The Gelfand transform, positive linear functionals, and positive-definite functions

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1 Introduction

In this note, unless we say otherwise every vector space or algebra we speak about is over $\mathbb{C}$.

If A is a Banach algebra and $e \in A$ satisfies $xe = x$ and $ex = x$ for all $x \in A$, and also $\|e\| = 1$, we say that e is *unity* and that A is *unital*.

If A is a unital Banach algebra and $x \in A$, the *spectrum of* x is the set $\sigma(x)$ of those $\lambda \in \mathbb{C}$ for which $\lambda e - x$ is not invertible. It is a fact that if A is a unital Banach algebra and $x \in A$, then $\sigma(x) \neq \emptyset$.[1]

If A and B are Banach algebras and $T : A \to B$ is a map, we say that T is an *isomorphism of Banach algebras* if T is an algebra isomorphism and an isometry.

Theorem 1 (Gelfand-Mazur)**.** If A is a Banach algebra and every nonzero element of A is invertible, then there is an isomorphism of Banach algebras $A \to \mathbb{C}$.

Proof. Let $x \in A$. $\sigma(x) \neq \emptyset$. If $\lambda_1, \lambda_2 \in \sigma(x)$, then neither $\lambda_1 e - x$ nor $\lambda_2 e - x$ is invertible, so they are both 0: $x = \lambda_1 e$ and $x = \lambda_2 e$, whence $\lambda_1 = \lambda_2$. Therefore $\sigma(x)$ has precisely one element, which we denote by $\lambda(x)$, and which satisfies

$$x = \lambda(x)e.$$

If $x, y \in A$, then $x + y = \lambda(x)e + \lambda(y)e = (\lambda(x) + \lambda(y))e$ and also $x + y = \lambda(x + y)e$, so $\lambda(x + y) = \lambda(x) + \lambda(y)$. If $x \in A$ and $\alpha \in \mathbb{C}$, then $\alpha x = \alpha\lambda(x)e$ and also $\alpha x = \lambda(\alpha x)e$, so $\lambda(\alpha x) = \alpha\lambda(x)$. Hence $x \mapsto \lambda(x)$ is linear. If $\lambda_0 \in \mathbb{C}$, then $\lambda(\lambda_0 e) = \lambda_0$, showing that $x \mapsto \lambda(x)$ is onto. If $\lambda(x) = \lambda(y)$ then $x = \lambda(x)e = \lambda(y)e = y$, showing that $x \mapsto \lambda(x)$ is one-to-one. Therefore $x \mapsto \lambda(x)$ is a linear isomorphism $A \to \mathbb{C}$.

If $x \in A$, then $x = \lambda(x)e$ gives

$$\|x\| = \|\lambda(x)e\| = |\lambda(x)| \, \|e\| = |\lambda(x)|,$$

showing that the map $x \mapsto \lambda(x)$ is an isometry $A \to \mathbb{C}$. □

[1] Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.

2 Complex homomorphisms

An ideal J of an algebra A is said to be *proper* if $J \neq A$. An ideal is called *maximal* if it is a maximal element in the collection of proper ideals of A ordered by set inclusion.

The following theorem, which is proved using the fact that a maximal ideal is closed, the fact that a quotient of a Banach algebra with a closed ideal is a Banach algebra, and the Gelfand-Mazur theorem, states some basic facts about algebra homomorphisms from a Banach algebra to $\mathbb{C}$.[2]

Theorem 2. If A is a commutative unital Banach algebra and Δ is the set of all nonzero algebra homomorphisms $A \to \mathbb{C}$, then:

1. If M is a maximal ideal of A then there is some $h \in \Delta$ for which $M = \ker h$.
2. If $h \in \Delta$ then $\ker h$ is a maximal ideal of A.
3. $x \in A$ is invertible if and only if $h(x) \neq 0$ for all $h \in \Delta$.
4. $x \in A$ is invertible if and only if x does not belong to any proper ideal of A.
5. $\lambda \in \sigma(x)$ if and only if there is some $h \in \Delta$ for which $h(x) = \lambda$.

3 The Gelfand transform and maximal ideals

Suppose that A is a commutative unital Banach algebra and that Δ is the set of all nonzero algebra homomorphisms $A \to \mathbb{C}$. For each $x \in A$, we define $\hat{x} : \Delta \to \mathbb{C}$ by

$$\hat{x}(h) = h(x), \qquad h \in \Delta.$$

We call $\hat{x}$ the *Gelfand transform of* x, and we call the map $\Gamma : A \to \mathbb{C}^\Delta$ defined by $\Gamma(x) = \hat{x}$ the *Gelfand transform*.

We define $\widehat{A} = \{\hat{x} : x \in A\}$, and we call the set Δ with the initial topology for $\widehat{A}$ the *maximal ideal space of* A. That is, the topology of Δ is the coarsest topology on Δ such that each $\hat{x} : \Delta \to \mathbb{C}$ is continuous. If X is a topological space, we denote by $C(X)$ the set of all continuous functions $X \to \mathbb{C}$. $C(X)$ is a commutative unital algebra, although it need not be a Banach algebra.

The *radical of* A, denoted $\operatorname{rad} A$, is the intersection of all maximal ideals of A. If $\operatorname{rad} A = \{0\}$, we say that A is *semisimple*.

The following theorem establishes some basic facts about the Gelfand transform and the maximal ideal space.[3]

Theorem 3. If A is a commutative unital Banach algebra and Δ is the maximal ideal space of A, then:

[2] Walter Rudin, *Functional Analysis*, second ed., p. 277, Theorem 11.5.
[3] Walter Rudin, *Functional Analysis*, second ed., p. 280, Theorem 11.9.

1. $\Gamma : A \to C(\Delta)$ is an algebra homomorphism with $\ker \Gamma = \operatorname{rad} A$.
2. If $x \in A$, then $\operatorname{im} \hat{x} = \sigma(x)$.
3. Δ is a compact Hausdorff space.

Proof. Let $x, y \in A$ and $\alpha \in \mathbb{C}$. For $h \in \Delta$,

$$\Gamma(\alpha x+y)(h) = h(\alpha x+y) = \alpha h(x)+h(y) = \alpha\Gamma(x)(h)+\Gamma(y)(h) = (\Gamma(x)+\Gamma(y)(h),$$

showing that $\Gamma(\alpha x + y) = \alpha\Gamma(x) + \Gamma(y)$, and

$$\Gamma(xy)(h) = h(xy) = h(x)h(y) = \Gamma(x)(h)\Gamma(y)(h) = (\Gamma(x)\Gamma(y))(h),$$

showing that $\Gamma(xy) = \Gamma(x)\Gamma(y)$. Therefore $\Gamma : A \to C(\Delta)$ is an algebra homomorphism. $x \in \ker \Gamma$ is equivalent to $h(x) = 0$ for all $h \in \Delta$, which is equivalent to $x \in \ker h$ for all $h \in \Delta$. But by Theorem 2, $\{\ker h : h \in \Delta\}$ is equal to the set of all maximal ideals of A, so $x \in \ker \Gamma$ is equivalent to $x \in \operatorname{rad} A$, i.e. $\ker \Gamma = \operatorname{rad} A$.

Let $x \in A$. If $\lambda \in \operatorname{im} \hat{x}$ then there is some $h \in \Delta$ for which $\hat{x}(h) = \lambda$, and by Theorem 2, this yields $\lambda \in \sigma(x)$. Hence $\operatorname{im} \hat{x} \subseteq \sigma(x)$. If $\lambda \in \sigma(x)$, then by Theorem 2 there is some $h \in \Delta$ for which $h(x) = \lambda$, i.e. there is some $h \in \Delta$ for which $\hat{x}(h) = \lambda$, i.e. $\lambda \in \operatorname{im} \hat{x}$. Hence $\sigma(x) \subseteq \operatorname{im} \hat{x}$. Therefore, $\operatorname{im} \hat{x} = \sigma(x)$.

It is straightforward to check that the topology of Δ is the subspace topology inherited from A^* with the weak-* topology; in particular, the topology of Δ is Hausdorff. Therefore, to prove that Δ is compact it suffices to prove that Δ is a weak-* compact subset of A^*. Let

$$K = \{\lambda \in A^* : \|\lambda\| \leq 1\}.$$

By the Banach-Alaoglu theorem, K is a weak-* compact subset of A^*. If $h \in \Delta$, then because h is an algebra homomorphism $A \to \mathbb{C}$ it follows that $\|h\| \leq 1$.[4] Thus, $\Delta \subset K$. Therefore, to prove that Δ is compact it suffices to prove that Δ is a weak-* closed subset of A^*.

Suppose that $h_i \in \Delta$ is a net that weak-* converges to $\lambda \in A^*$. Then $h_i(e) \to \lambda(e)$, i.e. $1 \to \lambda(e)$, so $\lambda(e) = 1$. Thus $\lambda \neq 0$. Let $x, y \in A$. On the one hand, $h_i(xy) \to \lambda(xy)$, and on the other hand, $h_i(x) \to \lambda(x)$ and $h_i(y) \to \lambda(y)$, so $h_i(x)h_i(y) \to \lambda(x)\lambda(y)$ and hence $h_i(xy) = h_i(x)h_i(y) \to \lambda(x)\lambda(y)$. Therefore, $\lambda(xy) = \lambda(x)\lambda(y)$, and because $\lambda \in A^*$ is linear, this shows that $\lambda : A \to \mathbb{C}$ is an algebra homomorphism, and hence that $\lambda \in \Delta$. Therefore Δ is a weak-* closed subset of A^*. $\square$

If A is a commutative unital Banach algebra, the above theorem shows that $\Gamma : A \to \widehat{A}$ is an algebra isomorphism if and only if $\operatorname{rad} A = \{0\}$, i.e., Γ is an algebra isomorphism if and only if A is semisimple.

