The heat kernel on $\mathbb{R}^n$

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1 Notation

For $f \in L^1(\mathbb{R}^n)$, we define $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = (\mathscr{F} f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx, \qquad \xi \in \mathbb{R}^n.$$

The statement of the Riemann-Lebesgue lemma is that $\hat{f} \in C_0(\mathbb{R}^n)$.

We denote by $\mathscr{S}_n$ the Fréchet space of Schwartz functions $\mathbb{R}^n \to \mathbb{C}$.

If α is a multi-index, we define

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n},$$

$$D_\alpha = i^{-|\alpha|} D^\alpha = \left(\frac{1}{i} D_1\right)^{\alpha_1} \cdots \left(\frac{1}{i} D_n\right)^{\alpha_n},$$

and

$$\Delta = D_1^2 + \cdots + D_n^2.$$

2 The heat equation

Fix n, and for $t > 0$, $x \in \mathbb{R}^n$, define

$$k_t(x) = k(t, x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

We call k the **heat kernel**. It is straightforward to check for any $t > 0$ that $k_t \in \mathscr{S}_n$. The heat kernel satisfies

$$k_t(x) = (t^{-1/2})^n k_1(t^{-1/2} x), \qquad t > 0, x \in \mathbb{R}^n.$$

For $a > 0$ and $f(x) = e^{-\pi a |x|^2}$, it is a fact that $\hat{f}(\xi) = a^{-n/2} e^{-\pi |\xi|^2 / a}$. Using this, for any $t > 0$ we get

$$\hat{k}_t(\xi) = e^{-4\pi^2 |\xi|^2 t}, \qquad \xi \in \mathbb{R}^n.$$

Thus for any $t > 0$,

$$\int_{\mathbb{R}^n} k_t(x)dx = \hat{k}_t(0) = 1.$$

Then the heat kernel is an **approximate identity**: if $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then $\|f * k_t - f\|_p \to 0$ as $t \to 0$, and if f is a function on $\mathbb{R}^n$ that is bounded and continuous, then for every $x \in \mathbb{R}^n$, $f * k_t(x) \to f(x)$ as $t \to 0$.[1] For each $t > 0$, because $k_t \in \mathscr{S}_n$ we have $f * k_t \in C^\infty(\mathbb{R}^n)$, and $D^\alpha(f * k_t) = f * D^\alpha k_t$ for any multi-index α.[2]

The **heat operator** is $D_t - \Delta$ and the **heat equation** is $(D_t - \Delta)u = 0$. It is straightforward to check that

$$(D_t - \Delta)k(t,x) = 0, \qquad t > 0, x \in \mathbb{R}^n,$$

that is, the heat kernel is a solution of the heat equation.

To get some practice proving things about solutions of the heat equation, we work out the following theorem from Folland.[3] In Folland's proof it is not apparent how the hypotheses on u and D_x are used, and we make this explicit.

Theorem 1. *Suppose that $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ is continuous, that u is C^2 on $(0,\infty) \times \mathbb{R}^n$, that*

$$(D_t - \Delta)u(t,x) = 0, \qquad t > 0, x \in \mathbb{R}^n,$$

and that $u(0,x) = 0$ for $x \in \mathbb{R}^n$. If for every $\epsilon > 0$ there is some C such that

$$|u(t,x)| \le Ce^{\epsilon|x|^2}, \qquad |D_x u(t,x)| \le Ce^{\epsilon|x|^2}, \qquad t > 0, x \in \mathbb{R}^n,$$

then $u = 0$.

Proof. If f and g are C^2 functions on some open set in $\mathbb{R} \times \mathbb{R}^n$, such as $(0,\infty) \times \mathbb{R}^n$, then

$$\begin{aligned} g(\partial_t f - \Delta f) + f(\partial_t g + \Delta g) &= \partial_t(fg) - g\sum_{j=1}^n \partial_j^2 f + f\sum_{j=1}^n \partial_j^2 g \\ &= \partial_t(fg) + \sum_{j=1}^n \partial_j(f\partial_j g - g\partial_j f) \\ &= \mathrm{div}_{t,x} F, \end{aligned}$$

where

$$F = (fg, f\partial_1 g - g\partial_1 f, \dots, f\partial_n g - g\partial_n f).$$

[1] k_1, and any k_t, belong merely to $\mathscr{S}_n$ and not to $\mathscr{D}(\mathbb{R}^n)$, which is demanded in the definition of an approximate identity in Rudin's *Functional Analysis*, second ed.

