Functions of bounded variation and a theorem of Khinchin

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For $q \geq 1$ let

$$\mathscr{A}_q = \left\{ \frac{a}{q} : 0 \leq a \leq q, \gcd(a,q) = 1 \right\}.$$

The sets $\mathscr{A}_q$ are pairwise disjoint. In particular $0 \in \mathscr{A}_1$ and $0 \notin \mathscr{A}_q$ for $q > 1$. We have

$$[0,1] \cap \mathbb{Q} = \bigcup_{q \geq 1} \mathscr{A}_q.$$

Write μ for Lebesgue measure on $[0,1]$.

The following is a version of a theorem of Khinchin about continued fractions.[1] In the literature on Diophantine approximation it is usually proved with the Borel-Cantelli lemma, rather than the machinery of bounded variation and almost everywhere differentiability.

Theorem 1 (Khinchin)**.** *Let $F : \mathbb{Z}_{\geq 1} \to \mathbb{R}_{>0}$ and let A be the set of those $x \in [0,1] \setminus \mathbb{Q}$ such that there are infinitely many q for which there is some $\frac{a}{q} \in \mathscr{A}_q$ satisfying*

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{qF(q)}.$$

If

$$\sum_{q=1}^{\infty} \frac{1}{F(q)} < \infty,$$

then $\mu(A) = 0$.

Proof. Define $f : [0,1] \to \mathbb{R}_{>0}$ by

$$f(x) = \begin{cases} \frac{1}{qF(q)} & x \in \mathscr{A}_q \\ 0 & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Let $N \geq 1$. If

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

[1] John J. Benedetto and Wojciech Czaja, *Integration and Modern Analysis*, p. 183, Theorem 4.3.3.

then

$$\sum_{j=0}^{N} f(t_j) = \sum_{q=1}^{\infty}\sum_{j=0}^{N} f(t_j)\cdot 1_{\mathscr{A}_q}(t_j)$$
$$= \sum_{q=1}^{\infty}\sum_{j=0}^{N} \frac{1}{qF(q)}\cdot 1_{\mathscr{A}_q}(t_j)$$
$$\leq \sum_{q=1}^{\infty}\frac{1}{F(q)},$$

and so

$$\sum_{j=1}^{N} |f(t_j)-f(t_{j-1})| \leq 2\sum_{j=1}^{N} f(t_j) \leq 2\sum_{q=1}^{\infty}\frac{1}{F(q)}.$$

Therefore

$$V(f) \leq 2\sum_{q=1}^{\infty}\frac{1}{F(q)} < \infty$$

by hypothesis, where $V(f)$ denotes the variation of f on $[0,1]$. The set D_f of points at which f is differentiable is a Borel set,[2] and because f has bounded variation, $\mu(D_f)=1$.[3] Let $E = D_f \setminus \mathbb{Q}$, whose measure is $\mu(E)=1$. Now let $x \in E$. There are $x_n \in [0,1]\setminus\mathbb{Q}$, $x_n \neq x$, $x_n \to x$, with which

$$f'(x) = \lim_{n\to\infty}\frac{f(x_n)-f(x)}{x_n - x} = \lim_{n\to\infty}\frac{0-0}{x_n-x} = 0.$$

If $\frac{a_n}{q_n} \to x$ with $\frac{a_n}{q_n} \in \mathscr{A}_{q_n}$, then

$$\frac{f(a_n/q_n)-f(x)}{\frac{a_n}{q_n}-x} = \frac{1}{q_nF(q_n)\left(\frac{a_n}{q_n}-x\right)} \to f'(x) = 0,$$

so $q_nF(q_n)\left|\frac{a_n}{q_n}-x\right| \to \infty$. There is thus some N such that if $n \geq N$ then

$$q_nF(q_n)\left|\frac{a_n}{q_n}-x\right| \geq 1,$$

i.e. if $n \geq N$ then

$$\left|x - \frac{a_n}{q_n}\right| \geq \frac{1}{q_nF(q_n)}.$$

Assume by contradiction that $x \in A$, so there are $\frac{a_n}{q_n} \in \mathscr{A}_{q_n}$, $\frac{a_n}{q_n} \neq \frac{a_m}{q_m}$ for $n \neq m$, with

$$\left|x - \frac{a_n}{q_n}\right| < \frac{1}{q_nF(q_n)},$$

[2] V. I. Bogachev, *Measure Theory*, volume 1, p. 371, Theorem 5.8.12.
[3] V. I. Bogachev, *Measure Theory*, volume 1, p. 335, Theorem 5.2.6.

and because $\sum_{q=1}^{\infty} \frac{1}{F(q)} < \infty$ it holds that $F(q) \to \infty$ and thus $\frac{1}{q_n F(q_n)} \to 0$. This means that $x - \frac{a_n}{q_n} \to 0$, which implies that $x \notin E$. We have shown that if $x \in A$ then $x \notin E$, so $A \subset [0,1] \setminus E$ and hence $\mu(A) \leq 1 - \mu(E) = 0$. $\square$