[4] Walter Rudin, *Functional Analysis*, second ed., p. 249, Theorem 10.7.

The above theorem tells us that if A is a commutative unital Banach algebra and $x \in A$, then $\operatorname{im} \hat{x} = \sigma(x)$. This gives us

$$\|\hat{x}\|_\infty = \rho(x), \tag{1}$$

where $\rho(x)$ is the *spectral radius of* x, defined by

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Therefore, $\hat{x} = 0$ is equivalent to $\rho(x) = 0$, and so by the above theorem, $x \in \operatorname{rad} A$ is equivalent to $\rho(x) = 0$. Moreover, it is a fact that $\rho(x) \leq \|x\|$.[5] Therefore,

$$\|\hat{x}\|_\infty \leq \|x\|. \tag{2}$$

In the proof of Theorem 3 we used the fact[6] that the norm of any algebra homomorphism from a Banach algebra to $\mathbb{C}$ is ≤ 1. In particular, this means that any algebra homomorphism from a Banach algebra to $\mathbb{C}$ is continuous. The following theorem shows that any algebra homomorphism from a Banach algebra to a commutative unital semisimple Banach algebra is continuous.[7]

Theorem 4. Suppose that A is a Banach algebra and that B is a commutative unital semisimple Banach algebra. If $\psi : A \to B$ is an algebra homomorphism, then ψ is continuous.

Proof. Because $\psi : A \to B$ is linear, to prove that ψ is continuous, by the closed graph theorem[8] it suffices to prove that

$$G = \{(x, \psi(x)) : x \in A\}$$

is closed in $A \times B$. To prove that G is closed in $A \times B$, it suffices to prove that if $(x_n, y_n) \in G$ converges to $(x, y) \in A \times B$ then $(x, y) \in G$.

Let $h \in \Delta_B$. Then $\phi = h \circ \psi : A \to \mathbb{C}$ is an algebra homomorphism. Because $h : B \to \mathbb{C}$ and $\phi : A \to \mathbb{C}$ are algebra homomorphisms with codomain $\mathbb{C}$, they are both continuous. Therefore, $h(y_n) \to h(y)$ and $\phi(x_n) \to \phi(x)$. Therefore,

$$h(y) = \lim h(y_n) = \lim h(\psi(x_n)) = \lim (h \circ \psi)(x_n) = \lim \phi(x_n) = \phi(x) = h(\psi(x)),$$

so $h(y - \psi(x)) = 0$. This is true for all $h \in \Delta_B$, hence $y - \psi(x) \in \operatorname{rad} B$. But B is semisimple, so $y - \psi(x) = 0$, i.e. $y = \psi(x)$, so $(x, y) \in G$. $\square$

If A is a commutative unital Banach algebra and $x \in A$, we recorded in (2) that $\|\hat{x}\|_\infty \leq \|x\|$. The following lemma[9] shows that if $\|x^2\| = \|x\|^2$ and $x \neq 0$, then $\inf \frac{\|\hat{x}\|_\infty}{\|x\|} \geq 1$, hence that $\|\hat{x}\|_\infty = \|x\|$. Therefore, if $\|x^2\| = \|x\|^2$ for all $x \in A$, then $\Gamma : A \to C(\Delta)$ is an isometry.

[5]Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.
[6]Walter Rudin, *Functional Analysis*, second ed., p. 249, Theorem 10.7.
[7]Walter Rudin, *Functional Analysis*, second ed., p. 281, Theorem 11.10.
[8]Walter Rudin, *Functional Analysis*, second ed., p. 51, Theorem 2.15.
[9]Walter Rudin, *Functional Analysis*, second ed., p. 282, Lemma 11.11.

Lemma 5. Let A be a commutative unital Banach algebra. If

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2}, \qquad s = \inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|},$$

then $s^2 \leq r \leq s$.

Theorem 3 shows that if A is a commutative unital Banach algebra, then $\Gamma : A \to C(\Delta)$ is an algebra homomorphism. Therefore $\Gamma(A) = \widehat{A}$ is a subalgebra of $C(\Delta)$. Moreover, Theorem 3 also shows that Δ is a compact Hausdorff space. Therefore, $C(\Delta)$ is a unital Banach algebra with the supremum norm. (If X is a topological space then $C(X)$ is an algebra, but need not be a Banach algebra.) For $\widehat{A}$ to be a Banach subalgebra of $C(\Delta)$ it is necessary and sufficient that $\widehat{A}$ be a closed subset of the Banach algebra $C(\Delta)$. The following theorem gives conditions under which this occurs.[10]

Theorem 6. If A is a commutative unital Banach algebra, then A is semisimple and $\widehat{A}$ is a closed subset of $C(\Delta)$ if and only if there exists some $K < \infty$ such that $\|x\|^2 \leq K \|x^2\|$ for all $x \in A$.

Proof. Suppose that there is some $0 < K < \infty$ such that $x \in A$ implies that $\|x\|^2 \leq K \|x^2\|$. Then

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2} \geq \inf_{x \neq 0} \frac{\|x^2\|}{K \|x^2\|} = \frac{1}{K}.$$

By Lemma 5, with $s = \inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|}$ we have

$$\frac{1}{K} \leq s,$$

hence $\|\hat{x}\|_\infty \geq \frac{1}{K} \|x\|$. Thus, if $x \in A$ then $\|\hat{x}\|_\infty \geq \frac{1}{K} \|x\|$, from which it follows that $\Gamma : A \to C(\Delta)$ is one-to-one. Since Γ is one-to-one, by Theorem 3 we get that A is semisimple. Suppose that $\hat{x}_n \in \widehat{A}$ converges to $\hat{x} \in \widehat{A}$, i.e. $\|\hat{x}_n - \hat{x}\|_\infty \to 0$, i.e. $\|\Gamma(x_n - x)\|_\infty \to 0$. But $\|\Gamma(x_n - x)\|_\infty \geq \frac{1}{K} \|x_n - x\|$, so $\|x_n - x\| \to 0$, showing that $\Gamma^{-1} : \widehat{A} \to A$ is bounded. Therefore $\Gamma : A \to \widehat{A}$ is bilipschitz, and so $\widehat{A}$ is a complete metric space, from which it follows that $\widehat{A}$ is a closed subset of $C(\Delta)$.

Suppose that A is semisimple and that $\widehat{A}$ is a closed subset of $C(\Delta)$. The fact that A is semisimple gives us by Theorem 3 that $\Gamma : A \to \widehat{A}$ is a bijection. The fact that $\widehat{A}$ is closed means that $\widehat{A}$ is a Banach algebra. Because $\Gamma : A \to \widehat{A}$ is continuous, linear, and a bijection, by the open mapping theorem[11] it follows that there are positive real numbers a, b such that if $x \in A$ then

$$a \|x\| \leq \|\Gamma x\|_\infty \leq b \|x\|.$$

Then $\inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|} \geq a$. By Lemma 5, it follows that $\inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2} \geq a^2$. Hence, for all $x \neq 0$ we have $\|x\|^2 \leq K \|x^2\|$, with $K = \frac{1}{a^2}$. $\square$

[10] Walter Rudin, *Functional Analysis*, second ed., p. 282, Theorem 11.12.
[11] Walter Rudin, *Functional Analysis*, second ed., p. 49, Corollary 2.12.

4 $\mathbf{L^1}$

Let $M(\mathbb{R}^n)$ denote the set of all complex Borel measures on $\mathbb{R}^n$, and let $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by $S(x, y) = x + y$. For $\mu_1, \mu_2 \in M(\mathbb{R}^n)$, we denote by $\mu_1 \times \mu_2$ the product measure on $\mathbb{R}^n \times \mathbb{R}^n$, and we define the *convolution of* μ_1 *and* μ_2 to be $\mu_1 * \mu_2 = S_*(\mu_1 \times \mu_2)$, the pushforward of $\mu_1 \times \mu_2$ with respect to S. That is, if E is a Borel subset of $\mathbb{R}^n$, then

$$\begin{aligned}(\mu_1 * \mu_2)(E) &= (S_*(\mu_1 \times \mu_2))(E) \\ &= (\mu_1 \times \mu_2)(S^{-1}(E)) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_E(x + y) d\mu_1(x) d\mu_2(y).\end{aligned}$$

With convolution as multiplication, $M(\mathbb{R}^n)$ is an algebra.

