Schwartz functions, Hermite functions, and the Hermite operator

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July 17, 2015

1 Schwartz functions

For $\phi \in C^\infty(\mathbb{R}, \mathbb{C})$ and $p \geq 0$, let

$$|\phi|_p = \sup_{0 \leq k \leq p} \sup_{u \in \mathbb{R}} (1+u^2)^{p/2} |\phi^{(k)}(u)|.$$

We define $\mathscr{S}$ to be the set of those $\phi \in C^\infty(\mathbb{R}, \mathbb{C})$ such that $|\phi|_p < \infty$ for all $p \geq 0$. $\mathscr{S}$ is a complex vector space and each $|\cdot|_p$ is a norm, and because each $|\cdot|_p$ is a norm, a fortiori $\{|\cdot|_p : p \geq 0\}$ is a separating family of seminorms. With the topology induced by this family of seminorms, $\mathscr{S}$ is a Fréchet space.[1] Furthermore, $D : \mathscr{S} \to \mathscr{S}$ defined by

$$(D\phi)(x) = \phi'(x), \qquad x \in \mathbb{R}$$

and $M : \mathscr{S} \to \mathscr{S}$ defined by

$$(M\phi)(x) = x\phi(x), \qquad x \in \mathbb{R}$$

are continuous linear maps.

Let $\mathscr{S}'$ be the collection of continuous linear maps $\mathscr{S} \to \mathbb{C}$. For $\phi \in \mathscr{S}$, define $e_\phi : \mathscr{S}' \to \mathbb{C}$ by

$$e_\phi(\omega) = \omega(\phi), \qquad \omega \in \mathscr{S}'.$$

The initial topology for the collection $\{e_\phi : \phi \in \mathscr{S}\}$ is called the **weak-* topology** on $\mathscr{S}'$. With this topology, $\mathscr{S}'$ is a locally convex space whose dual space is $\{e_\phi : \phi \in \mathscr{S}\}$.

2 L^2 norms

For $p \geq 0$ and $\phi, \psi \in \mathscr{S}$, let

$$[\phi, \psi]_p = \sum_{k=0}^{p} \int_{\mathbb{R}} (1+u^2)^p \phi^{(k)}(u) \overline{\psi^{(k)}(u)} du,$$

[1] Walter Rudin, *Functional Analysis*, second ed., p. 184, Theorem 7.4.

and let

$$[\phi]_p^2 = [\phi, \phi]_p = \sum_{k=0}^{p} \int_{\mathbb{R}} (1+u^2)^p |\phi^{(k)}(u)|^2 du.$$

Because $(1+u^2)^p \leq (1+u^2)^q$ when $p \leq q$, it is immediate that $[\phi]_p \leq [\phi]_q$ when $p \leq q$.

We relate the norms $|\cdot|_p$ and the norms $[\cdot]_p$.[2]

Lemma 1. For each $p \geq 1$, for all $\phi \in \mathscr{S}$,

$$\frac{1}{p\sqrt{\pi}} |\phi|_{p-1} \leq [\phi]_p \leq \sqrt{(p+1)\pi} |\phi|_{p+1}.$$

Proof. For $0 \leq k \leq p$,

$$\begin{aligned} \int_{\mathbb{R}} (1+u^2)^p |\phi^{(k)}(u)|^2 du &\leq \sup_{u \in \mathbb{R}} ((1+u^2)^{p+1} |\phi^{(k)}(u)|^2) \int_{\mathbb{R}} (1+u^2)^{-1} du \\ &= \sup_{u \in \mathbb{R}} ((1+u^2)^{p+1} |\phi^{(k)}(u)|^2) \cdot \pi \\ &\leq \pi |\phi|_{p+1}^2, \end{aligned}$$

hence

$$\begin{aligned} [\phi]_p^2 &= \sum_{k=0}^{p} \int_{\mathbb{R}} (1+u^2)^p |\phi^{(k)}(u)|^2 du \\ &\leq \sum_{k=0}^{p} \pi |\phi|_{p+1}^2 \\ &= (p+1)\pi |\phi|_{p+1}^2. \end{aligned}$$

For $0 \leq k \leq p-1$ and $u \in \mathbb{R}$, using the fundamental theorem of calculus

[2]Takeyuki Hida, *Brownian Motion*, p. 305, Lemma A.1.

