Gaussian measures, Hermite polynomials, and the Ornstein-Uhlenbeck semigroup

Jordan Bell

June 27, 2015

1 Definitions

For a topological space X, we denote by $\mathscr{B}_X$ the Borel σ-algebra of X.

We write $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. With the order topology, $\overline{\mathbb{R}}$ is a compact metrizable space, and $\mathbb{R}$ has the subspace topology inherited from $\overline{\mathbb{R}}$, namely the inclusion map is an embedding $\mathbb{R} \to \overline{\mathbb{R}}$. It follows that[1]

$$\mathscr{B}_{\mathbb{R}} = \{E \cap \mathbb{R} : E \in \mathscr{B}_{\overline{\mathbb{R}}}\}.$$

If $\mathscr{F}$ is a collection of functions $X \to \overline{\mathbb{R}}$ on a set X, we define $\bigvee \mathscr{F} : X \to \overline{\mathbb{R}}$ and $\bigwedge \mathscr{F} : X \to \overline{\mathbb{R}}$ by

$$\left(\bigvee \mathscr{F}\right)(x) = \sup\{f(x) : f \in \mathscr{F}\}, \qquad x \in X$$

and

$$\left(\bigwedge \mathscr{F}\right)(x) = \inf\{f(x) : f \in \mathscr{F}\}, \qquad x \in X.$$

If X is a measurable space and $\mathscr{F}$ is a countable collection of measurable functions $X \to \overline{\mathbb{R}}$, it is a fact that $\bigwedge \mathscr{F}$ and $\bigvee \mathscr{F}$ are measurable $X \to \overline{\mathbb{R}}$.

2 Kolmogorov's inequality

Kolmogorov's inequality is the following.[2]

Theorem 1 (Kolmogorov's inequality)**.** *Suppose that $(\Omega, \mathscr{S}, P)$ is a probability space, that $X_1, \ldots, X_n \in L^2(P)$, that $E(X_1) = 0, \ldots, E(X_n) = 0$, and that $X_1, \ldots, X_n$ are independent. Let*

$$S_k(\omega) = \sum_{j=1}^{k} X_j(\omega), \qquad \omega \in \Omega,$$

[1] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 138, Lemma 4.20.

[2] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 322, Theorem 10.11.

for $1 \le k \le n$. Then for any $\lambda > 0$,

$$P\left(\left\{\omega \in \Omega : \bigvee_{k=1}^{n} |S_k(\omega)| \ge \lambda\right\}\right) \le \frac{1}{\lambda^2}\sum_{j=1}^{n} V(X_j) = \frac{1}{\lambda^2} V(S_n).$$

3 Gaussian measures on R

For real a and $\sigma > 0$, one computes that

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right) dt = 1. \tag{1}$$

Suppose that γ is a Borel probability measure on $\mathbb{R}$. If

$$\gamma = \delta_a$$

for some $a \in \mathbb{R}$ or has density

$$p(t, a, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right), \qquad t \in \mathbb{R},$$

for some $a \in \mathbb{R}$ and some $\sigma > 0$, with respect to Lebesgue measure on $\mathbb{R}$, we say that γ is a **Gaussian measure**. We say that δ_a is a Gaussian measure with **mean** a and **variance** 0, and that a Gaussian measure with density $p(\cdot, a, \sigma^2)$ has **mean** a and **variance** σ^2. A Gaussian measure with mean 0 and variance 1 is said to be **standard**.

One calculates that the **characteristic function** of a Gaussian measure γ with density $p(\cdot, a, \sigma^2)$ is

$$\widetilde{\gamma}(y) = \int_{\mathbb{R}} \exp(iyx) d\gamma(x) = \exp\left(iay - \frac{1}{2}\sigma^2 y^2\right), \qquad y \in \mathbb{R}. \tag{2}$$

The **cumulative distribution function** of a standard Gaussian measure γ is, for $t \in \mathbb{R}$,

$$\Phi(t) = \gamma(-\infty, t] = \int_{-\infty}^{t} d\gamma(s) = \int_{-\infty}^{t} p(s, 0, 1) ds = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds.$$

We define $\Phi(-\infty) = 0$ and also define

$$\Phi(\infty) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds = 1,$$

using (1).

$\Phi : \overline{\mathbb{R}} \to [0, 1]$ is strictly increasing, thus $\Phi^{-1} : [0, 1] \to \overline{\mathbb{R}}$ makes sense, and is itself strictly increasing. Then $1 - \Phi$ is strictly decreasing. By (1),

$$\begin{aligned} 1 - \Phi(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds \\ &= \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds. \end{aligned}$$

The following lemma gives an estimate for $1 - \Phi(t)$ that tells us something substantial as $t \to +\infty$, beyond the immediate fact that $(1-\Phi)(\infty) = 1 - \Phi(\infty) = 0$.[3]

Lemma 2. For $t > 0$,

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{t} - \frac{1}{t^3}\right)e^{-t^2/2} \leq 1 - \Phi(t) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2}.$$

Proof. Integrating by parts,

$$\begin{aligned}
1 - \Phi(t) &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds \\
&= \int_t^\infty \frac{1}{s\sqrt{2\pi}} \cdot s\exp\left(-\frac{s^2}{2}\right) ds \\
&= -\frac{1}{s\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right)\Big|_t^\infty - \int_t^\infty \frac{1}{s^2\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds \\
&\leq \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right).
\end{aligned}$$