If $\mu \in M(\mathbb{R}^n)$, the *variation* of μ is the measure $|\mu| \in M(\mathbb{R}^n)$, where for a Borel subset E of $\mathbb{R}^n$, we define $|\mu|(E)$ to be the supremum of $\sum_{A \in \pi} |\mu(A)|$ over all partitions π of E into finitely many disjoint Borel subsets. The *total variation of* μ is $\|\mu\| = |\mu|(\mathbb{R}^n)$. One proves that $\|\cdot\|$ is a norm on $M(\mathbb{R}^n)$ and that with this norm, $M(\mathbb{R}^n)$ is a Banach algebra.[12]

Let m_n be Lebesgue measure on $\mathbb{R}^n$, let δ be the Dirac measure on $\mathbb{R}^n$, and let A be the set of those $\mu \in M(\mathbb{R}^n)$ for which there is some $f \in L^1(\mathbb{R}^n)$ and some $\alpha \in \mathbb{C}$ with which

$$d\mu = f dm_n + \alpha d\delta.$$

One proves that A is a Banach subalgebra of $M(\mathbb{R}^n)$. A is a unital Banach algebra, with unity δ. In particular, A is a unital Banach algebra that contains the Banach algebra $L^1(\mathbb{R}^n)$.

If $f + \alpha\delta, g + \beta\delta \in A$ (identifying $f \in L^1(\mathbb{R}^n)$ with the complex Borel measure whose Radon-Nikodym derivative with respect to m_n is f), then

$$(f + \alpha\delta) * (g + \beta\delta) = (f * g + \beta f + \alpha g) + \alpha\beta\delta, \tag{3}$$

where

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) dm_n(y).$$

If $t \in \mathbb{R}^n$, let $e_t(x) = \exp(it \cdot x)$, and if $f \in L^1(\mathbb{R}^n)$, define $\hat{f} : \mathbb{R}^n \to \mathbb{C}$, the *Fourier transform of* f, by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f e_{-t} dm_n, \qquad t \in \mathbb{R}^n.$$

If $t \in \mathbb{R}^n$, define $h_t : A \to \mathbb{C}$ by

$$h_t(f + \alpha\delta) = \hat{f}(t) + \alpha, \qquad f + \alpha\delta \in A,$$

and define $h_\infty : A \to \mathbb{C}$ by

$$h_\infty(f + \alpha\delta) = \alpha, \qquad f + \alpha\delta \in A.$$

[12]See Walter Rudin, *Real and Complex Analysis*, third ed., chapter 6.

By (3) it is apparent that for each $t \in \mathbb{R}^n \cup \{\infty\}$, the map h_t is a homomorphism of algebras. It can be proved that $\Delta = \{h_t : t \in \mathbb{R}^n\} \cup \{h_\infty\}$.[13] Let $\mathbb{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^n$, and define $T : \mathbb{R}^n \cup \{\infty\} \to \Delta$ by $T(t) = h_t$, which is a bijection.

Suppose that $t_k \to t$ in $\mathbb{R}^n$. If $f + \alpha\delta \in A$, then because $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is continuous, we have

$$T(t_k)(f + \alpha\delta) = h_{t_k}(f + \alpha\delta) = \hat{f}(t_k) + \alpha \to \hat{f}(t) + \alpha = T(t)(f + \alpha\delta).$$

Suppose that $t_k \to \infty$. If $f + \alpha\delta \in A$, then by the Riemann-Lebesgue lemma we have $\hat{f}(t_k) \to 0$, and hence

$$T(t_k)(f + \alpha\delta) = \hat{f}(t_k) + \alpha \to \alpha = h_\infty(f + \alpha\delta) = T(\infty)(f + \alpha\delta).$$

Therefore, $T : \mathbb{R}^n \cup \{\infty\} \to \Delta$ is continuous.

Suppose that $h_{t_k} \to h_t$ in Δ, $t_k, t \in \mathbb{R}^n$. If $f + \alpha\delta \in A$, then $h_{t_k}(f + \alpha\delta) \to h_t(f + \alpha\delta)$. But $h_{t_k}(f + \alpha\delta) = \hat{f}(t_k) + \alpha$ and $h_t(f + \alpha\delta) = \hat{f}(t) + \alpha$, so $\hat{f}(t_k) \to \hat{f}(t)$. Because this is true for all $f \in L^1(\mathbb{R}^n)$, it follows that $t_k \to t$. Suppose that $h_{t_k} \to h_\infty$ in Δ, $t_k \in \mathbb{R}^n$. If $f + \alpha\delta \in A$, then $h_{t_k}(f + \alpha\delta) \to h_\infty(f + \alpha\delta)$, i.e. $\hat{f}(t_k) + \alpha \to \alpha$, i.e. $\hat{f}(t_k) \to 0$. Because this is true for all $f \in L^1(\mathbb{R}^n)$, it follows that $t_k \to \infty$. Therefore, $T^{-1} : \Delta \to \mathbb{R}^n \cup \{\infty\}$ is continuous, and so Δ is homeomorphic to the one-point compactification of $\mathbb{R}^n$.

5 Involutions

If A is an algebra, an *involution of* A is a map $^* : A \to A$ satisfying

1. $(x + y)^* = x^* + y^*$
2. $(\alpha x)^* = \overline{\alpha} x^*$
3. $(xy)^* = y^* x^*$
4. $x^{**} = x$.

We say that x is *self-adjoint* if $x^* = x$.

Following Rudin, if A is a Banach algebra with an involution $^* : A \to A$ satisfying

$$\|xx^*\| = \|x\|^2, \qquad x \in A,$$

we say that A is a *B^*-algebra*.

The following theorem shows that a commutative unital B^*-algebra with maximal ideal space Δ is isomorphic as a B^*-algebra to $C(\Delta)$.[14] (An *isomorphism of B^*-algebras* is an isomorphism of Banach algebras that preserves the involution; the involution on $C(\Delta)$ is $(x \mapsto f(x)) \mapsto (x \mapsto \overline{f(x)})$.)

[13] Walter Rudin, *Functional Analysis*, second ed., p. 285.

[14] Walter Rudin, *Functional Analysis*, second ed., p. 289, Theorem 11.18.

Theorem 7 (Gelfand-Naimark)**.** If A is a commutative unital B^*-algebra, then $\Gamma : A \to C(\Delta)$ is an isomorphism of Banach algebras, and if $x \in A$ then $\Gamma(x^*) = \overline{\Gamma(x)}$.

Proof. Let $u \in A$ be self-adjoint, let $h \in \Delta$, and let $h(u) = \alpha + i\beta$. For $t \in \mathbb{R}$, put $z = u + ite$. We have

$$h(z) = h(u) + h(ite) = \alpha + i\beta + it = \alpha + i(\beta + t),$$

and

$$zz^* = (u + ite)(u - ite) = u^2 + t^2 e,$$

hence

$$\alpha^2 + (\beta + t)^2 = |h(z)|^2 \leq \|z\|^2 = \|zz^*\| = \left\|u^2 + t^2 e\right\| \leq \|u\|^2 + t^2,$$

i.e.

$$\alpha^2 + \beta^2 + 2\beta t \leq \|u\|^2 .$$

Because this is true for all $t \in \mathbb{R}$, it follows that $\beta = 0$. Therefore, if $u \in A$ is self-adjoint then $h(u) \in \mathbb{R}$.

Furthermore, if $x \in A$ then with $2u = x + x^*$ and $2v = i(x^* - x)$ we have $x = u + iv$ with u and v self-adjoint. Then $x^* = u - iv$, and so

$$h(x^*) = h(u - iv) = h(u) - ih(v) = \overline{h(x)}.$$

This shows that if $x \in A$ then $\Gamma(x^*) = \overline{\Gamma(x)}$. In particular, $\widehat{A}$ is closed under complex conjugation. If $h_1 \neq h_2$, then there is some $x \in A$ for which $h_1(x) \neq h_2(x)$, i.e. $\hat{x}(h_1) \neq \hat{x}(h_2)$, so $\widehat{A}$ separates points in Δ. Because $\widehat{A}$ is a unital Banach algebra, it follows from the Stone-Weierstrass theorem that $\widehat{A}$ is dense in $C(\Delta)$.

Let $x \in A$. With $y = xx^*$, we have $y^* = (xx^*)^* = x^{**}x^* = xx^* = y$, from which it follows that $\left\|y^2\right\| = \|y\|^2$. Assume by induction that $\|y^m\| = \|y\|^m$, for $m = 2^n$. Then, as $(y^m)^* = y^m$,

$$\left\|y^{2m}\right\| = \|y^m y^m\| = \|y^m (y^m)^*\| = \|y^m\|^2 = (\|y\|^m)^2 = \|y\|^{2m} .$$

The spectral radius formula[15] gives

$$\rho(y) = \lim \|y^n\|^{1/n} ,$$

and because $\|y^m\| = \|y\|^m$ for $m = 2^n$, we have $\lim \|y^m\|^{1/m} = \|y\|$. Because the limit of this subsequence is $\|y\|$, the limit of $\|y^n\|^{1/n}$ is also $\|y\|$, so we obtain

$$\rho(y) = \|y\| .$$

But (1) tells us $\|\hat{y}\|_\infty = \rho(y)$, so we have $\|\hat{y}\|_\infty = \|y\|$. Because $y = xx^*$, using $\Gamma(x^*) = \overline{\Gamma(x)}$ and the fact that Γ is an algebra homomorphism, we get

$$\Gamma(y) = \Gamma(xx^*) = \Gamma(x)\Gamma(x^*) = \Gamma(x)\overline{\Gamma(x)} = |\Gamma(x)|^2.$$

[15] Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.