and the Cauchy-Schwarz inequality,

$$\begin{aligned}
|(1+u^2)^{(p-1)/2}\phi^{(k)}(u)| &= \left|\int_{-\infty}^{u}((1+t^2)^{(p-1)/2}\phi^{(k)}(t))'dt\right| \\
&\leq \int_{\mathbb{R}} |(p-1)t(1+t^2)^{(p-1)/2-1}\phi^{(k)}(t)|dt \\
&+ \int_{\mathbb{R}} |(1+t^2)^{(p-1)/2}\phi^{(k+1)}(t)|dt \\
&\leq (p-1)\int_{\mathbb{R}}(1+t^2)^{-1/2}(1+t^2)^{(p-1)/2}|\phi^{(k)}(t)|dt \\
&+ \int_{\mathbb{R}}(1+t^2)^{-1/2}(1+t^2)^{p/2}|\phi^{(k+1)}(t)|dt \\
&\leq (p-1)\left(\int_{\mathbb{R}}(1+t^2)^{-1}dt\right)^{1/2}\left(\int_{\mathbb{R}}(1+t^2)^{p-1}|\phi^{(k)}(t)|^2dt\right)^{1/2} \\
&+ \left(\int_{\mathbb{R}}(1+t^2)^{-1}dt\right)^{1/2}\left(\int_{\mathbb{R}}(1+t^2)^{p}|\phi^{(k+1)}(t)|^2dt\right)^{1/2} \\
&\leq (p-1)\sqrt{\pi}[\phi]_{p-1} + \sqrt{\pi}[\phi]_p \\
&\leq p\sqrt{\pi}[\phi]_p,
\end{aligned}$$

which shows that

$$|\phi|_{p-1} \leq p\sqrt{\pi}[\phi]_p.$$

□

3 Hermite functions

Let λ be Lebesgue measure on $\mathbb{R}$, and let

$$(f,g)_{L^2} = \int_{\mathbb{R}} f\overline{g}d\lambda.$$

$L^2(\lambda)$ with the inner product $(\cdot,\cdot)_{L^2}$ is a separable Hilbert space. For $n \geq 0$, let

$$h_n(x) = (-1)^n(2^n n!\sqrt{\pi})^{-1/2}e^{x^2/2}D^n e^{-x^2},$$

the **Hermite functions**, the set of which is an orthonormal basis for $L^2(\lambda)$. We remark that the Hermite functions belong to $\mathscr{S}$. For $n < 0$ we define

$$h_n = 0,$$

to write some expressions in a uniform way.

We calculate that for $n \geq 0$,

$$Dh_n = \sqrt{\frac{n}{2}}h_{n-1} - \sqrt{\frac{n+1}{2}}h_{n+1}.$$

We define the **Hermite operator** $A : \mathscr{S} \to \mathscr{S}$ by

$$A = -D^2 + M^2 + 1.$$

A is a densely defined operator in $L^2(\lambda)$ that is symmetric and positive, and satisfies

$$Ah_n = (2n+2)h_n.$$

There is a unique bounded linear operator $T : L^2(\lambda) \to L^2(\lambda)$ satisfying

$$Th_n = A^{-1}h_n = (2n+2)^{-1}h_n, \qquad n \geq 0.$$

The operator norm of T is $\|T\| = \frac{1}{2}$, and T is self-adjoint. For $p \geq 1$, T^p is a Hilbert-Schmidt operator with Hilbert-Schmidt norm $\|T^p\|_{\mathrm{HS}} = 2^{-p}\sqrt{\zeta(2p)}$.

We define the **creation operator** $B : \mathscr{S} \to \mathscr{S}$ by

$$B = D + M$$

and we define the **annihilation operator** $C : \mathscr{S} \to \mathscr{S}$ by

$$C = -D + M,$$

which are continuous linear maps. They satisfy, for $n \geq 0$,

$$Bh_n = (2n)^{1/2}h_{n-1}, \qquad Ch_n = (2n+2)^{1/2}h_{n+1}.$$

(We remind ourselves that we have defined $h_{-1} = 0$.) It is immediate that $BC = A$ and that $B - C = 2D$. Using the creation operator, we can write the Hermite functions as

$$h_n = (2^n n!)^{-1/2}C^n h_0 = \pi^{-1/4}(2^n n!)^{-1/2}C^n(e^{-x^2/2}).$$

For $\phi, \psi \in \mathscr{S}$, using integration by parts,

$$(D\phi, \psi)_{L^2} = \int_{\mathbb{R}} \phi'(x)\overline{\psi(x)}dx = -\int_{\mathbb{R}} \phi(x)\overline{\psi'(x)}dx = (\phi, (-D)\psi)_{L^2},$$

and

$$(M\phi, \psi)_{L^2} = \int_{\mathbb{R}} x\phi(x)\overline{\psi(x)}dx = (\phi, M\psi)_{L^2}.$$

