Watson's lemma and Laplace's method

Jordan Bell

June 17, 2014

1 Watson's lemma

Our proof of **Watson's lemma** follows Miller.[1]

Theorem 1 (Watson's lemma)**.** *Suppose that $T > 0$, that $\phi : \mathbb{R} \to \mathbb{C}$ belongs to $L^1([0,T])$, that $\sigma > -1$, and that $g(t) = t^{-\sigma}\phi(t)$ is C^∞ on some neighborhood of 0. Then $F : (0,\infty) \to \mathbb{C}$ defined by*

$$F(\lambda) = \int_0^T e^{-\lambda t}\phi(t)dt$$

satisfies

$$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\sigma+n+1)}{n!\lambda^{\sigma+n+1}}, \qquad \lambda \to \infty.$$

Proof. Take g to be C^∞ on some interval with left endpoint < 0 and right endpoint s, $0 < s < T$. For p a nonnegative integer and $\lambda > 1$, define

$$F_p(\lambda) = \int_0^s e^{-\lambda t} t^{\sigma+p} dt,$$

which satisfies, doing the change of variable $\tau = \lambda t$,

$$\begin{aligned} F_p(\lambda) &= \int_0^\infty e^{-\lambda t} t^{\sigma+p} dt - \int_s^\infty e^{-\lambda t} t^{\sigma+p} dt \\ &= \lambda^{-(\sigma+p+1)} \int_0^\infty e^{-\tau}\tau^{\sigma+p} d\tau - \int_s^\infty e^{-\lambda t} t^{\sigma+p} dt \\ &= \lambda^{-(\sigma+p+1)}\Gamma(\sigma+p+1) - \int_s^\infty e^{-\lambda t} t^{\sigma+p} dt. \end{aligned}$$

[1]Peter D. Miller, *Applied Asymptotic Analysis*, p. 53, Proposition 2.1.

Using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_s^\infty e^{-\lambda t} t^{\sigma+p} dt &= \int_s^\infty e^{-\lambda t/2} e^{-\lambda t/2} t^{\sigma+p} dt \\
&\le \left(\int_s^\infty e^{-\lambda t} dt \right)^{1/2} \left(\int_s^\infty e^{-\lambda t} t^{2\sigma+2p} dt \right)^{1/2} \\
&= e^{-\lambda s/2} \lambda^{-1/2} \left(\int_s^\infty e^{-\lambda t} t^{2\sigma+2p} dt \right)^{1/2} \\
&< e^{-\lambda s/2} \left(\int_0^\infty e^{-t} t^{2\sigma+2p} dt \right)^{1/2} \\
&= e^{-\lambda s/2} \Gamma(2\sigma + 2p + 1)^{1/2}.
\end{aligned}$$

For any nonnegative integer m we have $e^{-\lambda s/2} = o_m(\lambda^{-(\sigma+m+1)})$ as $\lambda \to \infty$, hence, dealing with $\Gamma(2\sigma + 2p + 1)$ merely as a constant depending on p,

$$F_p(\lambda) = \lambda^{-(\sigma+p+1)} \Gamma(\sigma + p + 1) + o_{m,p}(\lambda^{-(\sigma+m+1)}) \tag{1}$$

as $\lambda \to \infty$.

Write

$$F(\lambda) = \int_0^s e^{-\lambda t} \phi(t) dt + \int_s^T e^{-\lambda t} \phi(t) dt.$$

One the one hand,

$$\left| \int_s^T e^{-\lambda t} \phi(t) dt \right| \le \int_s^T e^{-\lambda t} |\phi(t)| dt \le e^{-\lambda s} \int_s^T |\phi(t)| dt \le e^{-\lambda s} \left\| \phi \right\|_{L^1},$$

which shows that for any nonnegative integer n,

$$\int_s^T e^{-\lambda t} \phi(t) dt = o_n(\lambda^{-(\sigma+n+1)})$$

as $\lambda \to \infty$.

One the other hand, for each nonnegative integer N, **Taylor's theorem** tells us that the function $r_N : (r, s) \to \mathbb{C}$ defined by

$$r_N(t) = g(t) - \sum_{n=0}^N \frac{g^{(n)}(0)}{n!} t^n, \qquad t \in (r, s),$$

satisfies

$$|r_N(t)| \le \sup |g^{(N+1)}(\tau)| \cdot \frac{|t|^{N+1}}{(N+1)!},$$

where the supremum is over those τ strictly between 0 and t. Then for $t \in (0, s)$,

$$|r_N(t)| \le \sup_{0<\tau<s} |g^{(N+1)}(\tau)| \cdot \frac{t^{N+1}}{(N+1)!}.$$

Using the definition of r_N,

$$\begin{aligned}\int_0^s e^{-\lambda t}\phi(t)dt &= \int_0^s e^{-\lambda t}t^\sigma \sum_{n=0}^N \frac{g^{(n)}(0)}{n!}t^n dt + \int_0^s e^{-\lambda t}t^\sigma r_N(t)dt \\ &= \sum_{n=0}^N \frac{g^{(n)}(0)}{n!}F_n(\lambda) + \int_0^s e^{-\lambda t}t^\sigma r_N(t)dt\end{aligned}$$