That is, $\hat{y} = |\hat{x}|^2$ and with $\|\hat{y}\|_\infty = \|y\|$ we obtain

$$\|\hat{x}\|_\infty^2 = \|\hat{y}\|_\infty = \|y\| = \|xx^*\| = \|x\|^2,$$

i.e.

$$\|\hat{x}\|_\infty = \|x\|.$$

This shows that $\Gamma : A \to C(\Delta)$ is an isometry. In particular, Γ maps closed sets to closed sets, so $\widehat{A} = \Gamma(A)$ is a closed subset of $C(\Delta)$. We have already established that $\widehat{A}$ is dense in $C(\Delta)$, so $\widehat{A} = C(\Delta)$. The fact that Γ is an isometry yields that Γ is one-to-one, and the fact that $\widehat{A} = C(\Delta)$ means that that Γ is onto, hence Γ is a bijection, and therefore it is an isomorphism of algebras. Because Γ is an isometry, it is an isomorphism of Banach algebras. □

The following theorem states conditions under which a self-adjoint element of a unital Banach algebra with an involution has a square root.[16]

Theorem 8. Let A be a unital Banach algebra with an involution $^* : A \to A$. If $x \in A$ is self-adjoint and $\sigma(x)$ contains no real λ with $\lambda \leq 0$, then there is some self-adjoint $y \in A$ satisfying $y^2 = x$.

If A is a Banach algebra and $x \in A$, we say that $x \in A$ is *normal* if $xx^* = x^*x$. If A is a Banach algebra with involution $^* : A \to A$, by $x \geq 0$ we mean that x is self-adjoint and $\sigma(x) \subseteq [0, \infty)$, and we say that x is *positive*. The following theorem states basic facts about the spectrum of elements of a unital B^*-algebra.[17]

Theorem 9. If A is a unital B^*-algebra, then:

1. If x is self-adjoint, then $\sigma(x) \subseteq \mathbb{R}$.
2. If x is normal, then $\rho(x) = \|x\|$.
3. If $x \in A$, then $\rho(xx^*) = \|x\|^2$.
4. If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$.
5. If $x \in A$, then $xx^* \geq 0$.
6. If $x \in A$, then $e + xx^*$ is invertible.

6 Positive linear functionals

Suppose that A is a Banach algebra with an involution $^* : A \to A$. If $F : A \to \mathbb{C}$ is a linear map such that $F(xx^*)$ is real and ≥ 0 for all $x \in A$, we say that F is a *positive linear functional*. In particular, if $h \in \Delta$ and $x \in A$, then from Theorem 7 we have $h(x^*) = \overline{h(x)}$, and so $h(xx^*) = h(x)h(x^*) = h(x)\overline{h(x)} = |h(x)|^2 \geq 0$. Thus, the elements of Δ are positive linear functionals.

We shall use the following theorem to prove the theorem after it.[18]

[16] Walter Rudin, *Functional Analysis*, second ed., p. 294, Theorem 11.26.

[17] Walter Rudin, *Functional Analysis*, second ed., p. 294, Theorem 11.28.

[18] Walter Rudin, *Functional Analysis*, second ed., p. 137, Theorem 5.20.

Theorem 10. If X is a real or complex Banach space, X_1 and X_2 are closed subspaces of X, and $X = X_1 + X_2$, then there is some $\gamma < \infty$ such that for every $x \in X$ there are $x_1 \in X_1, x_2 \in X_2$ satisfying $x = x_1 + x_2$ and

$$\|x_1\| + \|x_2\| \leq \gamma \|x\|.$$

The following theorem establishes some basic properties of positive linear functionals on a unital Banach algebra with an involution.[19]

Theorem 11. Suppose that A is a unital Banach algebra with an involution $^* : A \to A$. If $F : A \to \mathbb{C}$ is a positive linear functional, then:

1. $F(x^*) = \overline{F(x)}$.
2. $|F(xy^*)|^2 \leq F(xx^*)F(yy^*)$.
3. $|F(x)|^2 \leq F(e)F(xx^*) \leq F(e)^2\rho(xx^*)$.
4. If x is normal, then $|F(x)| \leq F(e)\rho(x)$.
5. If A is commutative, then $\|F\| = F(e)$.
6. If there is some β such that $\|x^*\| \leq \beta \|x\|$ for all $x \in A$, then $\|F\| \leq \beta^{1/2}F(e)$.
7. F is a bounded linear map.

Proof. Suppose that $x, y \in A$. For any $\alpha \in \mathbb{C}$, we have on the one hand $F((x + \alpha y)(x + \alpha y)^*) \geq 0$, and on the other hand

$$F((x+\alpha y)(x+\alpha y)^*) = F((x+\alpha y)(x^*+\overline{\alpha}y^*)) = F(xx^*+\overline{\alpha}xy^*+\alpha yx^*+|\alpha|^2yy^*).$$

Therefore,

$$F(xx^*) + \overline{\alpha}F(xy^*) + \alpha F(yx^*) + |\alpha|^2 F(yy^*) \geq 0. \tag{4}$$

Applying (4) with $\alpha = 1$ gives

$$F(xx^*) + F(xy^*) + F(yx^*) + F(yy^*) \geq 0.$$

In particular, this expression is real, and because $F(xx^*)$ and $F(yy^*)$ are real we get that $F(xy^*) + F(yx^*)$ is real, so $\operatorname{Im} F(yx^*) = -\operatorname{Im} F(xy^*)$. Applying (4) with $\alpha = i$ gives

$$F(xx^*) - iF(xy^*) + iF(yx^*) + F(yy^*) \geq 0.$$

In particular, this expression is real, and so $-iF(xy^*) + iF(yx^*)$ is real, i.e. $F(xy^*) - F(yx^*)$ is imaginary, so $\operatorname{Re} F(yx^*) = \operatorname{Re} F(xy^*)$. Therefore $F(yx^*) = \overline{F(xy^*)}$. Using $y = e$ yields

$$F(x^*) = \overline{F(x)}.$$

[19] Walter Rudin, *Functional Analysis*, second ed., p. 296, Theorem 11.31.

Suppose that $x, y \in A$ and that $F(xy^*) \neq 0$. For any $t \in \mathbb{R}$, using (4) with $\alpha = \frac{t}{|F(xy^*)|} F(xy^*)$ gives

$$F(xx^*) + \frac{t}{|F(xy^*)|} \overline{F(xy^*)} F(xy^*) + \frac{t}{|F(xy^*)|} F(xy^*) F(yx^*) + t^2 F(yy^*) \geq 0,$$

i.e.

$$F(xx^*) + t|F(xy^*)| + \frac{t}{|F(xy^*)|} F(xy^*) F(yx^*) + t^2 F(yy^*) \geq 0,$$

and as $F(yx^*) = F((xy^*)^*) = \overline{F(xy^*)}$, we have

$$F(xx^*) + 2t|F(xy^*)| + t^2 F(yy^*) \geq 0.$$

For $t = -\frac{|F(xy^*)|}{F(yy^*)}$ this is

$$F(xx^*) - 2\frac{|F(xy^*)|^2}{F(yy^*)} + \frac{|F(xy^*)|^2}{F(yy^*)} \geq 0,$$

i.e.

$$|F(xy^*)|^2 \leq F(xx^*) F(yy^*).$$

Suppose that $x \in A$. Because $xe^* = x$ and $ee^* = e$, we have

$$|F(x)|^2 \leq F(e) F(xx^*).$$

We shall prove that $F(xx^*) \leq F(e)\rho(xx^*)$. Let $t > \rho(xx^*)$. It then follows that $\sigma(te - xx^*)$ is contained in the open right half-plane, and thus by Theorem 8 there is some self-adjoint $u \in A$ satisfying $u^2 = te - xx^*$. Then

$$F(te - xx^*) = F(u^2) = F(uu^*) \geq 0,$$

so

$$F(xx^*) \leq tF(e).$$

Because this is true for all $t > \rho(xx^*)$, we obtain

$$F(xx^*) \leq F(e)\rho(xx^*).$$

Suppose that x is normal. It is a fact that if x and y belong to a unital Banach algebra and $xy = yx$, then $\sigma(xy) \subseteq \sigma(x)\sigma(y)$.[20] Thus $\sigma(xx^*) \subseteq \sigma(x)\sigma(x^*)$, from which we get

$$\rho(xx^*) \leq \rho(x)\rho(x^*).$$

It is a fact that $\sigma(x^*) = \overline{\sigma(x)}$,[21], so we have $\rho(x) = \rho(x^*)$, and thus

$$\rho(xx^*) \leq \rho(x)^2.$$

[20] Walter Rudin, *Functional Analysis*, second ed., p. 293, Theorem 11.23.
[21] Walter Rudin, *Functional Analysis*, second ed., p. 288, Theorem 11.15.

