Notes on the KAM theorem

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1 Introduction

I hope eventually to expand these notes into a standalone presentation of KAM that presents a precise formulation of the theorem and gives detailed proofs of everything. There are few presentations of KAM in the literature that give a precise formulation of the theorem, and even those that give precise formulations such as [6] and [7] glide over some details. Gallavotti [4] explains the history of quasi-periodic phenomena in celestial mechanics.

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

For $x, y \in \mathbb{R}^n$, let $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$. Let $\|x\| = \sum_{j=1}^n x_j^2$ and let $\|x\|_\infty = \max_{1\leq j\leq n} |x_j|$. For $x, y \in \mathbb{R}^n$, we have $|\langle x, y\rangle| \leq n\|x\|_\infty \|y\|_\infty$.

If (M, ω) is a symplectic manifold and $H \in C^\infty(M)$, then the *Hamiltonian vector field* with energy function H is the vector field X_H on M uniquely determined by the condition $\omega_x(X_H(x), v) = (dH)(x)(v)$ for all points $x \in M$ and tangent vectors $v \in T_x M$.

We say that $(q^1, \dots, q^n, p_1, \dots, p_n)$ are *canonical coordinates* for (M, ω) if $\omega = \sum_{j=1}^n dq^j \wedge dp_j$. If $(q^1, \dots, q^n, p_1, \dots, p_n)$ are canonical coordinates for (M, ω) and $H \in C^\infty(M)$ then

$$X_H(x) = ((\partial_p H)(x), (-\partial_q H)(x))$$

for all $x \in M$, where

$$\partial_q H = \Big(\frac{\partial H}{\partial q^1}, \dots, \frac{\partial H}{\partial q^n}\Big), \quad \partial_p H = \Big(\frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}\Big).$$

Let ϕ be the flow of X_H on M. Then

$$\frac{d(q^j(\phi_t(x)))}{dt} = \frac{\partial H}{\partial p_j}(\phi_t(x)), \quad \frac{d(p_j(\phi_t(x)))}{dt} = -\frac{\partial H}{\partial q^j}(\phi_t(x)),$$

called *Hamilton's equations.*

2 Action-angle coordinates

Let (M, ω) be a $2n$-dimensional symplectic manifold. Let $f_1, \dots, f_n \in C^\infty(M)$. If $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq n$ (namely the functions are in involution) and if at each point in M the differentials of the functions are linearly independent in the cotangent space at that point, then we say that the set of functions is *completely integrable*.

We define the *momentum map* $F : M \to \mathbb{R}^n$ by $F = f_1 \times \cdots \times f_n$.

We say that F is *locally trivial* at a value y_0 in its range if there is a neighborhood U of y_0 such that for all $y \in U$ there is a smooth map $h_y : F^{-1}(U) \to F^{-1}(y_0)$ such that $F \times h_y$ is a diffeomorphism from $F^{-1}(U)$ to $U \times F^{-1}(y_0)$. The *bifurcation set* of F is the set Σ_F of $y_0 \in \mathbb{R}^n$ at which F fails to be locally trivial.

The following theorem is proved in [1, Theorem 5.2.21].

Theorem 1. *Let $U \subseteq \mathbb{R}^n$ be open. If $F|F^{-1}(U) : F^{-1}(U) \to U$ is a proper map then each of the vector fields $X_{f_i}|F^{-1}(U)$ is complete, $U \subseteq \mathbb{R}^n \setminus \Sigma_F$, and the fibers of the locally trivial fibration $F|F^{-1}(U)$ are disjoint unions of manifolds each diffeomorphic with $\mathbb{T}^n$.*

Let $\nu \in \mathbb{R}^n$, and define the linear flow F on $\mathbb{R}^n$ by $F_t(v) = v + t\nu$. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the projection map and let $\phi_t : \mathbb{T}^n \to \mathbb{T}^n$ be such that $\pi \circ F_t = \phi_t \circ \pi$; if $\pi(v_1) = \pi(v_2)$ then $\pi \circ F_t(v_1) = \pi \circ F_t(v_2)$, so such a map exists, and is clearly unique. A flow ϕ on $\mathbb{T}^n$ induced by a linear flow on $\mathbb{R}^n$ is called a *quasi-periodic* flow.