On the other hand, using the above work and again integrating by parts,

$$\begin{aligned}
1 - \Phi(t) &= \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) - \int_t^\infty \frac{1}{s^3\sqrt{2\pi}} \cdot s\exp\left(-\frac{s^2}{2}\right) ds \\
&= \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) + \frac{1}{s^3\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right)\Big|_t^\infty \\
&+ \int_t^\infty \frac{3}{s^4\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) ds \\
&\geq \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) - \frac{1}{t^3\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right).
\end{aligned}$$

□

The following theorem shows that if the variances of a sequence of independent centered random variables are summable then the sequence of random variables is summable almost surely.[4]

Theorem 3. Suppose that $\xi_j \in L^2(\Omega, \mathscr{S}, P)$, $j \geq 1$, are independent random variables each with mean 0. If $\sum_{j=1}^\infty V(\xi_j) < \infty$, then $\sum_{j=1}^\infty \xi_j$ converges almost surely.

Proof. Define $S_n : \Omega \to \mathbb{R}$ by

$$S_n(\omega) = \sum_{j=1}^n \xi_j(\omega),$$

[3] Vladimir I. Bogachev, *Gaussian Measures*, p. 2, Lemma 1.1.3.

[4] Karl R. Stromberg, *Probability for Analysts*, p. 58, Theorem 4.6.

define $Z_n : \Omega \to [0, \infty]$ by

$$Z_n = \bigvee_{j=1}^{\infty} |S_{n+j} - S_n|,$$

and define $Z : \Omega \to [0, \infty]$ by

$$Z = \bigwedge_{n=1}^{\infty} Z_n.$$

If $S_n(\omega)$ converges and $\epsilon > 0$, there is some n such that for all $j \geq 1$, $|S_{n+j}(\omega) - S_n(\omega)| < \epsilon$ and so $Z_n(\omega) \leq \epsilon$ and $Z(\omega) \leq \epsilon$. Therefore, if $S_n(\omega)$ converges then $Z(\omega) = 0$. On the other hand, if $Z(\omega) = 0$ and $\epsilon > 0$, there is some n such that $Z_n(\omega) < \epsilon$, hence $|S_{n+j}(\omega) - S_n(\omega)| < \epsilon$ for all $j \geq 1$. That is, $S_n(\omega)$ is a Cauchy sequence in $\mathbb{R}$, and hence converges. Therefore

$$\{\omega \in \Omega : S_n(\omega) \text{ converges}\} = \{\omega \in \Omega : Z(\omega) = 0\}. \tag{3}$$

Let $\epsilon > 0$. For any n and k, using Kolmogorov's inequality with $X_j = \xi_{n+j}$ for $j = 1, \ldots, k$,

$$P\left(\bigvee_{j=1}^{k} |S_{n+j} - S_n| \geq \epsilon\right) \leq \frac{1}{\epsilon^2} \sum_{j=1}^{k} V(X_j) \leq \frac{1}{\epsilon^2} \sum_{j=n+1}^{\infty} V(\xi_j).$$

Because this is true for each k, it follows that

$$P(Z_n \geq \epsilon) \leq \frac{1}{\epsilon^2} \sum_{j=n+1}^{\infty} V(\xi_j),$$

hence, for each n,

$$P(Z \geq \epsilon) \leq P(Z_n \geq \epsilon) \leq \frac{1}{\epsilon^2} \sum_{j=n+1}^{\infty} V(\xi_j).$$

Because $\sum_{j=1}^{\infty} V(\xi_j) < \infty$, $\sum_{j=n+1}^{\infty} V(\xi_j) \to 0$ as $n \to \infty$, so

$$P(Z \geq \epsilon) = 0.$$

Because this is true for all $\epsilon > 0$, we get $P(Z > 0) = 0$, i.e. $P(Z = 0) = 1$. By (3), this means that S_n converges almost surely. $\square$

The following theorem gives conditions under which the converse of the above theorem holds.[5]

[5] Karl R. Stromberg, *Probability for Analysts*, p. 59, Theorem 4.7.

Theorem 4. Suppose that $\xi_j \in L^2(\Omega, \mathscr{S}, P)$, $j \geq 1$, are independent random variables each with mean 0, and let $S_n = \sum_{j=1}^n \xi_j$. If

$$P\left(\bigvee_{n=1}^{\infty} |S_n| < \infty\right) > 0 \tag{4}$$

and there is some $\beta \in [0, \infty)$ such that $\bigvee_{j=1}^{\infty} |\xi_j| \leq \beta$ almost surely, then

$$\sum_{j=1}^{\infty} V(\xi_j) < \infty.$$

Proof. By (4), there is some $\alpha \in [0, \infty)$ such that $P(A) > 0$, for

$$A = \left\{\omega \in \Omega : \bigvee_{n=1}^{\infty} |S_n(\omega)| \leq \alpha\right\}.$$