But $|F(x)|^2 \leq F(e)^2\rho(xx^*)$, so we have $|F(x)|^2 \leq F(e)^2\rho(x)^2$, i.e.

$$|F(x)| \leq F(e)\rho(x).$$

Suppose that A is commutative, and let $x \in A$. Since A is commutative, x is normal and hence we have $|F(x)| \leq F(e)\rho(x)$, and as always we have $\rho(x) \leq \|x\|$. Therefore, for every $x \in A$ we have

$$|F(x)| \leq F(e)\,\|x\|\,.$$

This implies that $\|F\| \leq F(e)$, and because the above inequality is an equality for $x = e$, we have $\|F\| = F(e)$.

Suppose that there is some β such that $\|x^*\| \leq \beta\,\|x\|$ for all $x \in A$. We have $\rho(xx^*) \leq \|xx^*\| \leq \|x\|\,\|x^*\| \leq \beta\,\|x\|^2$. (We merely stipulated that A is a unital Banach algebra with an involution; if we had demanded that A be a B^*-algebra, then we would have $\|xx^*\| = \|x\|\,\|x^*\| = \|x\|^2$.) Using $|F(x)|^2 \leq F(e)^2\rho(xx^*)$ then gives us $|F(x)|^2 \leq \beta F(e)^2\,\|x\|^2$, hence

$$|F(x)| \leq \beta^{1/2}F(e)\,\|x\|\,.$$

If $F(e) = 0$, then $|F(x)|^2 \leq 0$ for all $x \in A$, and hence $F = 0$, which indeed is bounded. Otherwise, $F(e) > 0$, and F is bounded if and only if $\frac{1}{F(e)}F$ is bounded. Therefore, to prove that F is bounded it suffices to prove that F is bounded in the case where $F(e) = 1$.

Let H be the set of all self-adjoint elements of A. H and iH are real vector spaces. For any $x \in A$, defining $2u = x + x^*$ and $2v = i(x^* - x)$, we have $x = u + iv$, and u, v are self-adjoint. It follows that

$$A = H + iH.$$

Because the elements of H are self-adjoint, the restriction of F to H is a real-linear map $H \to \mathbb{R}$. For $u \in H$, because u is self-adjoint it is in particular normal, and so $|F(x)| \leq F(e)\rho(x) \leq F(e)\,\|x\| = \|x\|$, because $F(e) = 1$. Hence the restriction of F to H is a real-linear map $H \to \mathbb{R}$ with norm 1, and therefore there is a unique bounded real-linear map $\Phi : \overline{H} \to \mathbb{R}$ whose restriction to H is equal to the restriction of F to H, and $\|\Phi\| = 1$.

Suppose that $y \in \overline{H} \cap i\overline{H}$. There are $u_n \in H$ with $u_n \to y$ and there are $v_n \in H$ with $iv_n \to y$. Then $u_n^2 \to y^2$ and $-v_n^2 \to y$, or $v_n^2 \to -y^2$. Because $|F(u_n)|^2 \leq F(e)F(u_nu_n^*) = F(u_n^2)$, we have

$$|F(u_n)|^2 \leq F(u_n^2) \leq F(u_n^2 + v_n^2).$$

Because u_n and v_n are self-adjoint, $u_n^2 + v_n^2$ is normal and hence

$$|F(u_n^2 + v_n^2)| \leq F(e)\rho(u_n^2 + v_n^2) = \rho(u_n^2 + v_n^2) \leq \left\|u_n^2 + v_n^2\right\|,$$

and so we have

$$|F(u_n)|^2 \leq \left\|u_n^2 + v_n^2\right\|.$$

But $u_n^2 \to y$ and $v_n^2 \to -y$, so $\left\|u_n^2 + v_n^2\right\| \to \|y - y\| = 0$. Therefore, $F(u_n) \to 0$, and so

$$\Phi(y) = \lim F(u_n) \to 0.$$

That is, if $y \in \overline{H} \cap i\overline{H}$, then $F(y) = 0$.

Because $A = H + iH$, certainly $A = \overline{H} + i\overline{H}$, so by Theorem 10 there is some $\gamma < \infty$ such that for all $x \in A$, there are $x_1 \in \overline{H}$ and $x_2 \in \overline{H}$ satisfying

$$x = x_1 + ix_2, \qquad \|x_1\| + \|x_2\| \leq \gamma \|x\|.$$

Let $x \in A$ and let $x = x_1 + ix_2$, where x_1, x_2 satisfy the above, and let $x = u + iv$ with $u, v \in H$, namely $2u = x + x^*$ and $2v = i(x^* - x)$. Supposing that $x_1 - u$ and $x_2 - v \in \overline{H} \cap i\overline{H}$, which Rudin asserts but whose truth is not apparent to me, we obtain $F(x_1 - u) = 0$ and $F(x_2 - v) = 0$, or $F(x_1) = F(u)$ and $F(x_2) = F(v)$. Then,

$$F(x) = F(u + iv) = F(u) + iF(v) = F(x_1) + iF(x_2) = \Phi(x_1) + i\Phi(x_2),$$

and therefore, because $\|\Phi\| = 1$ and because $\|x_1\| + \|x_2\| \leq \gamma \|x\|$,

$$|F(x)| \leq |\Phi(x_1) + i\Phi(x_2)| \leq |\Phi(x_1)| + |\Phi(x_2)| \leq \|x_1\| + \|x_2\| \leq \gamma \|x\|,$$

showing that $\|F\| \leq \gamma$, and in particular that F is bounded. □

7 The Riesz-Markov theorem and extreme points

We say that a positive Borel measure μ on a compact Hausdorff space X is *regular* if for every Borel subset E of X we have

$$\mu(E) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq E\}$$

and

$$\mu(E) = \inf\{\mu(G) : G \text{ is open and } E \subseteq G\}.$$

We say that a complex Borel measure μ on a compact Hausdorff space is regular if the positive Borel measure $|\mu|$ is regular, and we write $\|\mu\| = |\mu|(X)$. The following is the Riesz-Markov theorem, stated for complex Borel measures on a compact Hausdorff space.[22]

Theorem 12 (Riesz-Markov)**.** Suppose that X is a compact Hausdorff space. If Λ is a bounded linear functional on $C(X)$, then there is one and only one regular complex Borel measure μ on X satisfying

$$\Lambda f = \int_X f \, d\mu, \qquad f \in C(X).$$

This measure μ satisfies $\|\mu\| = \|\Lambda\|$.

The following theorem uses the Riesz-Markov theorem to define a correspondence between positive linear functionals on a commutative unital Banach algebra with a symmetric involution and regular positive Borel measures on its maximal ideal space.[23]

[22]Walter Rudin, *Real and Complex Analysis*, third ed., p. 130, Theorem 6.19.
[23]Walter Rudin, *Functional Analysis*, second ed., p. 299, Theorem 11.33.

Theorem 13. Suppose that A is a commutative unital Banach algebra with an involution $^* : A \to A$ satisfying

$$h(x^*) = \overline{h(x)}, \qquad x \in A, h \in \Delta. \tag{5}$$

Let K be the set of all positive linear functionals $F : A \to \mathbb{C}$ satisfying $F(e) \leq 1$, and let M be the set of all regular positive Borel measures μ on Δ satisfying $\mu(\Delta) \leq 1$. K and M are convex sets. If $\mu \in M$, then $F : A \to \mathbb{C}$ defined by

$$F_\mu(x) = \int_\Delta \hat{x} d\mu, \qquad x \in A,$$

belongs to K, and this map $\mu \mapsto F_\mu$ is an isometric bijection $M \to K$.

Proof. If $F_1, F_2 \in K$ and $0 \leq t \leq 1$, then $(1-t)F_1 + tF_2$ is linear, and it is straightforward to check that it is positive. Moreover, $((1-t)F_1 + tF_2)(e) = (1-t)F_1(e) + tF_2(e) \leq (1-t) + t = 1$, so $(1-t)F_1 + tF_2 \in K$. Therefore K is a convex set.

Suppose that $\mu_1, \mu_2 \in M$, that a_1, a_2 are nonnegative real numbers, and let $\mu = a_1\mu_1 + a_2\mu_2$. If E is a Borel subset of Δ, then for any $\epsilon > 0$ there are compact subsets F_1, F_2 of Δ such that $\mu_1(E) < \mu_1(F_1) - \epsilon$ and $\mu_2(E) < \mu_2(F_2) - \epsilon$. With $F = F_1 \cup F_2$, we have

$$\begin{aligned}
\mu(F) &= a_1\mu_1(F) + a_2\mu_2(F) \\
&\geq a_1\mu_1(F_1) + a_2\mu_2(F_2) \\
&\geq a_1(\mu_1(E) + \epsilon) + a_2(\mu_2(E) + \epsilon) \\
&= \mu(E) + (a_1 + a_2)\epsilon.
\end{aligned}$$

It follows that $\mu(E) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq E\}$. If E is a Borel subset of Δ, then for any $\epsilon > 0$ there are open subsets G_1, G_2 of Δ such that $\mu_1(E) > \mu_1(G_1) - \epsilon$ and $\mu_2(E) > \mu_2(G_2) - \epsilon$. With $G = G_1 \cap G_2$, we have

$$\begin{aligned}
\mu(G) &= a_1\mu_1(G) + a_2\mu_2(G) \\
&\leq a_1\mu_1(G_1) + a_2\mu_2(G_2) \\
&< a_1(\mu_1(E) + \epsilon) + a_2(\mu_2(E) + \epsilon) \\
&= \mu(E) + (a_1 + a_2)\epsilon.
\end{aligned}$$

It follows that $\mu(E) = \inf\{\mu(G) : G \text{ is open and } E \subseteq G\}$. Therefore, $\mu = a_1\mu_1 + a_2\mu_2$ is a regular positive Borel measure. In particular, if $0 \leq t \leq 1$ and $a_1 = 1-t$, $a_2 = t$, then μ is a regular positive Borel measure. Finally, for $\mu = (1-t)\mu_1 + t\mu_2$, $0 \leq t \leq 1$, we have, because $\mu_1(\Delta) \leq 1$ and $\mu_2(\Delta) \leq 1$,

$$\mu(\Delta) = (1-t)\mu_1(\Delta) + t\mu_2(\Delta) \leq (1-t) + t = 1,$$

so $\mu \in M$, showing that M is a convex set.