Thus,

$$\begin{aligned}(B\phi, \psi)_{L^2} &= (D\phi, \psi)_{L^2} + (M\phi, \psi)_{L^2} \\ &= (\phi, (-D)\psi)_{L^2} + (\phi, M\psi)_{L^2} \\ &= (\phi, C\psi)_{L^2}\end{aligned}$$

and

$$(C\phi, \psi)_{L^2} = (\phi, B\psi)_{L^2}.$$

We shall use these calculations to obtain the following lemma.

Lemma 2. For $p \geq 0$ and for $\phi \in \mathscr{S}$,

$$B^p \phi = 2^{p/2} \sum_{n=0}^{\infty} \left(\frac{(n+p)!}{n!} \right)^{1/2} (\phi, h_{n+p})_{L^2} h_n$$

and

$$C^p \phi = 2^{p/2} \sum_{n=0}^{\infty} (\phi, h_{n-p})_{L^2} \left(\frac{n!}{(n-p)!} \right)^{1/2} h_n.$$

Proof. Because $Ch_n = (2n+2)^{1/2} h_{n+1}$,

$$(\phi, C^p h_n)_{L^2} = (\phi, h_{n+p})_{L^2} \prod_{j=n}^{n+p-1} (2j+2)^{1/2} = (\phi, h_{n+p})_{L^2} 2^{p/2} \left(\frac{(n+p)!}{n!} \right)^{1/2}.$$

With

$$\phi = \sum_{n=0}^{\infty} (\phi, h_n)_{L^2} h_n,$$

and because $(B\phi, \psi)_{L^2} = (\phi, C\psi)_{L^2}$, we have

$$\begin{aligned} B^p \phi &= \sum_{n=0}^{\infty} (B^p \phi, h_n)_{L^2} h_n \\ &= \sum_{n=0}^{\infty} (\phi, C^p h_n)_{L^2} h_n \\ &= \sum_{n=0}^{\infty} (\phi, h_{n+p})_{L^2} 2^{p/2} \left(\frac{(n+p)!}{n!} \right)^{1/2} h_n. \end{aligned}$$

Because $Bh_n = (2n)^{1/2} h_{n-1}$, and reminding ourselves that we define $h_n = 0$ for $n < 0$,

$$(\phi, B^p h_n)_{L^2} = (\phi, h_{n-p})_{L^2} \prod_{j=n-p+1}^{n} (2j)^{1/2} = (\phi, h_{n-p})_{L^2} 2^{p/2} \left(\frac{n!}{(n-p)!} \right)^{1/2}.$$

Because $(C\phi, \psi)_{L^2} = (\phi, B\psi)_{L^2}$, we have

$$\begin{aligned} C^p \phi &= \sum_{n=0}^{\infty} (C^p \phi, \psi)_{L^2} h_n \\ &= \sum_{n=0}^{\infty} (\phi, B^p \psi)_{L^2} h_n \\ &= \sum_{n=0}^{\infty} (\phi, h_{n-p})_{L^2} 2^{p/2} \left(\frac{n!}{(n-p)!} \right)^{1/2} h_n. \end{aligned}$$

□

We define the Fourier transform $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ by

$$(\mathscr{F}\phi)(\xi) = \int_{\mathbb{R}} \phi(x) e^{-i\xi x} \frac{dx}{(2\pi)^{1/2}}, \qquad \xi \in \mathbb{R}.$$

$\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is a continuous linear map, and satisfies

$$\mathscr{F} M = iD\mathscr{F}, \qquad \mathscr{F} D = iM\mathscr{F}.$$

From these we obtain

$$\mathscr{F} A = A\mathscr{F}, \qquad \mathscr{F} B = iB\mathscr{F}, \qquad \mathscr{F} C = -iC\mathscr{F},$$

and one proves the following using the above.