For $p \geq 1$, let

$$A_p = \left\{\omega \in \Omega : \bigvee_{n=1}^{p} |S_n(\omega)| \leq \alpha\right\},$$

which satisfies $A_p \downarrow A$ as $p \to \infty$. For each p, the random variables $\chi_{A_p} S_p$ and ξ_{p+1} are independent and the random variables χ_{A_p} and ξ_{p+1}^2 are independent, whence

$$\begin{aligned}
E(\chi_{A_p} S_{p+1}^2) &= E(\chi_{A_p}(S_p + \xi_{p+1})(S_p + \xi_{p+1})) \\
&= E(\chi_{A_p} S_p^2 + 2\chi_{A_p} S_p \xi_{p+1} + \chi_{A_p} \xi_{p+1}^2) \\
&= E(\chi_{A_p} S_p^2) + 2E(\chi_{A_p} S_p) E(\xi_{p+1}) + E(\chi_{A_p}) E(\xi_{p+1}^2) \\
&= E(\chi_{A_p} S_p^2) + P(A_p) V(\xi_{p+1}) \\
&\geq E(\chi_{A_p} S_p^2) + P(A) V(\xi_{p+1}).
\end{aligned}$$

Set $B_p = A_p \setminus A_{p+1}$. For $\omega \in A_p$, $|S_p(\omega)| \leq \alpha$, and for almost all $\omega \in \Omega$, $|\xi_{p+1}(\omega)| \leq \beta$, so for almost all $\omega \in B_p$,

$$|S_{p+1}(\omega)| \leq |S_p(\omega)| + |\xi_{p+1}(\omega) \leq \alpha + \beta,$$

hence

$$\begin{aligned}
P(A)V(\xi_{p+1}) &\leq E((\chi_{B_p} + \chi_{A_{p+1}}) S_{p+1}^2) - E(\chi_{A_p} S_p^2) \\
&= E(\chi_{B_p} S_{p+1}^2) + E(\chi_{A_{p+1}} S_{p+1}^2) - E(\chi_{A_p} S_p^2) \\
&\leq P(B_p)(\alpha + \beta)^2 + E(\chi_{A_{p+1}} S_{p+1}^2) - E(\chi_{A_p} S_p^2).
\end{aligned}$$

Adding the inequalities for $p = 1, 2, \ldots, n-1$, because B_p are pairwise disjoint,

$$\begin{aligned}
P(A) \sum_{p=1}^{n-1} V(\xi_{p+1}) &= (\alpha + \beta)^2 \sum_{p=1}^{n-1} P(B_p) + E(\chi_{A_n} S_n^2) - E(\chi_{A_1} S_1^2) \\
&\leq (\alpha + \beta)^2 + E(\chi_{A_n} S_n^2) \\
&\leq (\alpha + \beta)^2 + \alpha^2.
\end{aligned}$$

Because this is true for all n and $P(A) > 0$,

$$\sum_{p=1}^{\infty} V(\xi_{p+1}) < \infty,$$

and with $V(\xi_1) < \infty$ this completes the proof. $\square$

4 $\mathbf{R^n}$

If μ is a finite Borel measure on $\mathbb{R}^n$, we define the **characteristic function of** μ by

$$\widetilde{\mu}(y) = \int_{\mathbb{R}^n} e^{i\langle y,x\rangle} d\mu(x), \qquad y \in \mathbb{R}^n.$$

A Borel probability measure γ on $\mathbb{R}^n$ is said to be **Gaussian** if for each $f \in (\mathbb{R}^n)^*$, the pushforward measure $f_*\gamma$ on $\mathbb{R}$ is a Gaussian measure on $\mathbb{R}$, where

$$(f_*\gamma)(E) = \gamma(f^{-1}(E))$$

for E a Borel set in $\mathbb{R}$.

We now give a characterization of Gaussian measures on $\mathbb{R}^n$ and their densities.[6] In the following theorem, the vector $a \in \mathbb{R}^n$ is called the **mean of** γ and the linear transformation $K \in \mathscr{L}(\mathbb{R}^n)$ is called the **covariance operator of** γ. When $a = 0 \in \mathbb{R}^n$ and $K = \mathrm{id}_{\mathbb{R}^n}$, we say that γ is **standard**.

Theorem 5. A Borel probability measure γ on $\mathbb{R}^n$ is Gaussian if and only if there is some $a \in \mathbb{R}^n$ and some positive semidefinite $K \in \mathscr{L}(\mathbb{R}^n)$ such that

$$\widetilde{\gamma}(y) = \exp\left(i\langle y, a\rangle - \frac{1}{2}\langle Ky, y\rangle\right), \qquad y \in \mathbb{R}^n. \tag{5}$$

If γ is a Gaussian measure whose covariance operator K is positive definite, then the density of γ with respect to Lebesgue measure on $\mathbb{R}^n$ is

$$x \mapsto \frac{1}{\sqrt{(2\pi)^n \det K}} \exp\left(-\frac{1}{2}\left\langle K^{-1}(x-a), x-a\right\rangle\right), \qquad x \in \mathbb{R}^n.$$

Proof. Suppose that (5) is satisfied. Let $f \in (\mathbb{R}^n)^*$, i.e. a linear map $\mathbb{R}^n \to \mathbb{R}$, and put $\nu = f_*\gamma$. Using the change of variables formula, the characteristic function of ν is

$$\widetilde{\nu}(t) = \int_{\mathbb{R}} e^{its} d\nu(s) = \int_{\mathbb{R}^n} e^{itf(x)} d\gamma(x), \qquad t \in \mathbb{R}.$$

[6] Vladimir I. Bogachev, *Gaussian Measures*, p. 3, Proposition 1.2.2; Michel Simonnet, *Measures and Probabilities*, p. 303, Theorem 14.5.