[2] Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 11, Theorem 0.14.

[3] Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 144, Theorem 4.4.

Take $t_0 > 0$, $x_0 \in \mathbb{R}^n$, and let $f(t,x) = u(t,x)$ and $g(t,x) = k(t_0 - t, x - x_0)$ for $t > 0$, $x \in \mathbb{R}^n$. Let $0 < a < b < t_0$ and $r > 0$, and define

$$\Omega = \{(t,x) : |x| < r, a < t < b\}.$$

In Ω we check that $(\partial_t - \Delta)f = 0$ and $(\partial_t + \Delta)g = 0$, so by the divergence theorem,

$$\int_{\partial\Omega} F \cdot \nu = \int_\Omega \mathrm{div}_{t,x} F = \int_\Omega g(\partial_t f - \Delta f) + f(\partial_t g + \Delta g) = \int_\Omega g \cdot 0 + f \cdot 0 = 0.$$

On the other hand, as

$$\partial\Omega = \{(b,x) : |x| \leq r\} \cup \{(a,x) : |x| \leq r\} \cup \{(t,x) : a < t < b, |x| = r\},$$

we have

$$\begin{aligned}
\int_{\partial\Omega} F \cdot \nu &= \int_{|x|\leq r} F(b,x) \cdot (1,0,\dots,0) dx + \int_{|x|\leq r} F(a,x) \cdot (-1,0,\dots,0) dx \\
&\quad + \int_a^b \int_{|x|=r} F(t,x) \cdot \frac{x}{r} d\sigma(x) t^{n-1} dt \\
&= \int_{|x|\leq r} f(b,x) g(b,x) dx - \int_{|x|\leq r} f(a,x) g(a,x) dx \\
&\quad + \int_a^b \int_{|x|=r} \sum_{j=1}^n (f \partial_j g - g \partial_j f)(t,x) \frac{x_j}{r} d\sigma(x) t^{n-1} dt \\
&= \int_{|x|\leq r} u(b,x) k(t_0 - b, x - x_0) dx - \int_{|x|\leq r} u(a,x) k(t_0 - a, x - x_0) dx \\
&\quad + \int_a^b \int_{|x|=r} \sum_{j=1}^n \Big(u(t,x) \partial_j k(t_0 - t, x - x_0) \\
&\quad - k(t_0 - t, x - x_0) \partial_j u(t,x) \Big) \frac{x_j}{r} d\sigma(x) t^{n-1} dt,
\end{aligned}$$

where σ is surface measure on $\{|x| = r\} = rS^{n-1}$. As $r \to \infty$, the first two terms tend to

$$\int_{\mathbb{R}^n} u(b,x) k_{t_0-b}(x - x_0) dx = \int_{\mathbb{R}^n} u(b,x) k_{t_0-b}(x_0 - x) dx = u(b,\cdot) * k_{t_0-b}(x_0)$$

and

$$\int_{\mathbb{R}^n} u(a,x) k_{t_0-a}(x - x_0) dx = \int_{\mathbb{R}^n} u(a,x) k_{t_0-a}(x_0 - x) dx = u(a,\cdot) * k_{t_0-a}(x_0)$$

respectively. Let $\epsilon < \frac{1}{4(t_0-a)}$, and let C be as given in the statement of the theorem. Using $\partial_j k(t,x) = -\frac{x_j}{2t} k(t,x)$, for any $r > 0$ the third term is bounded by

$$n \int_a^b \int_{|x|=r} \Big(C e^{\epsilon r^2} \frac{|x - x_0|}{2t} k(t_0 - t, x - x_0) + k(t_0 - t, x - x_0) C e^{\epsilon r^2} \Big) d\sigma(x) t^{n-1} dt,$$

which is bounded by

$$n\int_a^b\int_{|x|=r} Ce^{\epsilon r^2}\left(\frac{|x_0|+r}{2a}+1\right)(4\pi(t_0-b))^{-n/2}\exp\left(-\frac{r^2}{4(t_0-a)}\right)d\sigma(x)t^{n-1}dt,$$

and writing $\eta = \frac{1}{4(t_0-a)} - \epsilon$ and $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, the surface area of the sphere of radius 1 in $\mathbb{R}^n$, this is equal to

$$(b-a)^n r^{n-1}\omega_n Ce^{-\eta r^2}\left(\frac{|x_0|+r}{2a}+1\right)(4\pi(t_0-b))^{-n/2},$$

which tends to 0 as $r\to\infty$. Therefore,

$$u(b,\cdot)*k_{t_0-b}(x_0) = u(a,\cdot)*k_{t_0-a}(x_0).$$

One checks that as $b\to t_0$, the left-hand side tends to $u(t_0,x_0)$, and that as $a\to 0$, the right-hand side tends to $u(0,x_0)=0$. Therefore,

$$u(t_0,x_0)=0.$$

This is true for any $t_0>0$, $x_0\in\mathbb{R}^n$, and as $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{C}$ is continuous, it follows that u is identically 0. □