Lemma 3. For $n \geq 0$,

$$\mathscr{F} h_n = (-i)^n h_n.$$

We further remark that for $\phi \in \mathscr{S}$,

$$\|\phi\|_\infty \leq 2^{-1/2}(\|\phi\|_{L^2}^2 + \|\phi'\|_{L^2}^2). \tag{1}$$

Finally, there is a unique Hilbert space isomorphism $\mathscr{F} : L^2(\lambda) \to L^2(\lambda)$ whose restriction to $\mathscr{S}$ is equal to $\mathscr{F}$ as already defined. Thus for $f \in L^2(\lambda)$, as

$$f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} h_n,$$

we get

$$\mathscr{F} f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} (-i)^n h_n.$$

4 Hermite operator

For $p \geq 0$ and $f \in L^2(\lambda)$, we define

$$\|f\|_p^2 = \sum_{n=0}^{\infty} (2n+2)^{2p} |(f, h_n)_{L^2}|^2.$$

We define

$$\mathscr{S}_p = \{f \in L^2(\lambda) : \|f\|_p < \infty\},$$

and for $f, g \in \mathscr{S}_p$ we define

$$(f, g)_p = \sum_{n=0}^{\infty} (2n+2)^{2p} (f, h_n)_{L^2} \overline{(g, h_n)_{L^2}},$$

for which

$$\|f\|_p^2 = (f, f)_p.$$

Lemma 4. For $\phi \in \mathscr{S}$, for each $p \geq 0$, $\phi \in \mathscr{S}_p$, and

$$\|\phi\|_p = \|A^p \phi\|_{L^2}.$$

Proof. $A^p\phi \in \mathscr{S}$, so $\|A^p\phi\|_{L^2} < \infty$. Because A is a symmetric operator and as $Ah_n = (2n+2)h_n$,

$$\begin{aligned}
\|A^p\phi\|_{L^2}^2 &= \sum_{n=0}^{\infty} |(A^p\phi, h_n)_{L^2}|^2 \\
&= \sum_{n=0}^{\infty} |(\phi, A^p h_n)_{L^2}|^2 \\
&= \sum_{n=0}^{\infty} (2n+2)^{2p} |(\phi, h_n)_{L^2}|^2 \\
&= \|\phi\|_p^2.
\end{aligned}$$

□

For $f, g \in L^2(\lambda)$, because T is self-adjoint,

$$\begin{aligned}
(T^p f, T^p g)_p &= \sum_{n=0}^{\infty} (2n+2)^{2p} (T^p f, h_n)_{L^2} \overline{(T^p f, h_n)_{L^2}} \\
&= \sum_{n=0}^{\infty} (2n+2)^{2p} (f, T^p h_n)_{L^2} \overline{(g, T^p h_n)_{L^2}} \\
&= \sum_{n=0}^{\infty} (2n+2)^{2p} (f, (2n+2)^{-p} h_n)_{L^2} \overline{(g, (2n+2)^{-p} h_n)_{L^2}} \\
&= \sum_{n=0}^{\infty} (f, h_n)_{L^2} \overline{(g, h_n)_{L^2}} \\
&= (f, g)_{L^2},
\end{aligned}$$

and so $\|T^p f\|_p = \|f\|_{L^2}$, which shows that

$$T^p L^2(\lambda) = \mathscr{S}_p.$$

If $f_i \in \mathscr{S}_p$ is a Cauchy sequence in the norm $\|\cdot\|_p$, then as $\|T^{-p} f_i - T^{-p} f_j\|_{L^2} = \|f_i - f_j\|_p$, $T^{-p} f_i$ is a Cauchy sequence in the norm $\|\cdot\|_{L^2}$ and so there is some $g \in L^2(\lambda)$ for which $\|T^{-p} f_i - g\|_{L^2} \to 0$. We have $T^p g \in \mathscr{S}_p$, and

$$\|f_i - T^p g\|_p = \left\|T^{-p} f_i - g\right\|_{L^2} \to 0,$$

thus $f_i \to T^p g$ in the norm $\|\cdot\|_p$, showing that $(\mathscr{S}_p, (\cdot,\cdot)_p)$ is a Hilbert space. Furthermore, $T^p : L^2(\lambda) \to \mathscr{S}_p$ is an isomorphism of Hilbert spaces, and thus $\{T^p h_n : n \geq 0\}$ is an orthonormal basis for $(\mathscr{S}_p, (\cdot,\cdot)_p)$.