Let v be the unique element of $\mathbb{R}^n$ such that $f(x) = \langle v, x\rangle$ for all $x \in \mathbb{R}^n$. Then

$$\widetilde{\nu}(t) = \int_{\mathbb{R}^n} e^{i\langle tv,x\rangle} d\gamma(x) = \widetilde{\gamma}(tv).$$

so by (5),

$$\widetilde{\nu}(t) = \exp\left(i\langle tv, a\rangle - \frac{1}{2}\langle Ktv, tv\rangle\right) = \exp\left(if(a)t - \frac{1}{2}\langle Kv, v\rangle t^2\right).$$

This implies that ν is a Gaussian measure on $\mathbb{R}$ with mean $f(a)$ and variance $\langle Kv, v\rangle$: if $\langle Kv, v\rangle = 0$ then $\nu = \delta_{f(a)}$, and if $\langle Kv, v\rangle > 0$ then ν has density

$$\frac{1}{\sqrt{\langle Kv, v\rangle}\sqrt{2\pi}} \exp\left(-\frac{(s - f(a))^2}{2\langle Kv, v\rangle}\right), \qquad s \in \mathbb{R},$$

with respect to Lebesgue measure on $\mathbb{R}$. That is, for any $f \in (\mathbb{R}^n)^*$, the pushforward measure $f_*\gamma$ is a Gaussian measure on $\mathbb{R}$, which is what it means for γ to be a Gaussian measure on $\mathbb{R}^n$.

Suppose that γ is Gaussian and let $f \in (\mathbb{R}^n)^*$. Then the pushforward measure $f_*\gamma$ is a Gaussian measure on $\mathbb{R}$. Let $a(f)$ be the mean of $f_*\gamma$ and let $\sigma^2(f)$ be the variance of $f_*\gamma$, and let v_f be the unique element of $\mathbb{R}^n$ such that $f(x) = \langle x, v_f\rangle$ for all $x \in \mathbb{R}^n$. Using the change of variables formula,

$$a(f) = \int_{\mathbb{R}} td(f_*\gamma)(t) = \int_{\mathbb{R}^n} f(x)d\gamma(x)$$

and

$$\begin{aligned}\sigma^2(f) &= \int_{\mathbb{R}} (t - a(f))^2 d(f_*\gamma)(t)\\ &= \int_{\mathbb{R}^n} (f(x) - a(f))^2 d\gamma(x)\\ &= \int_{\mathbb{R}^n} (f(x)^2 - 2f(x)a(f) + a(f)^2)d\gamma(x).\end{aligned}$$

Because $f \mapsto a(f)$ is linear $(\mathbb{R}^n)^* \to \mathbb{R}$, there is a unique $a \in \mathbb{R}^n = (\mathbb{R}^n)^{**}$ such that

$$a(f) = \langle v_f, a\rangle, \qquad f \in (\mathbb{R}^n)^*.$$

For $f, g \in (\mathbb{R}^n)^*$,

$$\begin{aligned}\sigma^2(f + g) = \int_{\mathbb{R}^n} &(f(x)^2 + 2f(x)g(x) + g(x)^2\\ &- 2f(x)a(f) - 2f(x)a(g) - 2g(x)a(f) - 2g(x)a(g)\\ &+ a(f)^2 + 2a(f)a(g) + a(g)^2)d\gamma(x),\end{aligned}$$

so

$$\sigma^2(f+g)-\sigma^2(f)-\sigma^2(g)=\int_{\mathbb{R}^n}(2f(x)g(x)-2f(x)a(g)-2g(x)a(f)\\+2a(f)a(g))d\gamma(x).$$

$B(f,g)=\frac{1}{2}(\sigma^2(f+g)-\sigma^2(f)-\sigma^2(g))$ is a symmetric bilinear form on $\mathbb{R}^n$, and

$$B(f,f)=2\int_{\mathbb{R}^n}(f(x)-a(f))^2d\gamma(x)\geq 0,$$

namely, B is positive semidefinite. It follows that there is a unique positive semidefinite $K\in\mathscr{L}(\mathbb{R}^n)$ such that $B(f,g)=\langle Kv_f,v_g\rangle$ for all $f,g\in(\mathbb{R}^n)^*$. For $y\in\mathbb{R}^n$ and for $v_f=y$, using the change of variables formula, using the fact that $f_*\gamma$ is a Gaussian measure on $\mathbb{R}$ with mean

$$a(f)=\langle v_f,a\rangle=\langle y,a\rangle$$

and variance

$$\sigma^2(f)=B(f,f)=\langle Kv_f,v_f\rangle=\langle Ky,y\rangle$$

and using (2),

$$\begin{aligned}\widetilde{\gamma}(y)&=\int_{\mathbb{R}^n}e^{if(x)}d\gamma(x)\\&=\int_{\mathbb{R}}e^{it}d(f_*\gamma)(t)\\&=\exp\left(i\langle y,a\rangle\cdot 1-\frac{1}{2}\langle Ky,y\rangle\cdot 1^2\right)\\&=\exp\left(i\langle y,a\rangle-\frac{1}{2}\langle Ky,y\rangle\right),\end{aligned}$$

which shows that (5) is satisfied.