3 Fundamental solutions

We extend k to $\mathbb{R}\times\mathbb{R}^n$ as

$$k(t,x)=\begin{cases}(4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right) & t>0, x\in\mathbb{R}^n\\ 0 & t\le 0, x\in\mathbb{R}^n.\end{cases}$$

This function is locally integrable in $\mathbb{R}\times\mathbb{R}^n$, so it makes sense to define $\Lambda_k\in\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ by

$$\Lambda_k\phi = \int_{\mathbb{R}}\int_{\mathbb{R}^n}\phi(t,x)k(t,x)dxdt, \qquad \phi\in\mathscr{D}(\mathbb{R}\times\mathbb{R}^n).$$

Suppose that P is a polynomial in n variables:

$$P(\xi)=\sum c_\alpha\xi^\alpha = \sum c_\alpha \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}.$$

We say that $E\in\mathscr{D}'(\mathbb{R}^n)$ is a **fundamental solution** of the differential operator

$$P(D)=\sum c_\alpha D_\alpha = \sum c_\alpha i^{-|\alpha|}D^\alpha$$

if $P(D)E=\delta$. If $E=\Lambda_f$ for some locally integrable f, $\Lambda_f\phi=\int_{\mathbb{R}^n}\phi(x)f(x)dx$, we also say that the function f is a fundamental solution of the differential operator $P(D)$. We now prove that the heat kernel extended to $\mathbb{R}\times\mathbb{R}^n$ in the above way is a fundamental solution of the heat operator.[4]

[4]Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 146, Theorem 4.6.

Theorem 2. *Λ_k is a fundamental solution of $D_t - \Delta$.*

Proof. For $\epsilon > 0$, define $K_\epsilon(t,x) = k(t,x)$ if $t > \epsilon$ and $K_\epsilon(t,x) = 0$ otherwise. For any $\phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)$,

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{\mathbb{R}^n}(k(t,x) - K_\epsilon(t,x))\phi(t,x)dxdt\right| &= \left|\int_0^\epsilon \int_{\mathbb{R}^n} k(t,x)\phi(t,x)dxdt\right| \\ &\leq \|\phi\|_\infty \int_0^\epsilon \int_{\mathbb{R}^n} k(t,x)dxdt \\ &= \|\phi\|_\infty \int_0^\epsilon dt \\ &= \|\phi\|_\infty \epsilon.\end{aligned}$$

This shows that $\Lambda_{K_\epsilon} \to \Lambda_k$ in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$, with the weak-* topology. It is a fact that for any multi-index, $E \mapsto D^\alpha E$ is continuous $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n) \to \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$, and hence $(D_t - \Delta)\Lambda_{K_\epsilon} \to (D_t - \Delta)\Lambda_k$ in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$. Therefore, to prove the theorem it suffices to prove that $(D_t - \Delta)\Lambda_{K_\epsilon} \to \delta$ (because $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$ with the weak-* topology is Hausdorff).

Let $\phi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)$. Doing integration by parts,

$$\begin{aligned}(D_t - \Delta)\Lambda_{K_\epsilon}(\phi) &= \Lambda_{K_\epsilon}((D_t - \Delta)\phi) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} K_\epsilon(t,x)(D_t\phi(t,x) - \Delta\phi(t,x))dxtx \\ &= \int_\epsilon^\infty \int_{\mathbb{R}^n} k(t,x)D_t\phi(t,x) - k(t,x)\Delta\phi(t,x)dxtx \\ &= \int_{\mathbb{R}^n}\left(k(\epsilon,x)\phi(\epsilon,x) - \int_\epsilon^\infty \phi(t,x)D_tk(t,x)dt\right)dx \\ &\quad + \int_\epsilon^\infty \int_{\mathbb{R}^n} \phi(t,x)\Delta k(t,x)dxdt \\ &= \int_{\mathbb{R}^n} k(\epsilon,x)\phi(\epsilon,x)dx \\ &\quad - \int_\epsilon^\infty \int_{\mathbb{R}^n} \phi(t,x)(D_t - \Delta)k(t,x)dtdx \\ &= \int_{\mathbb{R}^n} k(\epsilon,x)\phi(\epsilon,x)dx.\end{aligned}$$