Say that $\nu \neq \mu$, and let ϕ be the flow induced by ν and ψ be the flow induced by μ. Then for some i, $\nu_i \neq \mu_i$ and for any t such that $t(\nu_i - \mu_i) \notin \mathbb{Z}$, $\phi_t(\theta) \neq \psi_t(\theta)$ for any $\theta \in \mathbb{T}^n$. Hence $\phi \neq \psi$. Thus a quasi-periodic flow is induced by a unique vector $\nu \in \mathbb{R}^n$. We call ν the *frequency vector* of the flow ϕ.

We say that $\nu \in \mathbb{R}^n$ is *resonant* if there is some $0 \neq k \in \mathbb{Z}^n$ such that $\langle k, \nu \rangle = 0$, and we say that it is *nonresonant* otherwise.

Let ϕ be the quasi-periodic flow on $\mathbb{T}^n$ with frequency vector $\nu \in \mathbb{R}^n$. It can be shown that each orbit of ϕ is dense in $\mathbb{T}^n$ if and only if ν is nonresonant. This is proved in [1, pp. 818–820]; that each orbit of ϕ is dense in $\mathbb{T}^n$ if ν is nonresonant is proved in [5, Theorem 444].

Let $H = f_1$; we call this distinguished function the Hamiltonian, and we are concerned with the flow of the Hamiltonian vector field X_H.

The following theorem is proved in [1, Theorem 5.2.24].

Theorem 2. *Let c be in the range of F, let I_c^0 denote a connected component of $F^{-1}(c)$, and let ϕ be the flow of X_H. Then there is a quasiperiodic flow ψ on $\mathbb{T}^n$ and a diffeomorphism $g : \mathbb{T}^n \to I_c^0$ such that $g \circ \psi_t = \phi_t | I_c^0 \circ g$.*

Let $\mathbb{R}^{2n} = \{q^1, \dots, q^n, p_1, \dots, p_n\}$ and let $\omega = \sum_{j=1}^n dq^j \wedge dp_j$. Let $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, where I is the $n \times n$ identity matrix. For $u, v \in \mathbb{R}^{2n}$ we have that $\omega(u, v) = \langle u, Jv \rangle$.

Let B^n be an open ball in $\mathbb{R}^n$. $B^n \times \mathbb{T}^n$ is a symplectic submanifold of $\mathbb{R}^{2n}$. We define coordinates $I^j = q^j$ and $\theta_j = p_j + \mathbb{Z}$, $j = 1, \dots, n$. If $H \in C^\infty(B^n \times \mathbb{T}^n)$ does not depend on $\theta_1, \dots, \theta_n$ then we say that it has *action-angle coordinates* in $B^n \times \mathbb{T}^n$.

If $H \in C^\infty(B^n \times \mathbb{T}^n)$ admits action-angle coordinates (I, θ) then for all $x \in B^n \times \mathbb{T}^n$ we have

$$\frac{d(I^j(\phi_t(x)))}{dt} = \frac{\partial H}{\partial \theta_j}(\phi_t(x)) = 0,$$

i.e. $I^j(\phi_t(x)) = I^j(x)$ for all t, and as H depends only on I this gives

$$\frac{d(\theta_j(\phi_t(x)))}{dt} = -\frac{\partial H}{\partial I^j}(\phi_t(x)) = -\frac{\partial H}{\partial I^j}(x) = \nu_j,$$

where $\nu = \nu(I(x))$. We integrate this equation from 0 to t and get

$$\theta_j(\phi_t(x)) - \theta_j(x) = t\nu_j.$$

Thus for $x \in B^n \times \mathbb{T}^n$, given $I(x)$ the trajectory $\phi_t(x)$ of x under the Hamiltonian flow of H can be explicitly seen if we know $\nu(I(x))$. We say that a value of I determines an *invariant torus* for the Hamiltonian flow of H.

If (M, ω) is a symplectic manifold and $H \in C^\infty(M)$, we say that H *admits action-angle coordinates* (I, θ) on an open set $U \subset M$ if there exists a symplectic diffeomorphism $\psi : U \to B^n \times \mathbb{T}^n$ such that $H \circ \psi^{-1}$ has action-angle coordinates (I, θ) in $B^n \times \mathbb{T}^n$. If H admits action-angle coordinates, then one can check that the push-forward $\psi_* X_H$ is the Hamiltonian vector field $X_{H \circ \psi^{-1}}$, so that

$$\psi_* X_H = -\sum_{j=1}^{n} \frac{\partial (H \circ \psi^{-1})}{\partial I_j} \frac{\partial}{\partial \theta_j}.$$

Let $f_1, \ldots, f_n \in C^\infty(\mathbb{R}^{2n})$. If the set $\{f_1, \ldots, f_n\}$ is completely integrable, with $H = f_1$, then for any open set $U \subseteq \mathbb{R}^{2n} \setminus \Sigma_F$ for which $F^{-1}(c) = \mathbb{T}^n$ for all $c \in U$, Abraham and Marsden [1, pp. 398–400] find action-angle coordinates in U. Here $F = f_1 \times \cdots f_n$, the momentum map. This construction is also explained by Arnold [2, pp. 282–284].