Let $\mu \in M$. It is apparent that $F_\mu : A \to \mathbb{C}$ is linear. For $x \in A$, we have $\Gamma(xx^*) = \Gamma(x)\Gamma(x^*)$, and as $\Gamma(x^*) = \overline{\Gamma(x)}$ by (5), we get $\Gamma(xx^*) = |\Gamma(x)|^2$. As

$|\Gamma(x)|^2(h) \geq 0$ for all $h \in \Delta$, we have

$$F_\mu(xx^*) = \int_\Delta \Gamma(xx^*) d\mu = \int_\Delta |\Gamma(x)|^2 d\mu \geq 0,$$

showing that F_μ is a positive linear functional. Furthermore, $\hat{e}(h) = h(e) = 1$ for all $h \in \Delta$, so

$$F_\mu(e) = \mu(\Delta) \leq 1,$$

showing that $F_\mu \in K$.

If $x \in \operatorname{rad} A$, then $\rho(x) = 0$ by (1), and so $F(x) = 0$ by Theorem 11. We define $\widehat{F} : \widehat{A} \to \mathbb{C}$

$$\widehat{F}(\hat{x}) = F(x);$$

this makes sense because if $\hat{x} = \hat{y}$ then $\Gamma(x - y) = 0$, and so by Theorem 3 we have $x - y \in \operatorname{rad} A$ and hence $F(x - y) = 0$, i.e. so $F(x) = F(y)$. For $x \in A$, x is normal because A is commutative so we have by Theorem 11 that

$$|\widehat{F}(\hat{x})| = |F(x)| \leq F(e)\rho(x)$$

and by (1) we have $\rho(x) = \|\hat{x}\|_\infty$, so

$$|\widehat{F}(\hat{x})| \leq F(e) \|\hat{x}\|_\infty .$$

As $\widehat{F}(\hat{e}) = F(e)$, it follows that $\left\|\widehat{F}\right\| = F(e)$. By (5) and because $\widehat{A}$ separates points in Δ, applying the Stone-Weierstrass we obtain that $\widehat{A}$ is dense in $C(\Delta)$. Because $\widehat{F}$ is a continuous linear functional on the dense subspace $\widehat{A}$ of $C(\Delta)$, there is a unique continuous linear functional Λ on $C(\Delta)$ such that $\Lambda = \widehat{F}$ on $\widehat{A}$, and $\|\Lambda\| = \left\|\widehat{F}\right\|$. Applying Theorem 12, there is one and only one regular complex Borel measure μ on X that satisfies

$$\Lambda f = \int_\Delta f d\mu, \qquad f \in C(\Delta), \tag{6}$$

and $\|\mu\| = \|\Lambda\| = \left\|\widehat{F}\right\| = F(e)$. It follows that $\mu \mapsto F_\mu$ is one-to-one. Because $\hat{e}(h) = 1$ for all $h \in \Delta$,

$$\mu(\Delta) = \int_\Delta \chi_\Delta d\mu = \int_\Delta \hat{e} d\mu = \Lambda\hat{e} = \widehat{F}(\hat{e}) = F(e) = \|\mu\| = |\mu|(\Delta).$$

The fact that $\mu(\Delta) = |\mu|(\Delta)$ implies that μ is a positive measure. The above equalities also state $\mu(\Delta) = F(e)$, and since $F \in K$ we have $F(e) \leq 1$, hence $\mu(\Delta) \leq 1$. Therefore, $\mu \in M$. For $x \in A$, as $\hat{x} \in C(\Delta)$ we have by (6) that

$$F_\mu(x) = \int_\Delta \hat{x} d\mu = \Lambda\hat{x} = \widehat{F}(\hat{x}) = F(x),$$

showing that $F = F_\mu$. This shows that $\mu \mapsto F_\mu$ is onto, and therefore $\mu \mapsto F_\mu$ is a bijection $M \to K$. $\square$

Because the map $\mu \mapsto F_\mu$ in the above theorem is an isometric bijection $M \to K$, it follows that that μ is an extreme point of M if and only if F_μ is an extreme point of K.

It is a fact that the set of extreme points of the set of regular Borel probability measures on a compact Hausdorff space X is $\{\delta_x : x \in X\}$.[24] Given this, one proves that the set of extreme points of M is $\{0\} \cup \{\delta_h : h \in \Delta\}$. For $x \in A$, $F_0(x) = 0$, i.e. $F_0 = 0$. For $h \in \Delta$ and $x \in A$,

$$F_{\delta_h}(x) = \int_\Delta \hat{x} d\delta_h = \hat{x}(h) = h(x),$$

so $F_{\delta_h} = h$. Therefore, the extreme points of K are $\{0\} \cup \Delta$, that is, the set of algebra homomorphisms $A \to \mathbb{C}$.

Corollary 14. Suppose that A is a commutative unital Banach algebra with involution $^* : A \to A$ satisfying

$$h(x^*) = \overline{h(x)}, \qquad x \in A, h \in \Delta.$$

If K is the set of all positive linear functionals $F : A \to \mathbb{C}$ satisfying $F(e) \leq 1$, then

$$\operatorname{ext} K = \{0\} \cup \Delta.$$

Moreover, it is straightforward to check that the set K in the above corollary is a weak-* closed subset of A^*: if $F_i \in K$ is a net that weak-* converges to $\Lambda \in A^*$, one checks that $\Lambda(xx^*) \geq 0$ for all $x \in A$ and that $\Lambda e \leq 1$. By the Banach-Alaoglu theorem, the set $B = \{\Lambda \in A^* : \|\Lambda\| \leq 1\}$ is weak-* compact, and if $F \in K$ then $\|F\| = F(e)$ by Theorem 11 and $F(e) \leq 1$, so $K \subseteq B$. Hence, K is a weak-* compact subset of A^*. Therefore, the Krein-Milman theorem[25] tells us that K is equal to the weak-* closure of the convex hull of the set of its extreme points, and by the above corollary this means that K is equal to the weak-* closure of the convex hull of $\{0\} \cup \Delta$.

8 Positive definite functions

A function $\phi : \mathbb{R}^n \to \mathbb{C}$ is said to be *positive-definite* if $r \geq 1$, $x_1, \ldots, x_r \in \mathbb{R}^n, c_1, \ldots, c_r \in \mathbb{C}$ imply that

$$\sum_{i,j=1}^r c_i \overline{c_j} \phi(x_i - x_j) \geq 0;$$

in particular, for ϕ to be positive-definite demands that the left-hand side of this inequality is real.

A positive-definite function need not be measurable. For example, $\mathbb{R}$ is a vector space over $\mathbb{Q}$, and if $\psi : \mathbb{R} \to \mathbb{R}$ is a vector space automorphism of $\mathbb{R}$ over

[24] Barry Simon, *Convexity: An Analytic Viewpoint*, p. 128, Example 8.16.
[25] Walter Rudin, *Functional Analysis*, second ed., p. 75, Theorem 3.23.

$\mathbb{Q}$, one proves that $x \mapsto e^{i\psi(x)}$ is a positive-definite function $\mathbb{R} \to \mathbb{C}$, and that there are ψ for which $x \mapsto e^{i\psi(x)}$ is not measurable.

The following theorem states some basic facts about positive-definite functions. More material on positive-definite functions is presented in Bogachev; for example, if $\phi : \mathbb{R}^n \to \mathbb{C}$ is a measurable positive-definite function, then there is a continuous positive-definite function $\mathbb{R}^n \to \mathbb{C}$ that is equal to ϕ almost everywhere.[26]

Theorem 15. If $\phi : \mathbb{R}^n \to \mathbb{C}$ is positive-definite, then

$$\phi(0) \geq 0, \tag{7}$$

and for all $x \in \mathbb{R}^n$ we have

$$\overline{\phi(x)} = \phi(-x) \tag{8}$$

and

$$|\phi(x)| \leq \phi(0). \tag{9}$$

Proof. Using $r = 1$ and $c_1 = 1$, we have for all $x_1 \in \mathbb{R}^n$ that $\phi(x_1 - x_1) \geq 0$, i.e. $\phi(0) \geq 0$.