So, using $k_t(x) = k_t(-x)$ and writing $\phi_t(x) = \phi(t,x)$,

$$\begin{aligned}(D_t - \Delta)\Lambda_{K_\epsilon}(\phi) &= \int_{\mathbb{R}^n} k_\epsilon(-x)\phi_\epsilon(x)dx \\ &= k_\epsilon * \phi_\epsilon(0) \\ &= k_\epsilon * \phi_0(0) + k_\epsilon * (\phi_\epsilon - \phi_0)(0).\end{aligned}$$

Using the definition of convolution, the second term is bounded by

$$\sup_{x\in\mathbb{R}^n} |\phi_\epsilon(x) - \phi_0(x)|\|k_\epsilon\|_1 = \sup_{x\in\mathbb{R}^n} |\phi_\epsilon(x) - \phi_0(x)|,$$

which tends to 0 as $\epsilon \to 0$. Because k is an approximate identity, $k_\epsilon * \phi_0(0) \to \phi_0(0)$ as $\epsilon \to 0$. That is,

$$(D_t - \Delta)\Lambda_{K_\epsilon}(\phi) \to \phi_0(0) = \delta(\phi)$$

as $\epsilon \to 0$, showing that $(D_t - \Delta)\Lambda_{K_\epsilon} \to \delta$ in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$ and completing the proof. □

4 Functions of the Laplacian

This section is my working through of material in Folland.[5] For $f \in \mathscr{S}_n$ and for any nonnegative integer k, doing integration by parts we get

$$\mathscr{F}((-\Delta)^k f)(\xi) = \int_{\mathbb{R}^n} ((-\Delta)^k f)(x) e^{-2\pi i \xi x} dx = (4\pi^2|\xi|^2)^k (\mathscr{F} f)(\xi), \qquad \xi \in \mathbb{R}^n.$$

Suppose that P is a polynomial in one variable: $P(x) = \sum c_k x^k$. Then, writing $P(-\Delta) = \sum c_k (-\Delta)^k$, we have

$$\begin{aligned} \mathscr{F}(P(-\Delta)f)(\xi) &= \sum c_k \mathscr{F}((-\Delta)^k f)(\xi) \\ &= \sum c_k (4\pi^2|\xi|^2)^k (\mathscr{F} f)(\xi) \\ &= (\mathscr{F} f)(\xi) P(4\pi^2|\xi|^2). \end{aligned}$$

We remind ourselves that **tempered distributions** are elements of $\mathscr{S}_n'$, i.e. continuous linear maps $\mathscr{S}_n \to \mathbb{C}$. The Fourier transform of a tempered distribution Λ is defined by $\widehat{\Lambda} f = (\mathscr{F}\Lambda) f = \Lambda \hat{f}$, $f \in \mathscr{S}_n$. It is a fact that the Fourier transform is an isomorphism of locally convex spaces $\mathscr{S}_n' \to \mathscr{S}_n'$.[6]

Suppose that $\psi : (0, \infty) \to \mathbb{C}$ is a function such that

$$\Lambda f = \int_{\mathbb{R}^n} f(\xi) \psi(4\pi^2|\xi|^2) d\xi, \qquad f \in \mathscr{S}_n,$$

is a tempered distribution. We define $\psi(-\Delta) : \mathscr{S}_n \to \mathscr{S}_n'$ by

$$\psi(-\Delta) f = \mathscr{F}^{-1}(\hat{f}\Lambda), \qquad f \in \mathscr{S}_n.$$

Define $\check{f}(x) = f(-x)$; this is **not** the inverse Fourier transform of f, which we denote by $\mathscr{F}^{-1}$. As well, write $\tau_x f(y) = f(y - x)$. For $u \in \mathscr{S}_n'$ and $\phi \in \mathscr{S}_n$, we **define** the convolution $u * \phi : \mathbb{R}^n \to \mathbb{C}$ by

$$(u * \phi)(x) = u(\tau_x \check{\phi}), \qquad x \in \mathbb{R}^n.$$

One proves that $u * \phi \in C^\infty(\mathbb{R}^n)$, that

$$D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$$

[5] Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., pp. 149–152, §4B.