Suppose that $H \in C^\infty(B^n \times \mathbb{T}^n)$ has action-angle coordinates (I, θ), and assume that for all $I \in B^n$,

$$\det(\partial_I^2 H(I)) \neq 0.$$

Then by the inverse function theorem, for every $I \in B^n$ there is a neighborhood U of I and a neighborhood V of $\nu = \partial_I H(I)$ such that $\partial_I H : U \to V$ is a diffeomorphism. In $U \times \mathbb{T}^n$ we can use ν and θ as coordinates.

For $\nu \in \mathbb{R}^n$, let $g_\nu = \{k \in \mathbb{Z}^n : \langle \nu, k \rangle = 0\}$, and let $\mathrm{rank}(g_\nu)$ be the rank of the $\mathbb{Z}$-module g_ν, i.e. the maximal number of elements of g_ν that are linearly independent over $\mathbb{Z}$. The proof of the following theorem follows [8, Proposition 2.1].

Theorem 3. *Let $\nu \in \Omega$ and let $r = \mathrm{rank}(g_\nu)$. In the torus with frequency ν, each trajectory is dense in some $(n-r)$-dimensional subtorus and the n-dimensional torus is foliated by these $(n-r)$-dimensional tori.*

Proof. There exists a basis $k_1, \dots, k_r$ of g_ν and vectors $k_1^*, \dots, k_{n-r}^* \in \mathbb{Z}^n$ such that the $n \times n$ matrix K_0 with rows $k_1^*, \dots, k_{n-r}^*, k_1, \dots, k_r$, has determinant 1. (I should show why such a basis exists.) Let $K_0 = \begin{bmatrix} K^* \\ K \end{bmatrix}$. K^* is an $(n-r) \times n$ matrix and K is an $r \times n$ matrix.

Let $q = K_0\theta$. Since $\det(K_0) = 1$, K_0 is invertible over $\mathbb{Z}$. The coordinate θ is only determined up to $\mathbb{Z}^n$, and for $q_1 - q_2 \in \mathbb{Z}^n$ then also $\theta_1 - \theta_2 \in \mathbb{Z}^n$. Thus $q = K_0\theta$ are coordinates on $\mathbb{T}^n$. The equation $\dot{\theta} = \nu$ can be written using the q coordinates as $\dot{q} = K_0\nu$. Then

$$K_0\nu = \begin{bmatrix} K^* \\ K \end{bmatrix} \nu = \begin{bmatrix} K^*\nu \\ K\nu \end{bmatrix} = \begin{bmatrix} K^*\nu \\ 0 \end{bmatrix}.$$

Let $\nu^* = K^*\nu$.

We see that $\{l \in \mathbb{Z}^n : l_1 = \dots = l_{n-r} = 0\} \subseteq g_{K_0\nu}$; since they both have rank r, they are equal. It follows that $\nu^* \in \mathbb{R}^{n-r}$ is nonresonant. Hence any trajectory on the n-dimensional torus with frequency ν is dense in the r-dimensional torus $\{q \in \mathbb{T}^n : q_{n-r+1} = \dots = q_n = \text{constant}\}$. □

3 Diophantine frequency vectors

For $c > 0$ and $\gamma \geq 0$ we define

$$D_n(c, \gamma) = \{\nu \in \mathbb{R}^n : |\langle k, \nu \rangle| \geq \frac{1}{c\|k\|_\infty^\gamma} \quad \text{for all } k \in \mathbb{Z}^n\}.$$

We further define $D_n(\gamma) = \bigcup_{c>0} D_n(c, \gamma)$.