Using $r = 2$, for $x_1, x_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{C}$ we have

$$c_1\overline{c_1}\phi(x_1 - x_1) + c_1\overline{c_2}\phi(x_1 - x_2) + c_2\overline{c_1}\phi(x_2 - x_1) + c_2\overline{c_2}\phi(x_2 - x_2) \geq 0.$$

Take $x_1 = x$ and $x_2 = 0$, with which

$$|c_1|^2\phi(0) + c_1\overline{c_2}\phi(x) + c_2\overline{c_1}\phi(-x) + |c_2|^2\phi(0) \geq 0;$$

in particular, the left-hand side is real, and because $\phi(0)$ is real by (7), this implies that $c_1\overline{c_2}\phi(x) + c_2\overline{c_1}\phi(-x)$ is real. That is, it is equal to its complex conjugate:

$$c_1\overline{c_2}\phi(x) + c_2\overline{c_1}\phi(-x) = \overline{c_1}c_2\overline{\phi(x)} + \overline{c_2}c_1\overline{\phi(-x)}.$$

The fact that this holds every $c_1, c_2 \in \mathbb{C}$ implies that $\overline{\phi(x)} = \phi(-x)$.

Again using that

$$c_1\overline{c_1}\phi(x_1 - x_1) + c_1\overline{c_2}\phi(x_1 - x_2) + c_2\overline{c_1}\phi(x_2 - x_1) + c_2\overline{c_2}\phi(x_2 - x_2) \geq 0,$$

with $x_1 = x, x_2 = 0$ and $c_2 = 1$ we get

$$|c_1|^2\phi(0) + c_1\phi(x) + \overline{c_1}\phi(-x) + \phi(0) \geq 0.$$

Applying (8) gives

$$|c_1|^2\phi(0) + c_1\phi(x) + \overline{c_1\phi(x)} + \phi(0) \geq 0.$$

For $c_1 \in \mathbb{C}$ such that $|c_1| = 1$,

$$2\phi(0) + 2\mathrm{Re}\ (c_1\phi(x)) \geq 0,$$

[26] Vladimir I. Bogachev, *Measure Theory*, vol. 1, p. 221, Theorem 3.10.20. See also Anthony W. Knapp, *Basic Real Analysis*, p. 406.

or

$$-\text{Re }(c_1\phi(x)) \leq \phi(0).$$

Thus, taking $c_1 \in \mathbb{C}$ such that $|c_1| = 1$ and for which $-\text{Re }(c_1\phi(x)) = |\phi(x)|$, we get $|\phi(x)| \leq \phi(0)$. □

The following lemma about positive-definite functions follows a proof in Bogachev.[27]

Lemma 16. If $\phi : \mathbb{R}^n \to \mathbb{C}$ is a measurable positive-definite function and $f \in L^1(\mathbb{R}^n)$ is nonnegative, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x-y)f(x)f(y)dm_n(x)dm_n(y) \geq 0.$$

Proof. For $r \geq 2$ and for any $x_1, \ldots, x_r$ and $c_1 = 1, \ldots, c_r = 1$, we have

$$\sum_{j,k=1}^{r} \phi(x_j - x_k) \geq 0,$$

or

$$r\phi(0) + \sum_{j \neq k} \phi(x_j - x_k) \geq 0.$$

By (9), ϕ is bounded. It follows that we can integrate both sides of the above inequality over $(\mathbb{R}^n)^r$ with respect to the positive measure

$$f(x_1)\cdots f(x_r)dm_n(x_1)\cdots dm_n(x_r).$$

Writing

$$I = \int_{\mathbb{R}^n} f(x)dm_n(x),$$

we obtain

$$r\phi(0)I^r + \sum_{j \neq k} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \phi(x_j - x_k)f(x_1)\cdots f(x_r)dm_n(x_1)\cdots dm_n(x_r) \geq 0,$$

and so, writing

$$J = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x-y)f(x)f(y)dm_n(x)dm_n(y),$$

we have

$$r\phi(0)I^r + \sum_{j \neq k} JI^{r-2} \geq 0,$$

or

$$r\phi(0)I^r + r(r-1)JI^{r-2} \geq 0.$$

[27] Vladimir I. Bogachev, *Measure Theory*, vol. 1, p. 221, Lemma 3.10.19.

If $I = 0$, then because f is nonnegative it follows that f is 0 almost everywhere, in which case $J = 0$, so the claim is true. If $I > 0$, then dividing by $r(r-1)I^{r-2}$ we obtain

$$\frac{1}{r-1}\phi(0)I^2 + J \geq 0.$$

This inequality holds for all $r \geq 2$, so taking $r \to \infty$ yields

$$J \geq 0,$$

which is the claim. □

For $f, g \in L^1(\mathbb{R}^n)$,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dm_n(y).$$

The *support* of a function $f : \mathbb{R}^n \to \mathbb{C}$, denoted supp$f$, is the closure of the set $\{x \in \mathbb{R}^n : f(x) \neq 0\}$. We denote by $C_c(\mathbb{R}^n)$ the set of all continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ such that supp$f$ is a compact set. It is straightforward to check that an element of $C_c(\mathbb{R}^n)$ is uniformly continuous on $\mathbb{R}^n$. The following theorem is similar to the previous lemma, but applies to functions that need not be nonnegative.[28]

Theorem 17. For $f : \mathbb{R}^n \to \mathbb{C}$, define $\widetilde{f} : \mathbb{R}^n \to \mathbb{C}$ by $\widetilde{f}(x) = \overline{f(-x)}$. If $\phi : \mathbb{R}^n \to \mathbb{C}$ is a continuous positive-definite function, then for all $f \in C_c(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} (f * \widetilde{f})\psi \geq 0.$$

If μ is a complex Borel measure on $\mathbb{R}^n$, the *Fourier transform of* μ is the function $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e_{-\xi} d\mu, \qquad \xi \in \mathbb{R}^n.$$

One proves using the dominated convergence theorem that $\hat{\mu}$ is continuous.

Theorem 18. If μ is a finite positive Borel measure on $\mathbb{R}^n$, then $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$ is positive-definite.

[28] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 85, Proposition 3.35.

Proof. For $\xi_1, \ldots, \xi_r \in \mathbb{R}^n$ and $c_1, \ldots, c_r \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{j,k=1}^r c_j \overline{c_k} \hat{\mu}(\xi_j - \xi_k) &= \sum_{j,k=1}^r c_j \overline{c_k} \int_{\mathbb{R}^n} e^{-i(\xi_j - \xi_k)\cdot x} d\mu(x) \\
&= \int_{\mathbb{R}^n} \sum_{j,k=1}^r c_j e^{-i\xi_j \cdot x} \overline{c_k e^{-i\xi_k \cdot x}} d\mu(x) \\
&= \int_{\mathbb{R}^n} \left(\sum_{j=1}^r c_j e^{-i\xi_j \cdot x} \right) \overline{\left(\sum_{k=1}^r c_k e^{-i\xi_k \cdot x} \right)} d\mu(x) \\
&= \int_{\mathbb{R}^n} \left| \sum_{j=1}^r c_j e^{-i\xi_j \cdot x} \right|^2 d\mu(x) \\
&\geq 0.
\end{aligned}$$

□

The following proof of Bochner's theorem follows an exercise in Rudin.[29]

Theorem 19 (Bochner)**.** If $\phi : \mathbb{R}^n \to \mathbb{C}$ is continuous and positive-definite, then there is some finite positive Borel measure ν on $\mathbb{R}^n$ for which $\phi = \hat{\nu}$.

Proof. Let A be the Banach algebra defined in §4, whose elements are those complex Borel measures μ on $\mathbb{R}^n$ for which there is some $f \in L^1(\mathbb{R}^n)$ and some $\alpha \in \mathbb{C}$ such that

$$d\mu = f dm_n + \alpha d\delta,$$

where m_n is Lebesgue measure on $\mathbb{R}^n$. For $f + \alpha\delta, g + \beta\delta \in A$, we have

$$(f + \alpha\delta) * (g + \beta\delta) = (f * g + \beta f + \alpha g) + \alpha\beta\delta;$$

we are identifying $f \in L^1(\mathbb{R}^n)$ with the complex Borel measure whose Radon-Nikodym derivative with respect to m_n is f. The norm on A is the total variation norm of a complex measure; one checks that for $f + \alpha\delta$ this is

$$\|f + \alpha\delta\| = \|f\| + |\alpha|,$$

where $\|f\| = \int_{\mathbb{R}^n} |f(x)| dm_n(x)$.

For $f \in L^1(\mathbb{R}^n)$, we define $\widetilde{f} \in L^1(\mathbb{R}^n)$ by $\widetilde{f}(x) = \overline{f(-x)}$, and we define $^* : A \to A$ by

$$(f + \alpha\delta)^* = \widetilde{f} + \overline{\alpha}\delta, \qquad f + \alpha\delta \in A.$$

[29] Walter Rudin, *Functional Analysis*, second ed., p. 303, Exercise 14. Other references on Bochner's theorem are the following: Barry Simon, *Convexity: An Analytic Viewpoint*, p. 153, Theorem 9.17; Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis*, vol. II, p. 293, Theorem 33.3; Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 220, Theorem 3.9.16; Walter Rudin, *Fourier Analysis on Groups*, p. 19, Theorem 1.4.3; Yitzhak Katznelson, *An Introduction to Harmonic Analysis*, third ed., p. 170; Vladimir I. Bogachev, *Measure Theory*, vol. II, p. 121, Theorem 7.13.1; Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 95, Theorem 4.18.