Suppose that γ is a Gaussian measure and further that the covariance operator K is positive definite. By the spectral theorem, there is an orthonormal basis $\{e_1,\dots,e_n\}$ for $\mathbb{R}^n$ such that $\langle Ke_j,e_j\rangle>0$ for each $1\leq j\leq n$. Write $\langle Ke_j,e_j\rangle=\sigma_j^2$, and for $y\in\mathbb{R}^n$ set $y_j=\langle y,e_j\rangle$, with which $y=y_1e_1+\cdots+y_ne_n$ and then

$$\begin{aligned}\langle Ky,y\rangle&=\langle y_1Ke_1+\cdots+y_nKe_n,y_1e_1+\cdots+y_ne_n\rangle\\&=\left\langle y_1\sigma_1^2e_1+\cdots+y_n\sigma_n^2e_n,y_1e_1+\cdots+y_ne_n\right\rangle\\&=\sigma_1^2y_1^2+\cdots+\sigma_n^2y_n^2.\end{aligned}$$

And

$$\langle y,a\rangle=\langle y_1e_1+\cdots+y_ne_n,a_1e_1+\cdots+a_ne_n\rangle=a_1y_1+\cdots+a_ny_n.$$

Let γ_j be the Gaussian measure on $\mathbb{R}$ with mean a_j and variance σ_j^2. Because $\sigma_j^2 > 0$, the measure γ_j has density $p(\cdot, a_j, \sigma_j^2)$ with respect to Lebesgue measure on $\mathbb{R}$, and thus

$$\begin{aligned}
\widetilde{\gamma}(y) &= \exp\left(i\langle y, a\rangle - \frac{1}{2}\langle Ky, y\rangle\right) \\
&= \exp\left(i\sum_{j=1}^n a_j y_j - \frac{1}{2}\sum_{j=1}^n \sigma_j^2 y_j^2\right) \\
&= \prod_{j=1}^n \exp\left(ia_j y_j - \frac{1}{2}\sigma_j^2 y_j^2\right) \\
&= \prod_{j=1}^n \widetilde{\gamma_j}(y_j) \\
&= \prod_{j=1}^n \int_{\mathbb{R}} \exp(iy_j t)\, d\gamma_j(t) \\
&= \prod_{j=1}^n \int_{\mathbb{R}} \exp(iy_j t) p(t, a_j, \sigma_j^2)\, dt \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^n \exp(iy_j x_j) p(x_j, a_j, \sigma_j^2)\, dx \\
&= \int_{\mathbb{R}^n} e^{i\langle y, x\rangle} \prod_{j=1}^n p(x_j, a_j, \sigma_j^2)\, dx.
\end{aligned}$$

This implies that γ has density

$$x \mapsto \prod_{j=1}^n p(x_j, a_j, \sigma_j^2), \qquad x \in \mathbb{R}^n,$$

with respect to Lebesgue measure on $\mathbb{R}^n$. Moreover,

$$\begin{aligned}
\left\langle K^{-1}(x-a), x-a\right\rangle &= \left\langle \sum_{j=1}^n \sigma_j^{-2}(x_j - a_j)e_j, \sum_{j=1}^n (x_j - a_j)e_j \right\rangle \\
&= \sum_{j=1}^n \frac{(x_j - a_j)^2}{\sigma_j^2},
\end{aligned}$$

so we have, as $\det K = \prod_{j=1}^n \sigma_j^2$,

$$\begin{aligned}
\prod_{j=1}^n p(x_j, a_j, \sigma_j^2) &= \prod_{j=1}^n \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left(-\frac{(x_j - a_j)^2}{2\sigma_j^2}\right) \\
&= \frac{1}{\sqrt{(2\pi)^n \det K}} \exp\left(-\frac{1}{2}\left\langle K^{-1}(x-a), x-a\right\rangle\right).
\end{aligned}$$

$\square$

Because $\mathbb{R}$ is a second-countable topological space, the Borel σ-algebra $\mathscr{B}_{\mathbb{R}^n}$ is equal to the product σ-algebra $\bigotimes_{j=1}^n \mathscr{B}_{\mathbb{R}}$. The density of the standard Gaussian measure γ_n with respect to Lebesgue measure on $\mathbb{R}^n$ is, by Theorem 5,

$$x \mapsto \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\langle x, x\rangle\right), \qquad x \in \mathbb{R}^n.$$

It follows that γ_n is equal to the product measure $\prod_{j=1}^n \gamma_1$, and thus that the probability space $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n}, \gamma_n)$ is equal to the product $\prod_{j=1}^n (\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \gamma_1)$.