Theorem 4. *For any $\nu \in \mathbb{R}^n$ and for any positive integer K, there is some $0 \neq k \in \mathbb{Z}^n$ with $\|k\|_\infty \leq 2K$ such that*

$$|\langle k, \nu \rangle| \leq \frac{n\|\nu\|_\infty}{(2K)^{n-1}}.$$

Proof. Let $B_K = \{k \in \mathbb{Z}^n : 0 < \|k\|_\infty \leq K\}$. The set B_K has $(2K+1)^n - 1$ elements. For $k \in B_K$ we have

$$|\langle k, \nu \rangle| \leq n\|k\|_\infty \|\nu\|_\infty \leq nK\|\nu\|_\infty.$$

Let $A = nK\|\nu\|_\infty$.

Let $M = (2K+1)^n - 2$. In the set $\{|\langle k, \nu \rangle| : k \in B_K\}$, there are two elements that are in same interval $[\frac{(j-1)A}{M}, \frac{jA}{M}]$, $j = 1, \dots, M$, since B_K has $M+1$ elements and there are M such intervals. That is, there are $k', k'' \in B_K$ such that $|\langle k', \nu \rangle|, |\langle k'', \nu \rangle| \in [\frac{(j-1)A}{M}, \frac{jA}{M}]$ for some j. Hence $|\langle k', \nu \rangle - \langle k'', \nu \rangle| \leq \frac{A}{M} = \frac{nK\|\nu\|_\infty}{(2K+1)^n - 2}$.

One can show by induction that for all $n \geq 1$, $\frac{K}{(2K+1)^n - 2} \leq \frac{1}{(2K)^{n-1}}$. Therefore for $k = k' - k''$ we have

$$|\langle k, \nu \rangle| \leq \frac{n\|\nu\|_\infty}{(2K)^{n-1}},$$

Finally, $\|k\|_\infty \leq \|k'\|_\infty + \|k''\|_\infty \leq 2K$. □

Corollary 5. *If $\gamma < n-1$ then $D_n(\gamma) = \emptyset$.*

Proof. Let $c > 0$. Suppose that there is some $\nu \in D_n(c,\gamma)$. Let K be the least integer such that $(2K)^{n-1-\gamma}$ is greater than $2cn\|\nu\|_\infty$; since $n-1-\gamma > 0$ such a K exists.

By Theorem 4, there is some $0 \neq k \in \mathbb{Z}^n$ with

$$|\langle k, \nu\rangle| \leq \frac{n\|\nu\|_\infty}{(2K)^{n-1}}.$$

Then

$$\begin{aligned} |\langle k, \nu\rangle| &\leq \frac{n\|\nu\|_\infty (2K)^{-\gamma}}{(2K)^{n-1-\gamma}} \\ &\leq \frac{n\|\nu\|_\infty (2K)^{-\gamma}}{2cn\|\nu\|_\infty} \\ &= \frac{1}{2c(2K)^\gamma} \\ &\leq \frac{1}{2c(4\|k\|_\infty)^\gamma} \\ &< \frac{1}{c\|k\|_\infty^\gamma}, \end{aligned}$$

contradicting that $\nu \in D_n(c,\gamma)$. Therefore for all $c > 0$, $D_n(c,\gamma) = \emptyset$. □

Treschev and Zubelevich give a construction for points in $D_n(c, n-1)$ for sufficiently large c [8, Theorem 9.2]. Thus there is some $C(n)$ such that for all $c \geq C(n)$, $D_n(c, n-1) \neq \emptyset$. It is clear that for $\gamma' \geq \gamma$ we have the inclusion $D_n(c,\gamma) \subseteq D_n(c,\gamma')$. Hence this construction also shows that $D_n(c,\gamma) \neq \emptyset$ for all $\gamma \geq n-1$ and $c \geq C(n)$. However this construction does not show that $m(D_n(c, n-1)) > 0$ for $c \geq C(n)$. Indeed, one can show that $m(D_n(n-1)) = 0$, but also that the set $D_n(n-1)$ has Hausdorff dimension n [7, p. 5].

Our proof of the following theorem expands on [8, Theorem 9.3]. Let $Q_n(L) = \{\nu \in \mathbb{R}^n : \|\nu\|_\infty \leq \frac{L}{2}\}$, the cube in $\mathbb{R}^n$ of edge length L. Let m be n-dimensional Lebesgue measure. We will use the fact that the maximal $n-1$ dimensional area of the intersection of $Q_n(L)$ and a hyperplane is $\sqrt{2}L^{n-1}$ [3].