For $p \le q$,

$$\|f\|_p \le \|f\|_q ,$$

so $\mathscr{S}_q \subset \mathscr{S}_p$. For $p \ge q$, let $i_{q,p} : \mathscr{S}_q \to \mathscr{S}_p$ be the inclusion map.[3]

Theorem 5. For $p < q$, the inclusion map $i_{q,p} : \mathscr{S}_q \to \mathscr{S}_p$ is a Hilbert-Schmidt operator, with Hilbert-Schmidt norm

$$\|i_{q,p}\|_{\mathrm{HS}} = 2^{-q+p}\sqrt{\zeta(2q-2p)}.$$

Proof. $\{T^q h_n : n \ge 0\}$ is an orthonormal basis for $(\mathscr{S}_q, (\cdot,\cdot)_q)$, and

$$\begin{aligned}
\|i_{q,p}\|_{\mathrm{HS}}^2 &= \sum_{n=0}^{\infty} \|i_{q,p} T^q h_n\|_p^2 \\
&= \sum_{n=0}^{\infty} \|T^q h_n\|_p^2 \\
&= \sum_{n=0}^{\infty} \left\|(2n+2)^{-q} h_n\right\|_p^2 \\
&= \sum_{n=0}^{\infty} (2n+2)^{-2q}(2n+2)^{2p} \\
&= 2^{-2q+2p}\zeta(2q-2p).
\end{aligned}$$

□

5 The Hilbert spaces S_p

For $f \in L^2(\lambda)$,

$$f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} h_n,$$

and for $N \ge 0$ we define $f_N : \mathbb{R} \to \mathbb{C}$ by

$$f_N(x) = \sum_{n=0}^{N} (f, h_n)_{L^2} h_n(x), \qquad x \in \mathbb{R},$$

which belongs to $\mathscr{S}$.

For $k \ge 0$, we define $C_b^k(\mathbb{R})$ to be the set of those functions $\mathbb{R} \to \mathbb{C}$ that are k-times differentiable and such that for each $0 \le j \le k$, $f^{(j)}$ is continuous and bounded. With the norm

$$\|f\|_{C_b^k} = \sum_{j=0}^{k} \left\|f^{(j)}\right\|_\infty$$

this is a Banach space. Because the Hermite functions belong to $\mathscr{S}$, for $f \in L^2(\lambda)$ and for any k and N, the function f_N belongs to $C_b^k(\mathbb{R})$.

[3] Hui-Hsiung Kuo, *White Noise Distribution Theory*, p. 18, Lemma 3.3.

Lemma 6. If $p \geq 1$ and $f \in \mathscr{S}_p$, then there is some $F \in C_b^{p-1}(\mathbb{R})$ such that f is equal almost everywhere to F.

Proof. **Cramér's inequality** states that there is a constant K_0 such that for all n, $\|h_n\|_\infty \leq K_0$. For $M < N$, using this and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|f_N - f_M\|_{C_b^0} &= \left\| \sum_{n=M+1}^{N} (f, h_n)_{L^2} h_n \right\|_\infty \\
&\leq K_0 \sum_{n=M+1}^{N} |(f, h_n)_{L^2}| \\
&= K_0 \sum_{n=M+1}^{N} (2n+2)^{-1}(2n+2)|(f, h_n)_{L^2}| \\
&\leq \left(\sum_{n=M+1}^{N} (2n+2)^{-2} \right)^{1/2} \left(\sum_{n=M+1}^{N} (2n+2)^2 |(f, h_n)_{L^2}|^2 \right)^{1/2} \\
&= \left(\sum_{n=M+1}^{N} (2n+2)^{-2} \right)^{1/2} \|f_N - f_M\|_1 .
\end{aligned}$$

Because $f \in \mathscr{S}_p \subset \mathscr{S}_1$, f_N is a Cauchy sequence in $\mathscr{S}_1$, hence f_N is a Cauchy sequence in $C_b^0(\mathbb{R})$, so there is some $F \in C_b^0(\mathbb{R})$ such that f_N converges to F in $C_b^0(\mathbb{R})$. We assert that $F = f$ as elements of $L^2(\lambda)$.