For $f_1, \ldots, f_n \in L^2(\gamma_1)$, we define $f_1 \otimes \cdots \otimes f_n \in L^2(\gamma_n)$, called the **tensor product of** $f_1, \ldots, f_n$, by

$$(f_1 \otimes \cdots \otimes f_n)(x) = \prod_{j=1}^n f_j(x_j), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

It is straightforward to check that for $f_1, \ldots, f_n, g_1, \ldots, g_n \in L^2(\gamma_1)$,

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n \rangle_{L^2(\gamma_n)} = \prod_{j=1}^n \langle f_j, g_j \rangle_{L^2(\gamma_1)}.$$

One proves that the linear span of the collection of all tensor products is dense in $L^2(\gamma_n)$, and that $\{v_k : k \geq 0\}$ is an orthonormal basis for $L^2(\gamma_1)$, then

$$\{v_{k_1} \otimes \cdots \otimes v_{k_n} : (k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n\} \tag{6}$$

is an orthonormal basis for $L^2(\gamma_n)$.

We will later use the following statement about centered Gaussian measures.[7]

Theorem 6. Let γ be a Gaussian measure on $\mathbb{R}^n$ with mean 0 and let $\theta \in \mathbb{R}$. Then the pushforward of the product measure $\gamma \times \gamma$ on $\mathbb{R}^n \times \mathbb{R}^n$ under the mapping $(u, v) \mapsto u \sin\theta + v \cos\theta$, $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, is equal to γ.

Proof. Let μ be the pushforward of $\gamma \times \gamma$ under the above mapping. and let $K \in \mathscr{L}(\mathbb{R}^n)$ be the covariance operator of γ. For $y \in \mathbb{R}^n$, using the change of variables formula,

$$\begin{aligned}
\int_{\mathbb{R}^n} \exp(i\langle y, x\rangle)\, d\mu(x) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \exp(i\langle y, u\sin\theta + v\cos\theta\rangle)\, d(\gamma \times \gamma)(u, v) \\
&= \left(\int_{\mathbb{R}^n} \exp(i\langle y\sin\theta, u\rangle)\, d\gamma(u)\right) \\
&\quad \cdot \left(\int_{\mathbb{R}^n} \exp(i\langle y\cos\theta, v\rangle)\, d\gamma(v)\right) \\
&= \widetilde{\gamma}(y\sin\theta)\widetilde{\gamma}(y\cos\theta).
\end{aligned}$$

[7] Vladimir I. Bogachev, *Gaussian Measures*, p. 5, Lemma 1.2.5.

By Theorem 5,

$$\begin{aligned}
\widetilde{\gamma}(y\sin\theta)\widetilde{\gamma}(y\cos\theta) &= \exp\left(-\frac{1}{2}\left\langle Ky\sin\theta, y\sin\theta\right\rangle\right)\exp\left(-\frac{1}{2}\left\langle Ky\cos\theta, y\cos\theta\right\rangle\right) \\
&= \exp\left(-\frac{1}{2}\sin^2(\theta)\left\langle Ky, y\right\rangle - \frac{1}{2}\cos^2(\theta)\left\langle Ky, y\right\rangle\right) \\
&= \exp\left(-\frac{1}{2}\left\langle Ky, y\right\rangle\right).
\end{aligned}$$

Thus, the characteristic function of μ is

$$\widetilde{\mu}(y) = \exp\left(-\frac{1}{2}\left\langle Ky, y\right\rangle\right), \qquad y \in \mathbb{R}^n,$$

which implies that μ is equal to the Gaussian measure with mean 0 and covariance operator K, i.e., $\mu = \gamma$. □

5 Hermite polynomials

For $k \geq 0$, we define the **Hermite polynomial** H_k by

$$H_k(t) = \frac{(-1)^k}{\sqrt{k!}}\exp\left(\frac{t^2}{2}\right)\frac{d^k}{dt^k}\exp\left(-\frac{t^2}{2}\right), \qquad t \in \mathbb{R}.$$

It is apparent that $H_k(t)$ is a polynomial of degree k.

Theorem 7. For real λ and t,

$$\exp\left(\lambda t - \frac{1}{2}\lambda^2\right) = \sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}}H_k(t)\lambda^k.$$

Proof. For $u \in \mathbb{C}$, let $g(u) = \exp\left(-\frac{1}{2}u^2\right)$. For $t \in \mathbb{R}$,

$$\begin{aligned}
g(u) &= \sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{k!}(u-t)^k \\
&= \sum_{k=0}^{\infty}\frac{\sqrt{k!}}{(-1)^k}\exp\left(-\frac{t^2}{2}\right)H_k(t)\frac{1}{k!}(u-t)^k \\
&= \exp\left(-\frac{t^2}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{k!}}H_k(t)(u-t)^k.
\end{aligned}$$

Therefore, for real λ and t,

$$\begin{aligned}\exp\left(\lambda t - \frac{1}{2}\lambda^2\right) &= \exp\left(\frac{1}{2}t^2 - \frac{1}{2}(\lambda - t)^2\right)\\ &= \exp\left(\frac{1}{2}t^2\right) g(\lambda - t)\\ &= \exp\left(\frac{1}{2}t^2\right) g(t - \lambda)\\ &= \exp\left(\frac{1}{2}t^2\right) \exp\left(-\frac{t^2}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k!}} H_k(t)(-\lambda)^k\\ &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} H_k(t)\lambda^k.\end{aligned}$$

□

Theorem 8. Let γ_1 be the standard Gaussian measure on $\mathbb{R}$, with density $p(t,0,1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)$. Then

$$\{H_k : k \geq 0\}$$

is an orthonormal basis for $L^2(\gamma_1)$.