and using the inequality for $r_N(t)$,

$$\begin{aligned}\left|\int_0^s e^{-\lambda t}t^\sigma r_N(t)dt\right| &\le \int_0^s e^{-\lambda t}t^\sigma |r_N(t)|dt \\ &\le \sup_{0<\tau<s}|g^{(N+1)}(\tau)| \cdot \frac{1}{(N+1)!}\int_0^s e^{-\lambda t}t^{\sigma+N+1}dt \\ &= \sup_{0<\tau<s}|g^{(N+1)}(\tau)| \cdot \frac{1}{(N+1)!}F_{N+1}(\lambda)dt.\end{aligned}$$

Using this and (1),

$$\int_0^s e^{-\lambda t}t^\sigma r_N(t)dt = O_N(\lambda^{-(\sigma+N+2)}).$$

Putting together what we have shown, for any nonnegative integer N, as $\lambda \to \infty$,

$$\begin{aligned}F(\lambda) &= \sum_{n=0}^N \frac{g^{(n)}(0)}{n!}F_n(\lambda) + O_N(\lambda^{-(\sigma+N+2)}) + O_N(\lambda^{-(\sigma+N+2)}) \\ &= \sum_{n=0}^N \frac{g^{(n)}(0)}{n!}\lambda^{-(\sigma+n+1)}\Gamma(\sigma+n+1) + \sum_{n=0}^N \frac{g^{(n)}(0)}{n!} \cdot o_{N,n}(\lambda^{-(\sigma+N+1)}) \\ &\quad + O_N(\lambda^{-(\sigma+N+2)}) \\ &= \sum_{n=0}^N \frac{g^{(n)}(0)}{n!}\lambda^{-(\sigma+n+1)}\Gamma(\sigma+n+1) + o_N(\lambda^{-(\sigma+N+1)}),\end{aligned}$$

which proves the claim. □

2 Laplace's method for an interval

Theorem 2. *Suppose that $a < b$, that $f \in C^2([a,b],\mathbb{R})$, and that there is a unique $x_0 \in [a,b]$ at which f is equal to its supremum over $[a,b]$. Suppose also that $a < x_0 < b$ and that $f''(x_0) < 0$. Then*

$$\int_a^b e^{Mf(x)}dx \sim e^{Mf(x_0)}\sqrt{\frac{2\pi}{-Mf''(x_0)}}.$$

as $M \to \infty$.

Proof. We remark first that $f'(x_0) = 0$ because f is equal to its supremum over $[a, b]$ at this point, which is not a boundary point. The claim says that a ratio has limit 1 as $M \to \infty$. We shall prove that the liminf and the limsup of this ratio are both 1, which will prove the claim. Let $\epsilon > 0$. Because $f'' : [a, b] \to \mathbb{R}$ is continuous, there is some $\delta > 0$ such that $|x - x_0| < \delta$ implies $f''(x) \geq f''(x_0) - \epsilon$; we take δ small enough that $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Writing

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_1(x) = f(x_0) + R_1(x), \qquad x \in [a, b],$$

Taylor's theorem tells us that for each $x \in [a, b]$ there is some ξ_x strictly between x_0 and x such that

$$R_1(x) = \frac{f''(\xi_x)}{2}(x - x_0)^2.$$

Thus for $|x - x_0| < \delta$ we have $|\xi_x - x_0| < \delta$, so

$$f(x) \geq f(x_0) + \frac{f''(x_0) - \epsilon}{2}(x - x_0)^2.$$

Using this inequality, which applies for any $x \in (x_0 - \delta, x_0 + \delta)$, and because the integrand in the following integral is positive,

$$\begin{aligned} \int_a^b e^{Mf(x)} dx &\geq \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)} dx \\ &\geq \int_{x_0-\delta}^{x_0+\delta} e^{M\left(f(x_0) + \frac{f''(x_0)-\epsilon}{2}(x-x_0)^2\right)} dx \\ &= e^{Mf(x_0)} \int_{x_0-\delta}^{x_0+\delta} e^{-M\frac{-f''(x_0)+\epsilon}{2}(x-x_0)^2} dx. \end{aligned}$$

Changing variables, keeping in mind that $f''(x_0) < 0$,

$$\int_{x_0-\delta}^{x_0+\delta} e^{-M\frac{-f''(x_0)+\epsilon}{2}(x-x_0)^2} dx = \int_{-\delta\sqrt{M\frac{-f''(x_0)+\epsilon}{2}}}^{\delta\sqrt{M\frac{-f''(x_0)+\epsilon}{2}}} e^{-y^2} \left(M\frac{-f''(x_0)+\epsilon}{2}\right)^{-1/2} dy.$$