Using

$$Dh_n = \sqrt{\frac{n}{2}} h_{n-1} - \sqrt{\frac{n+1}{2}} h_{n+1},$$

we calculate

$$\begin{aligned}
f_N' &= -\sqrt{\frac{N}{2}} (f, h_{N-1})_{L^2} h_N - \sqrt{\frac{N+1}{2}} (f, h_N)_{L^2} h_{N+1} \\
&\quad + \sum_{n=0}^{N-1} \left(\sqrt{\frac{n+1}{2}} (f, h_{n+1})_{L^2} - \sqrt{\frac{n}{2}} (f, h_{n-1})_{L^2} \right) h_n,
\end{aligned}$$

hence for $M < N$,

$$\begin{aligned}
f_N' - f_M' &= -\sqrt{\frac{N}{2}} (f, h_{N-1})_{L^2} h_N - \sqrt{\frac{N+1}{2}} (f, h_N)_{L^2} h_{N+1} \\
&\quad + \sqrt{\frac{M}{2}} (f, h_{M-1})_{L^2} h_M + \sqrt{\frac{M+1}{2}} (f, h_M)_{L^2} h_{M+1} \\
&\quad + \sum_{n=M}^{N-1} \left(\sqrt{\frac{n+1}{2}} (f, h_{n+1})_{L^2} - \sqrt{\frac{n}{2}} (f, h_{n-1})_{L^2} \right) h_n,
\end{aligned}$$

and for $N \geq M+2$,

$$\begin{aligned}\|f_N' - f_M'\|_1 &= (2N+2)^2 \frac{N}{2} |(f, h_{N-1})|_{L^2}^2 + (2N+4)^2 \frac{N+1}{2} |(f, h_{N-1})|_{L^2}^2 \\ &(2M+2)^2 \frac{M+1}{2} |(f, h_{M+1})|_{L^2}^2 + (2M+4)^2 \frac{M+2}{2} |(f, h_{M+1})|_{L^2}^2 \\ &+ \sum_{n=M+2}^{N-1} (2n+2)^2 \left| \sqrt{\frac{n+1}{2}} (f, h_{n+1})_{L^2} - \sqrt{\frac{n}{2}} (f, h_{n-1})_{L^2} \right|^2 \\ &= O(\|f_N - f_M\|_2),\end{aligned}$$

whence f_N' is a Cauchy sequence in $C_b^0(\mathbb{R})$, and so f_N is a Cauchy sequence in $C_b^1(\mathbb{R})$. □

We prove that for $p \geq 1$ the derivatives of the partial sums f_N are a Cauchy sequence in $L^2(\lambda)$.[4]

Lemma 7. For $p \geq 1$ and $f \in \mathscr{S}_p$, f_N' is a Cauchy sequence in $L^2(\lambda)$.

Proof. Because $f_N \in \mathscr{S}$,

$$f_N' = Df_N = \frac{B-C}{2} f_N.$$

Then

$$\|f_N' - f_M'\|_{L^2} \leq \frac{1}{2} \|Bf_N - Bf_M\|_{L^2} + \frac{1}{2} \|Cf_N - Cf_M\|_{L^2}.$$

For $M < N$, as $Bh_n = (2n)^{1/2} h_{n-1}$,

$$\begin{aligned}\|Bf_N - Bf_M\|_{L^2}^2 &= \left\| B \sum_{n=M+1}^{N} (f, h_n)_{L^2} h_n \right\|_{L^2}^2 \\ &= \left\| \sum_{n=M+1}^{N} (f, h_n)_{L^2} (2n)^{1/2} h_{n-1} \right\|_{L^2}^2 \\ &= \sum_{n=M+1}^{N} |(f, h_n)_{L^2}|^2 (2n) \\ &\leq \sum_{n=M+1}^{N} (2n+2)^2 |(f, h_n)_{L^2}|^2,\end{aligned}$$

[4] Jeremy J. Becnel and Ambar N. Sengupta, *The Schwartz space: a background to white noise analysis*, `https://www.math.lsu.edu/~preprint/2004/as20041.pdf`, Lemma 7.1.

and as $Ch_n = (2n+2)^{1/2} h_{n+1}$,

$$\begin{aligned}
\|Cf_N - Cf_M\|_{L^2}^2 &= \left\| C \sum_{n=M+1}^{N} (f, h_n)_{L^2} h_n \right\|_{L^2}^2 \\
&= \left\| \sum_{n=M+1}^{N} (f, h_n)_{L^2} (2n+2)^{1/2} h_{n+1} \right\|_{L^2}^2 \\
&= \sum_{n=M+1}^{N} |(f, h_n)_{L^2}|^2 (2n+2) \\
&\leq \sum_{n=M+1}^{N} (2n+2)^2 |(f, h_n)_{L^2}|^2.
\end{aligned}$$