Proof. For $\lambda, \mu \in \mathbb{R}$, on the one hand, using (1) with $a = \lambda + \mu$ and $\sigma = 1$,

$$\begin{aligned}&\int_{\mathbb{R}} \exp\left(\lambda t - \frac{1}{2}\lambda^2\right) \exp\left(\mu t - \frac{1}{2}\mu^2\right) d\gamma_1(t)\\ =&e^{\lambda\mu} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(t - (\lambda + \mu))^2\right) dt\\ =&e^{\lambda\mu}.\end{aligned}$$

On the other hand, using Theorem 7,

$$\begin{aligned}&\int_{\mathbb{R}} \exp\left(\lambda t - \frac{1}{2}\lambda^2\right) \exp\left(\mu t - \frac{1}{2}\mu^2\right) d\gamma_1(t)\\ =&\int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} H_k(t)\lambda^k\right) \left(\sum_{l=0}^{\infty} \frac{1}{\sqrt{l!}} H_l(t)\mu^l\right) d\gamma_1(t)\\ =&\int_{\mathbb{R}} \sum_{k,l\geq 0} \frac{1}{\sqrt{k!l!}} \lambda^k \mu^l H_k(t) H_l(t) d\gamma_1(t)\\ =&\sum_{k,l\geq 0} \frac{1}{\sqrt{k!l!}} \lambda^k \mu^l \langle H_k, H_l \rangle_{L^2(\gamma_1)}.\end{aligned}$$

Therefore

$$\sum_{k,l\geq 0} \frac{1}{\sqrt{k!l!}} \lambda^k \mu^l \langle H_k, H_l \rangle_{L^2(\gamma_1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \mu^k.$$

From this, we get that if $k \neq l$ then $\frac{1}{\sqrt{k!l!}} \langle H_k, H_l \rangle_{L^2(\gamma_1)} = 0$, i.e.

$$\langle H_k, H_l \rangle_{L^2(\gamma_1)} = 0.$$

If $k = l$, then $\frac{1}{\sqrt{k!l!}} \langle H_k, H_l \rangle_{L^2(\gamma_1)} = \frac{1}{k!}$, i.e.

$$\langle H_k, H_k \rangle_{L^2(\gamma_1)} = 1.$$

Therefore, $\{H_k : k \geq 0\}$ is an orthonormal set in $L^2(\gamma_1)$.

Suppose that $f \in L^2(\gamma_1)$ satisfies $\langle f, H_k \rangle_{L^2(\gamma_1)} = 0$ for each $k \geq 0$. Because $H_k(t)$ is a polynomial of degree k, for each $k \geq 0$ we have

$$\operatorname{span}\{H_0, H_1, H_2, \ldots, H_k\} = \operatorname{span}\{1, t, t^2, \ldots, t^k\}.$$

Hence for each $k \geq 0$, $\langle f, t^k \rangle_{L^2(\gamma_1)} = 0$. One then proves that $\operatorname{span}\{1, t, t^2, \ldots\}$ is dense in $L^2(\gamma_1)$, from which it follows that the linear span of the Hermite polynomials is dense in $L^2(\gamma_1)$ and thus that they are an orthonormal basis. □

Lemma 9. For $k \geq 1$,

$$H_k'(t) = \sqrt{k} H_{k-1}(t), \qquad H_k'(t) = t H_k(t) - \sqrt{k+1} H_{k+1}(t).$$

Proof. Theorem 7 says

$$\exp\left(\lambda t - \frac{1}{2}\lambda^2\right) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} H_k(t) \lambda^k.$$

On the one hand,

$$\begin{aligned} \frac{d}{dt} \exp\left(\lambda t - \frac{1}{2}\lambda^2\right) &= \lambda \exp\left(\lambda t - \frac{1}{2}\lambda^2\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} H_k(t) \lambda^{k+1} \\ &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{(k-1)!}} H_{k-1}(t) \lambda^k. \end{aligned}$$

On the other hand,

$$\frac{d}{dt} \exp\left(\lambda t - \frac{1}{2}\lambda^2\right) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} H_k'(t) \lambda^k.$$

Therefore, $H_0'(t) = 0$, and for $k \geq 1$,

$$\frac{1}{\sqrt{(k-1)!}} H_{k-1}(t) = \frac{1}{\sqrt{k!}} H_k'(t),$$

i.e.,

$$H_k'(t) = \sqrt{k} H_{k-1}(t).$$

□

For $\alpha = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$, we define the **Hermite polynomial** H_α by

$$H_\alpha(x) = H_{k_1}(x_1) \cdots H_{k_n}(x_n), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Because the collection of all Hermite polynomials H_k is an orthonormal basis for the Hilbert space $L^2(\gamma_1)$, following (6) we have that the collection of all Hermite polynomials H_α is an orthonormal basis for the Hilbert space $L^2(\gamma_n)$.

Theorem 10. For γ_n the standard Gaussian measure on $\mathbb{R}^n$, with mean $0 \in \mathbb{R}^n$ and covariance operator $\mathrm{id}_{\mathbb{R}^n}$, the collection

$$\{H_\alpha : \alpha \in \mathbb{Z}^n_{\geq 0}\}$$

is an orthonormal basis for $L^2(\gamma_n)$.