On the one hand,

$$\begin{aligned}((f+\alpha\delta)*(g+\beta\delta))^* &= ((f*g+\beta f+\alpha g)+\alpha\beta\delta)^* \\ &= \widetilde{f*g}+\overline{\beta}\widetilde{f}+\overline{\alpha}\widetilde{g}+\overline{\alpha\beta}\delta \\ &= \widetilde{f*g}+\overline{\beta}\widetilde{f}+\overline{\alpha}\widetilde{g}+\overline{\alpha\beta}\delta,\end{aligned}$$

and

$$\widetilde{f*g}(x)=\overline{\int_{\mathbb{R}^n} f(y)g(-x-y)dm_n(y)}.$$

On the other hand,

$$\begin{aligned}(g+\beta\delta)^* * (f+\alpha\delta)^* &= (\widetilde{g}+\overline{\beta}\delta)*(\widetilde{f}+\overline{\alpha}\delta) \\ &= (\widetilde{g}*\widetilde{f}+\overline{\alpha}\widetilde{g}+\overline{\beta}\widetilde{f})+\overline{\beta\alpha}\delta,\end{aligned}$$

and

$$\begin{aligned}(\widetilde{g}*\widetilde{f})(x) &= \int_{\mathbb{R}^n}\widetilde{g}(y)\widetilde{f}(x-y)dm_n(y) \\ &= \int_{\mathbb{R}^n}\overline{g(-y)f(-x+y)}dm_n(y) \\ &= \overline{\int_{\mathbb{R}^n} g(-y-x)f(y)dm_n(y)}.\end{aligned}$$

Therefore we have

$$((f+\alpha\delta)*(g+\beta\delta))^* = (g+\beta\delta)^* * (f+\alpha\delta)^*.$$

Thus $^*: A\to A$ is an involution (the other properties demanded of an involution are immediate).

We define $F: A\to\mathbb{C}$ by

$$F(f+\alpha\delta)=\int_{\mathbb{R}^n} f\phi dm_n+\alpha\phi(0), \qquad f+\alpha\delta\in A.$$

It is apparent that F is linear, and because $|\phi(x)|\leq\phi(0)$ for all x,

$$\begin{aligned}|F(f+\alpha\delta)| &\leq \left|\int_{\mathbb{R}^n} f\phi dm_n\right|+|\alpha|\phi(0) \\ &\leq \int_{\mathbb{R}^n}|f||\phi|dm_n+|\alpha|\phi(0) \\ &\leq \phi(0)\int_{\mathbb{R}^n}|f|dm_n+|\alpha|\phi(0) \\ &= \phi(0)\left\|f+\alpha\delta\right\|,\end{aligned}$$

from which it follows that $\|F\|=\phi(0)$, and in particular that F is bounded. Let $A_0=\{f+\alpha\delta\in A: f\in C_c(\mathbb{R}^n)\}$. Because $F: A\to\mathbb{C}$ is bounded and A_0

is a dense subset of A, to prove that F is a positive linear functional it suffices to prove that for all $f+\alpha\delta \in A_0$ we have $F((f+\alpha\delta)*(f+\alpha\delta)^*) \geq 0$.

For $g \in C_c(\mathbb{R}^n)$, by Theorem 17 we obtain

$$F(g*g^*) = F(g*\widetilde{g}) = \int_{\mathbb{R}^n} (g*\widetilde{g})\phi dm_n \geq 0. \tag{10}$$

Define $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

and for $\epsilon > 0$, define $\eta_\epsilon : \mathbb{R}^n \to \mathbb{R}$ by $\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$. Let $f+\alpha\delta \in A_0$ and define $g_\epsilon = f + \alpha\eta_\epsilon \in C_c(\mathbb{R}^n)$. From (10) we have $F(g_\epsilon * g_\epsilon^*) \geq 0$ for any $\epsilon > 0$. On the other hand,

$$\begin{aligned} F(g_\epsilon * g_\epsilon^*) &= F((f+\alpha\eta_\epsilon)*(\widetilde{f}+\overline{\alpha}\eta_\epsilon)) \\ &= F(f*\widetilde{f} + \overline{\alpha} f*\eta_\epsilon + \alpha\eta_\epsilon * \widetilde{f} + |\alpha|^2 \eta_\epsilon * \eta_\epsilon) \\ &= F(f*\widetilde{f}) + \overline{\alpha}\int_{\mathbb{R}^n}(f*\eta_\epsilon)\phi dm_n + \alpha\int_{\mathbb{R}^n}(\eta_\epsilon*\widetilde{f})\phi dm_n \\ &\quad + |\alpha|^2\int_{\mathbb{R}^n}(\eta_\epsilon*\eta_\epsilon)\phi dm_n. \end{aligned}$$

We take as granted that

$$\int_{\mathbb{R}^n}(f*\eta_\epsilon)\phi dm_n \to \int_{\mathbb{R}^n} f\phi dm_n$$

as $\epsilon \to 0$, that

$$\int_{\mathbb{R}^n}(\eta_\epsilon*\widetilde{f})\phi dm_n \to \int_{\mathbb{R}^n} \widetilde{f}\phi dm_n$$

as $\epsilon \to 0$, and that

$$\int_{\mathbb{R}^n}(\eta_\epsilon*\eta_\epsilon)\phi dm_n \to \phi(0)$$

as $\epsilon \to 0$. Furthermore,

$$\begin{aligned} F((f+\alpha\delta)*(f+\alpha\delta)^*) &= F((f+\alpha\delta)*(\widetilde{f}+\overline{\alpha}\delta)) \\ &= F(f*\widetilde{f}+\overline{\alpha}f+\alpha\widetilde{f}+|\alpha|^2) \end{aligned}$$

Thus

$$F(g_\epsilon * g_\epsilon^*) \to F((f+\alpha\delta)*(f+\alpha\delta)^*)$$

as $\epsilon \to 0$. Since $F(g_\epsilon * g_\epsilon^*) \geq 0$ for all $\epsilon > 0$, it follows that

$$F((f+\alpha\delta)*(f+\alpha\delta)^*) \geq 0.$$

Therefore, $F : A \to \mathbb{C}$ is a positive linear functional.

Because F is a positive linear functional and $F(e) = F(\delta) = 1$, we can apply Theorem 12, according to which there is a regular positive Borel measure μ on Δ satisfying

$$F(f + \alpha\delta) = \int_\Delta \Gamma(f + \alpha\delta) d\mu, \qquad f + \alpha\delta \in A,$$

and hence, from the definition of F,

$$\int_{\mathbb{R}^n} f\phi dm_n + \alpha\phi(0) = \int_\Delta \Gamma(f + \alpha\delta) d\mu, \qquad f + \alpha \in A.$$

We state the following again from §4 for easy access. If $t \in \mathbb{R}^n$, define $h_t : A \to \mathbb{C}$ by

$$h_t(f + \alpha\delta) = \hat{f}(t) + \alpha, \qquad f + \alpha\delta \in A,$$

and also define $h_\infty : A \to \mathbb{C}$ by

$$h_\infty(f + \alpha\delta) = \alpha, \qquad f + \alpha\delta \in A.$$

Let $\mathbb{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^n$. We proved in §4 that the map $T : \mathbb{R}^n \cup \{\infty\} \to \Delta$ defined by $T(t) = h_t$ is a homeomorphism. With $\nu = (T^{-1})_*\mu$ we have $\mu = T_*\nu$, and then

$$\begin{aligned}
\int_\Delta \Gamma(f + \alpha\delta) d\mu &= \int_\Delta \Gamma(f) d\mu + \alpha \int_\Delta \Gamma(\delta) d\mu \\
&= \int_\Delta \Gamma(f) d(T_*\nu) + \alpha \int_\Delta \chi_\Delta d\mu \\
&= \int_{\mathbb{R}^n \cup \{\infty\}} \Gamma(f) \circ T d\nu + \alpha\mu(\Delta) \\
&= \int_{\mathbb{R}^n \cup \{\infty\}} \Gamma(f) \circ T(t) d\nu(t) + \alpha F(\delta) \\
&= \int_{\mathbb{R}^n \cup \{\infty\}} h_t(f) d\nu(t) + \alpha\phi(0) \\
&= \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t) + \alpha\phi(0).
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^n} f\phi dm_n + \alpha\phi(0) = \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t) + \alpha\phi(0),$$

i.e.

$$\int_{\mathbb{R}^n} f\phi dm_n = \int_{\mathbb{R}^n} \hat{f}(t) d\nu(t).$$

As

$$\begin{aligned}
\int_{\mathbb{R}^n} \hat{f}(t) d\nu(t) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-it\cdot x} f(x) dm_n(x) \right) d\nu(t) \\
&= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{-ix\cdot t} d\nu(t) dm_n(x) \\
&= \int_{\mathbb{R}^n} f(x) \hat{\nu}(x) dm_n(x),
\end{aligned}$$

we have

$$\int_{\mathbb{R}^n} f\phi dm_n = \int_{\mathbb{R}^n} f\hat{\nu} dm_n.$$

This is true for all $f \in L^1(\mathbb{R}^n)$, from which it follows that $\phi = \hat{\nu}$. $\square$