Theorem 6. *Let $L > 0$. For $\gamma > n-1$ and $c > 0$,*

$$m(Q_n(L) \setminus D_n(c,\gamma)) \leq \frac{4\sqrt{2}n(3L)^{n-1}}{c}\Big(1 - \frac{1}{\gamma - n + 1}\Big).$$

Proof. Let $Q_n = Q_n(L)$. Let $\Pi_k = \{\nu \in \mathbb{R}^n : |\langle \nu, k\rangle| < \frac{1}{c\|k\|_\infty^\gamma}\}$. Let $\nu \in Q_n \setminus D_n(c,\gamma)$. Then there is some $k \neq 0$ such that $|\langle k, \nu\rangle| < \frac{1}{c\|k\|_\infty^\gamma}$, and so $\nu \in \Pi_k$. Thus

$$Q_n \setminus D_n(c,\gamma) \subseteq \bigcup_{k\neq 0} (Q_n \cap \Pi_k),$$

so

$$m(Q_n \setminus D_n(c,\gamma)) \leq \sum_{k \neq 0} m(Q_n \cap \Pi_k).$$

Let $k \neq 0$. Π_k is the region bounded by the two hyperplanes $\pi_1 = \{\nu \in \mathbb{R}^n : \langle \nu, k\rangle = \frac{1}{c\|k\|_\infty^\gamma}\}$ and $\pi_2 = \{\nu \in \mathbb{R}^n : \langle \nu, k\rangle = -\frac{1}{c\|k\|_\infty^\gamma}\}$. Let $p_1 = \frac{k}{c\|k\|_\infty^\gamma \|k\|} \in \pi_1$ and $p_2 = -\frac{k}{c\|k\|_\infty^\gamma \|k\|}\pi_2$. For any two points $\nu_1, \nu_2 \in \pi_1$ we can check that $\langle p_1 - p_2, \nu_1 - \nu_2\rangle = 0$, and for any two points $\nu_1, \nu_2 \in \pi_2$ we can check that $\langle p_1 - p_2, \nu_1 - \nu_2\rangle = 0$. Thus the vector $p_1 - p_2$ is orthogonal to each of the hyperplanes π_1 and π_2. It follows that the distance between the hyperplanes π_1 and π_2 is the distance between the points p_1 and p_2, which is $2 \cdot \frac{\|k\|}{c\|k\|_\infty^\gamma \|k\|^2}$. Since $\|k\| \geq \|k\|_\infty$, this is $\leq \frac{2}{c\|k\|_\infty^{\gamma+1}}$. Therefore

$$m(Q_n \cap \Pi_k) \leq \frac{2}{c\|k\|_\infty^{\gamma+1}} \cdot \sqrt{2}L^{n-1},$$

where we use the fact that the maximal $n-1$ dimensional area of the intersection of $Q_n = Q_n(L)$ and a hyperplane is $\sqrt{2}L^{n-1}$ [3].

For each positive integer l, the hypercube $\{k \in \mathbb{Z}^n : \|k\|_\infty = l\}$ has $2n$ faces, on each of which there are $(2l+1)^{n-1}$ points with integer coordinates. Hence for each integer positive integer l, we have $\#\{k \in \mathbb{Z}^n : \|k\|_\infty = l\} \leq 2n(2l+1)^{n-1}$.

Therefore

$$\begin{aligned}
m(Q_n \setminus D_n(c,\gamma)) &\leq \sum_{k\neq 0} m(Q_n \cap \Pi_k) \\
&\leq \sum_{k \neq 0} \frac{2\sqrt{2}L^{n-1}}{c\|k\|_\infty^{\gamma+1}} \\
&= \sum_{l=1}^\infty \sum_{\|k\|_\infty = l} \frac{2\sqrt{2}L^{n-1}}{cl^{\gamma+1}} \\
&\leq \sum_{l=1}^\infty 2n(2l+1)^{n-1} \frac{2\sqrt{2}L^{n-1}}{cl^{\gamma+1}} \\
&\leq \sum_{l=1}^\infty 2n(3l)^{n-1} \frac{2\sqrt{2}L^{n-1}}{cl^{\gamma+1}} \\
&= \frac{4\sqrt{2}n(3L)^{n-1}}{c} \sum_{l=1}^\infty \frac{1}{l^{\gamma-n+2}}.
\end{aligned}$$

Since the terms in the sum are positive and decreasing, we can estimate the sum using an integral:

$$\sum_{l=1}^{\infty} \frac{1}{l^{\gamma-n+2}} \leq 1 + \int_{1}^{\infty} \frac{dx}{x^{\gamma-n+2}} = 1 + \frac{1}{\gamma-n+1},$$

finishing the proof. □

Corollary 7. *If $\gamma > n-1$ then $m(\mathbb{R}^n \setminus D_n(\gamma)) = 0$.*

