned}$$

Write $w = u + iv$. Then we have

$$d^2\mathscr{E}[R][w] = 2\int -u\Delta u - v\Delta v + u^2 + v^2 - R^{p-1}(pu^2 + v^2)$$

Define

$$L_+ = -\Delta + 1 - pR^{p-1} \qquad L_- = -\Delta + 1 - R^{p-1},$$

which gives

$$(L_+u, u)_{L^2} = \int -u\Delta u + u^2 - pu^2R^{p-1}$$

and

$$(L_-v, v)_{L^2} = \int -v\Delta v + v^2 - v^2R^{p-1}.$$

Thus

$$d^2\mathscr{E}[R][w] = 2(L_+u,u)_{L^2} + 2(L_-v,v)_{L^2}.$$

And we assert that the remainder term of the Taylor series is $O(\int |w|^3)$, because R is bounded. Therefore

$$\Delta\mathscr{E} = (L_+u,u)_{L^2} + (L_-v,v)_{L^2} + O\Big(\int |w|^3\Big).$$

We can bound $\int |w|^3$ using the Gagliardo-Nirenberg inequality, which gives us (for $n = 3$)

$$\|w\|_{L^3}^3 \le C_0\|\nabla w\|_{L^2}^{3/2}\|w\|_{L^2}^{3/2} \le C_0\|w\|_{H^1}^3,$$

for some C_0 that doesn't depend on w. Therefore

$$\Delta\mathscr{E} = (L_+u,u)_{L^2} + (L_-v,v)_{L^2} + O(\|w\|_{H^1}^3),$$

so there is some C such that

$$\Delta\mathscr{E} \ge (L_+u,u)_{L^2} + (L_-v,v)_{L^2} - C\|w\|_{H^1}^3.$$