[6] Walter Rudin, *Functional Analysis*, second ed., p. 192, Theorem 7.15.

for any multi-index, that $u * \phi$ is a tempered distribution, that $\mathscr{F}(u * \phi) = \hat{\phi}\hat{u}$, and that $\hat{u} * \hat{\phi} = \mathscr{F}(\phi u)$.[7]

We can also write $\psi(-\Delta)$ in the following way. There is a unique $\kappa_\psi \in \mathscr{S}_n'$ such that

$$\mathscr{F}\kappa_\psi = \Lambda.$$

For $f \in \mathscr{S}_n$, we have $\mathscr{F}(\kappa_\psi * f) = \hat{f}\hat{\kappa}_\psi = \hat{f}\Lambda$, but, using the definition of $\psi(-\Delta)$ we also have $\mathscr{F}(\psi(-\Delta)f) = \mathscr{F}\mathscr{F}^{-1}(\hat{f}\Lambda) = \hat{f}\Lambda$, so

$$\kappa_\psi * f = \psi(-\Delta)f.$$

Moreover, $\kappa_\psi * f \in C^\infty(\mathbb{R}^n)$; this shows that $\psi(-\Delta)f$ can be interpreted as a tempered distribution or as a function. We call κ_ψ the **convolution kernel** of $\psi(-\Delta)$.

For a fixed $t > 0$, define $\psi(s) = e^{-ts}$. Then $\Lambda : \mathscr{S}_n \to \mathbb{C}$ defined by

$$\Lambda f = \int_{\mathbb{R}^n} f(\xi)\psi(4\pi^2|\xi|^2)d\xi = \int_{\mathbb{R}^n} f(\xi)\exp\left(-4\pi^2|\xi|^2 t\right)d\xi = \int_{\mathbb{R}^n} f(\xi)\hat{k}_t(\xi)d\xi$$

is a tempered distribution. Using the Plancherel theorem, we have

$$\Lambda f = \int_{\mathbb{R}^n} \hat{f}(\xi)k_t(\xi)d\xi.$$

With $\kappa_\psi \in \mathscr{S}_n'$ such that $\mathscr{F}\kappa_\psi = \Lambda$, we have

$$\Lambda f = (\mathscr{F}\kappa_\psi)(f) = \kappa_\psi(\hat{f}).$$

Because $f \mapsto \hat{f}$ is a bijection $\mathscr{S}_n \to \mathscr{S}_n$, this shows that for any $f \in \mathscr{S}_n$ we have

$$\kappa_\psi(f) = \int_{\mathbb{R}^n} f(\xi)k_t(\xi)d\xi.$$

Hence,

$$e^{t\Delta}f = \kappa_\psi * f = k_t * f, \qquad t > 0, f \in \mathscr{S}_n. \tag{1}$$

Suppose that $\phi : (0,\infty) \to \mathbb{C}$ and $\omega : (0,\infty) \to (0,\infty)$ are functions and that

$$\psi(s) = \int_0^\infty \phi(\tau)e^{-s\omega(\tau)}d\tau, \qquad s > 0.$$

Manipulating symbols suggests that it may be true that

$$\psi(-\Delta) = \int_0^\infty \phi(\tau)e^{\omega(\tau)\Delta}d\tau,$$

and then, for $f \in \mathscr{S}_n$,

$$\psi(-\Delta)f = \int_0^\infty \phi(\tau)e^{\omega(\tau)\Delta}f d\tau = \int_0^\infty \phi(\tau)(k_{\omega(\tau)} * f)d\tau,$$

[7] Walter Rudin, *Functional Analysis*, second ed., p. 195, Theorem 7.19.

and hence

$$\kappa_\psi(x) = \int_0^\infty \phi(\tau) k_{\omega(\tau)}(x) d\tau, \qquad x \in \mathbb{R}^n. \tag{2}$$

Take $\psi(s) = s^{-\beta}$ with $0 < \operatorname{Re}\beta < \frac{n}{2}$. Because $\operatorname{Re}\beta < \frac{n}{2}$, one checks that

$$\Lambda f = \int_{\mathbb{R}^n} f(\xi)(4\pi^2|\xi|^2)^{-\beta} d\xi$$

is a tempered distribution. As $\operatorname{Re}\beta > 0$, we have

$$s^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-s\tau} d\tau$$

and writing $\phi(\tau) = \frac{\tau^{\beta-1}}{\Gamma(\beta)}$ and $\omega(\tau) = \tau$, we suspect from (2) that the convolution kernel of $(-\Delta)^{-\beta}$ is

$$\kappa_\psi(x) = \int_0^\infty \frac{\tau^{\beta-1}}{\Gamma(\beta)} k_\tau(x) d\tau,$$

which one calculates is equal to

$$\frac{\Gamma\left(\frac{n}{2} - \beta\right)}{\Gamma(\beta) 4^\beta \pi^{n/2} |x|^{n-2\beta}}. \tag{3}$$