Proof. Let $L > 0$. For every $c > 0$, $m(Q_n(L) \setminus D_n(\gamma)) \leq m(Q_n(L) \setminus D_n(c,\gamma))$. By Theorem 6, $m(Q_n(L) \setminus D_n(c,\gamma)) \to 0$ as $c \to \infty$. Hence $m(Q_n(L) \setminus D_n(\gamma)) = 0$. But then

$$m(\mathbb{R}^n \setminus D_n(\gamma)) = \lim_{L\to\infty} m(Q_n(L) \setminus D_n(\gamma)) = \lim_{L\to\infty} 0 = 0.$$

□

Fix $\gamma > n-1$. Let $\alpha = \frac{1}{c}$. Let A_α be an α-neighborhood of the boundary of Ω. We will make whatever assumption about $\partial\Omega$ we need in order to get $m(A_\alpha) = O(\alpha)$.

Suppose that L is sufficiently large so that $\Omega \subseteq Q_n(L)$. Then Theorem 6 gives us that $m(\Omega \setminus D_n(c,\gamma)) = O(\alpha)$.

Let $\Omega_\alpha = D_n(c,\gamma) \cap (\Omega \setminus A_\alpha)$. Since $\Omega \setminus \Omega_\alpha = (\Omega \setminus D_n(c,\gamma)) \cup (\Omega \cap A_\alpha)$, we have $m(\Omega \setminus \Omega_\alpha) = O(\alpha)$.

4 Statement of KAM

If we have a Hamiltonian system which admits action-angle coordinates in $B^n \times \mathbb{T}^n$, then the trajectories of points in phase space are constrained to lie on invariant tori. Moreover, on these tori the dynamics of the system are quasi-periodic; a priori we don't have a reason to expect that the dynamics should be so nice just because the trajectories lie on tori. But a generic Hamiltonian on the same phase space (I would like to make this notion precise) does not admit action-angle coordinates. The KAM theorem is a statement about the dynamics induced by making a sufficiently small change to a Hamiltonian. If we perturb a Hamiltonian which admits action-angle coordinates to one which probably does not, if the perturbation is sufficiently small, then most of the trajectories of points under the flow of the new Hamiltonian will also lie on tori. In some sense which I want to clarify, the invariant tori of the new Hamiltonian are close to the invariant tori of the Hamiltonian that admits action-angle coordinates. It is not clear to me how an invariant torus of the old Hamiltonian transforms into an invariant torus of the new Hamiltonian; in what sense does an invariant torus for the old Hamiltonian become an invariant torus for the new Hamiltonian?

In particular, a consequence of the KAM theorem is that if we make a small perturbation of a Hamiltonian system that admits action-angle coordinates then the trajectories of most points will not be dense on a hypersurface in phase space, since they are constrained to lie on n-dimensional tori. In other words, the new Hamiltonian system is not ergodic, since the invariant tori have lower dimension than $n-1$, and so have $n-1$-dimensional measure 0.

Let's explain the KAM theorem in another way. Suppose that we have a symplectic manifold M and a Lagrangian foliation $\mathscr{F}_0$ whose leaves are tori, and suppose that the leaves of $\mathscr{F}_0$ are invariant tori for a Hamiltonian H_0. That is, the Hamiltonian vector field X_{H_0} is tangent to all the leaves in $\mathscr{F}_0$. Now let $H = H_0 + \epsilon H_1$. The leaves of the foliation $\mathscr{F}_0$ will not be invariant under the flow of H. We would like to obtain a symplectomorphism $\Phi : M \to M$ such that the Hamiltonian vector field X_H is tangent to most leaves in the foliation $\mathscr{F} = \Phi(\mathscr{F}_0)$. Here we mean most in a measure theoretic sense that depends on the magnitude ϵ of the perturbation away from the Hamiltonian that admits action-angle coordinates.

How do we construct a diffeomorphism? Often the best way is to demand that it be the time 1 flow of a vector field, so $\Phi = \Phi_1$ for some Φ_t, and to see if such a vector field exists. Suppose that f is a function such that if Φ_t is the flow of X_f then $\Phi_1 = \Phi$.

5 Normal forms

Normal forms of vector fields, homological equation [9].

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