Thus

$$\frac{\int_a^b e^{Mf(x)} dx}{e^{Mf(x_0)} \left(\frac{-Mf''(x_0)}{2}\right)^{-1/2}} \tag{2}$$

is lower bounded by

$$\left(\frac{-f''(x_0)+\epsilon}{-f''(x_0)}\right)^{-1/2} \int_{-\delta\sqrt{M\frac{-f''(x_0)+\epsilon}{2}}}^{\delta\sqrt{M\frac{-f''(x_0)+\epsilon}{2}}} e^{-y^2} dy,$$

so we get that the liminf of (2) as $M \to \infty$ is lower bounded by

$$\left(\frac{-f''(x_0)+\epsilon}{-f''(x_0)}\right)^{-1/2} \sqrt{\pi}.$$

But this is true for all $\epsilon > 0$ and (2) and its liminf do not depend on ϵ, so the liminf of (2) as $M \to \infty$ is lower bounded by $\sqrt{\pi}$. In other words,

$$\liminf_{M\to\infty} \frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\left(\frac{-Mf''(x_0)}{2\pi}\right)^{-1/2}} \geq 1.$$

Let $\epsilon > 0$ with $f''(x_0) + \epsilon < 0$; this is possible because $f''(x_0) < 0$. Because $f'' : [a, b] \to \mathbb{R}$ is continuous there is some $\delta > 0$ such that $|x - x_0| < \delta$ implies that $f''(x) \leq f''(x_0) + \epsilon$; we take $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Taylor's theorem tells us that for any $x \in [a, b]$ there is some ξ_x strictly between x_0 and x such that

$$f(x) = f(x_0) + \frac{f''(\xi_x)}{2}(x - x_0)^2.$$

Therefore, as $|x - x_0| < \delta$ implies that $|\xi_x - x_0| < \delta$,

$$f(x) \leq f(x_0) + \frac{f''(x_0) + \epsilon}{2}(x - x_0)^2. \tag{3}$$

Furthermore, $f : [a, b] \to \mathbb{R}$ is continuous, so it makes sense to define

$$C = \sup_{x\in[a,x_0-\delta]\cup[x_0+\delta,b]} f(x).$$

Because x_0 is not in this union of intervals, by hypothesis we know that $C < f(x_0)$, and we define $\eta = f(x_0) - C > 0$. This means that for all $x \in [a, x_0 - \delta] \cup [x_0 + \delta, b]$, $f(x) \leq f(x_0) - \eta$. Then

$$\begin{aligned}
\int_a^b e^{Mf(x)}dx &= \int_a^{x_0-\delta} e^{Mf(x)}dx + \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}dx + \int_{x_0+\delta}^b e^{Mf(x)}dx \\
&\leq \int_a^{x_0-\delta} e^{MC}dx + \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}dx + \int_{x_0+\delta}^b e^{MC}dx \\
&= (b - a - 2\delta)e^{MC} + \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}dx \\
&< (b - a)e^{MC} + \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}dx.
\end{aligned}$$

For the integral over $(x_0 - \delta, x_0 + \delta)$,

$$\begin{aligned}
\int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}dx &\leq \int_{x_0-\delta}^{x_0+\delta} e^{M\left(f(x_0)+\frac{f''(x_0)+\epsilon}{2}(x-x_0)^2\right)}dx \\
&= e^{Mf(x_0)} \int_{x_0-\delta}^{x_0+\delta} e^{M\frac{f''(x_0)+\epsilon}{2}(x-x_0)^2}dx \\
&< e^{Mf(x_0)} \int_{-\infty}^{\infty} e^{M\frac{f''(x_0)+\epsilon}{2}(x-x_0)^2}dx.
\end{aligned}$$

Changing variables, and keeping in mind that $f''(x_0)+\epsilon<0$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{M\frac{f''(x_0)+\epsilon}{2}(x-x_0)^2}dx &= \int_{-\infty}^{\infty} e^{-y^2}\left(-\frac{M}{2}(f''(x_0)+\epsilon)\right)^{-1/2}dy \\ &= \left(-\frac{M}{2\pi}(f''(x_0)+\epsilon)\right)^{-1/2}.\end{aligned}$$

Therefore

$$\int_a^b e^{Mf(x)}dx < (b-a)e^{MC} + e^{Mf(x_0)}\left(-\frac{M}{2\pi}(f''(x_0)+\epsilon)\right)^{-1/2},$$

which we rearrange as

$$\frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\left(-\frac{M}{2\pi}(f''(x_0)+\epsilon)\right)^{-1/2}} < (b-a)e^{-M\eta}\left(-\frac{M}{2\pi}(f''(x_0)+\epsilon)\right)^{1/2}+1.$$

As $M\to\infty$ the first term on the right-hand side tends to 0, because $\eta>0$. Therefore,

$$\limsup_{M\to\infty}\frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\left(-\frac{M}{2\pi}(f''(x_0)+\epsilon)\right)^{-1/2}}\le 1.$$

This is true for all $\epsilon>0$, so it holds that

$$\limsup_{M\to\infty}\frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\left(\frac{-Mf''(x_0)}{2\pi}\right)^{-1/2}}\le 1,$$

completing the proof. □