Thus

$$\|f'_N - f'_M\|_{L^2} \leq \frac{1}{2} \|f_N - f_M\|_1 + \frac{1}{2} \|f_N - f_M\|_1 = \|f_N - f_M\|_1 .$$

Because $f \in \mathscr{S}_p$ and $p \geq 1$, the series $\sum_{n=0}^{\infty} (2n+2)^2 |(f, h_n)_{L^2}|^2$ converges, from which the claim follows. $\square$

Now we establish that if $p \geq 1$ and $f \in \mathscr{S}_p$ then there is some $F \in C_b^0(\mathbb{R})$ such that f is equal almost everywhere to F, F is differentiable almost everywhere, and $F' \in \mathscr{S}_{p-1}$.[5]

Theorem 8. For $p \geq 1$ and $f \in \mathscr{S}_p$, there is some $F \in C_b^0(\mathbb{R})$ such that f is equal almost everywhere to F, F is differentiable almost everywhere, f'_N converges to F' in the norm $\|\cdot\|_{L^2}$, and $F' \in \mathscr{S}_{p-1}$.

Proof. Lemma 7 tells us that f'_N is a Cauchy sequence in the norm $\|\cdot\|_{L^2}$, and hence there is some $g \in L^2(\lambda)$ to which f'_N converges in the norm $\|\cdot\|_{L^2}$. For $x \leq y$, by the fundamental theorem of calculus,

$$f_N(y) = f_N(x) + \int_0^1 f'_N(x + t(y-x)) \cdot (y-x) dt.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_0^1 |f'_N(x + t(y-x)) \cdot (y-x) - g(x + t(y-x)) \cdot (y-x)| dt \\
=& \int_x^y |f'_N(u) - g(u)| du \\
\leq& \sqrt{y-x} \, \|f'_N - g\|_{L^2} .
\end{aligned}$$

[5] Jeremy J. Becnel and Ambar N. Sengupta, *The Schwartz space: a background to white noise analysis*, `https://www.math.lsu.edu/~preprint/2004/as20041.pdf`, Theorem 7.3.

Because $\|f_N' - g\|_{L^2} \to 0$ as $N \to \infty$,

$$\int_0^1 f_N'(x + t(y-x)) \cdot (y-x)dt \to \int_0^1 g(x + t(y-x)) \cdot (y-x)dt.$$

Then by Lemma 6, taking $N \to \infty$, for any $y > x$ we have

$$F(y) = F(x) + \int_0^1 g(x + t(y-x)) \cdot (y-x)dt = F(x) + \frac{1}{y-x}\int_x^y g(s)ds.$$

By the **Lebesgue differentiation theorem**, for almost all $x \in \mathbb{R}$,

$$\frac{1}{y-x}\int_x^y g(s)ds \to g(x), \qquad y \to x.$$

Therefore for almost all $x \in \mathbb{R}$,

$$F'(x) = g(x).$$

Thus $F' = g$ in $L^2(\lambda)$, and as $f_N' \to g$ in $L^2(\lambda)$,

$$\begin{aligned}
F' &= \lim_{N\to\infty} f_N' \\
&= \lim_{N\to\infty} \left(\frac{B-C}{2}\right) \sum_{n=0}^{N} (f, h_n)_{L^2} h_n \\
&= \frac{1}{2}\sum_{n=0}^{\infty} (f, h_n)_{L^2}((2n)^{1/2} h_{n-1} - (2n+2)^{1/2} h_{n+1}) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \left((2n+2)^{1/2}(f, h_n)_{L^2} - (2n)^{1/2}(f, h_{n-1})_{L^2}\right) h_n,
\end{aligned}$$

for which

$$\begin{aligned}
\|F'\|_{p-1}^2 &= \frac{1}{4}\sum_{n=0}^{\infty} (2n+2)^{2p-2} \left|(2n+2)^{1/2}(f, h_n)_{L^2} - (2n)^{1/2}(f, h_{n-1})_{L^2}\right|^2 \\
&\leq \frac{1}{2}\sum_{n=0}^{\infty} (2n+2)^{2p-2} \left((2n+2)|(f, h_n)_{L^2}|^2 + 2n|(f, h_{n-1})_{L^2}|^2\right),
\end{aligned}$$

which is finite because $f \in \mathscr{S}_p$. Therefore $F' \in \mathscr{S}_{p-1}$. □