What we have written so far does not prove that this is the convolution kernel of $(-\Delta)^{-\beta}$ because it used (2), but it is straightforward to calculate that indeed the convolution kernel of $(-\Delta)^{-\beta}$ is (3). This calculation is explained in an exercise in Folland.[8]

Taking $\alpha = 2\beta$ and defining

$$R_\alpha(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{n/2} |x|^{n-\alpha}}, \qquad 0 < \operatorname{Re}\alpha < n, x \in \mathbb{R}^n,$$

we call R_α the **Riesz potential** of order α. Taking as granted that (3) is the convolution kernel of $(-\Delta)^{-\beta}$, we have

$$(-\Delta)^{-\alpha/2} f = R_\alpha * f, \qquad f \in \mathscr{S}_n.$$

Then, if $n > 2$ and $\alpha = 2$ satisfies $0 < \operatorname{Re}\alpha < n$, we work out that

$$R_2(x) = \frac{1}{(n-2)\omega_n |x|^{n-2}},$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, and hence

$$(-\Delta)^{-1} f = R_2 * f, \qquad f \in \mathscr{S}_n,$$

[8]Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 154, Exercise 1.

and applying $-\Delta$ we obtain

$$f = -\Delta(R_2 * f) = (-\Delta R_2) * f,$$

hence $-\Delta R_2 = \delta$. That is, R_2 is the fundamental solution for $-\Delta$.

Suppose that $\operatorname{Re}\beta > 0$. Then, using the definition of $\Gamma(\beta)$ as an integral, with $\psi(s) = (1+s)^{-\beta}$, we have

$$\psi(s) = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-(1+s)\tau} d\tau, \qquad s > 0.$$

Manipulating symbols suggests that

$$\psi(-\Delta) = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} e^{\tau\Delta} d\tau,$$

and using (1), assuming the above is true we would have for all $f \in \mathscr{S}_n$,

$$\psi(-\Delta)f = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} e^{\tau\Delta} f d\tau = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} (k_\tau * f) d\tau,$$

whose convolution kernel is

$$\frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} e^{-\tau} k_\tau d\tau.$$

We write $\alpha = 2\beta$ and define, for $\operatorname{Re}\alpha > 0$,

$$B_\alpha(x) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right) (4\pi)^{n/2}} \int_0^\infty \tau^{\frac{\alpha-n}{2}-1} e^{-\tau - \frac{|x|^2}{4\tau}} d\tau, \qquad x \neq 0.$$

We call B_α the **Bessel potential** of order α. It is straightforward to show, and shown in Folland, that $\|B_\alpha\|_1 < \infty$, so $B_\alpha \in L^1(\mathbb{R}^n)$. Therefore we can take the Fourier transform of B_α, and one calculates that it is

$$\widehat{B}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}, \qquad \xi \in \mathbb{R}^n,$$

and then

$$\psi(-\Delta) = (1-\Delta)^{-\alpha/2} f = B_\alpha * f, \qquad f \in \mathscr{S}_n.$$

5 Gaussian measure

If μ is a measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{C}$ is a function such that for every $x \in \mathbb{R}^n$ the integral $\int_{\mathbb{R}^n} f(x-y) d\mu(y)$ converges, we define the **convolution** $\mu * f : \mathbb{R}^n \to \mathbb{C}$ by

$$(\mu * f)(x) = \mu(\tau_x \check{f}) = \int_{\mathbb{R}^n} (\tau_x \check{f})(y) d\mu(y) = \int_{\mathbb{R}^n} \check{f}(y-x) d\mu(y) = \int_{\mathbb{R}^n} f(x-y) d\mu(y).$$

Let ν_t be the measure on $\mathbb{R}^n$ with density k_t. We call ν_t **Gaussian measure**. It satisfies

$$\nu_t * f(x) = \int_{\mathbb{R}^n} f(x-y) d\nu_t(y) = \int_{\mathbb{R}^n} f(x-y) k_t(y) dy = f * k_t(x), \qquad x \in \mathbb{R}^n.$$