For $\alpha = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$, write $|\alpha| = k_1 + \cdots + k_n$. For $k \geq 0$, we define

$$\mathcal{X}_k = \mathrm{span}\{H_\alpha : |\alpha| = k\},$$

which is a subspace of $L^2(\gamma_n)$ of dimension

$$\binom{k+n-1}{k}.$$

As $\mathcal{X}_k$ is a finite dimensional subspace of $L^2(\gamma_n)$, it is closed. $L^2(\gamma_n)$ is equal to the orthogonal direct sum of the $\mathcal{X}_k$:

$$L^2(\gamma_n) = \bigoplus_{k=0}^\infty \mathcal{X}_k.$$

Let

$$I_k : L^2(\gamma_n) \to \mathcal{X}_k$$

be the orthogonal projection onto $\mathcal{X}_k$.

6 Ornstein-Uhlenbeck semigroup

Let γ be a Gaussian measure on $\mathbb{R}^n$ with mean 0 and covariance operator K. For $t \geq 0$, we define $M_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$M_t(u, v) = e^{-t}u + \sqrt{1 - e^{-2t}}v, \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By Theorem 6, $M_{t*}(\gamma \times \gamma) = \gamma$. Therefore, for $p \geq 1$ and $f \in L^p(\gamma)$, using the change of variables formula,

$$\int_{\mathbb{R}^n} |f(x)|^p d\gamma(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(M_t(u, v))|^p d(\gamma \times \gamma)(u, v).$$

Applying Fubini's theorem, the function

$$u \mapsto \int_{\mathbb{R}^n} |f(M_t(u,v))|^p d\gamma(v) = \int_{\mathbb{R}^n} |f(e^{-t}u + \sqrt{1-e^{-2t}}v)|^p d\gamma(v)$$

belongs to $L^1(\gamma)$. We define the **Ornstein-Uhlenbeck semigroup** $\{T_t : t \geq 0\}$ **on** $L^p(\gamma)$, $p \geq 1$, by

$$T_t(f)(u) = \int_{\mathbb{R}^n} f(M_t(u,v))d\gamma(v) = \int_{\mathbb{R}^n} f\left(e^{-t}u + \sqrt{1-e^{-2t}}v\right) d\gamma(v),$$

for $u \in \mathbb{R}^n$.

Theorem 11. Let γ be a Gaussian measure on $\mathbb{R}^n$ with mean 0. If $f \in L^1(\gamma)$, then

$$\int_{\mathbb{R}^n} (T_t f)(x) d\gamma(x) = \int_{\mathbb{R}^n} f(x) d\gamma(x).$$

Proof. Using Fubini's theorem, then the change of variables formula, then Theorem 6,

$$\begin{aligned}
\int_{\mathbb{R}^n} (T_t f)(u) d\gamma(u) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(M_t(u,v)) d\gamma(v) \right) d\gamma(u) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(M_t(u,v)) d(\gamma \times \gamma)(u,v) \\
&= \int_{\mathbb{R}^n} f(x) d(M_{t*}(\gamma \times \gamma))(x) \\
&= \int_{\mathbb{R}^n} f(x) d\gamma(x).
\end{aligned}$$

□

Theorem 12. Let γ be a Gaussian measure on $\mathbb{R}^n$ with mean 0. For $p \geq 1$ and $t \geq 0$, T_t is a bounded linear operator $L^p(\gamma) \to L^p(\gamma)$ with operator norm 1.

Proof. For $f \in L^p(\gamma)$, using Jensen's inequality and then Theorem 11,

$$\begin{aligned}
\|T_t f\|_{L^p(\gamma)}^p &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(M_t(u,v)) d\gamma(v) \right|^p d\gamma(u) \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(M_t(u,v))|^p d\gamma(v) \right) d\gamma(u) \\
&= \int_{\mathbb{R}^n} T_t(|f|^p)(u) d\gamma(u) \\
&= \int_{\mathbb{R}^n} |f|^p(u) d\gamma(u) \\
&= \|f\|_{L^p(\gamma)}^p,
\end{aligned}$$

i.e. $\|T_t f\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)}$. This shows that the operator norm of T_t is ≤ 1. But, as γ is a probability measure,

$$T_t 1 = \int_{\mathbb{R}^n} 1 d\gamma(v) = 1,$$

so T_t has operator norm 1. $\square$

For a Banach space E, we denote by $\mathscr{B}(E)$ the set of bounded linear operators $E \to E$. The **strong operator topology on** E is the coarsest topology on E such that for each $x \in E$, the map $A \mapsto Ax$ is continuous $\mathscr{B}(E) \to \mathbb{E}$. To say that a map $Q : [0, \infty) \to \mathscr{B}(E)$ is **strongly continuous means** that for each $t \in [0, \infty)$, $Q(s) \to Q_t$ in the strong operator topology as $s \to t$, i.e., for each $x \in E$, $Q(s)x \to Q(t)x$ in E.

A **one-parameter semigroup in** $\mathscr{B}(E)$ is a map $Q : [0, \infty) \to \mathscr{B}(E)$ such that (i) $Q(0) = \mathrm{id}_E$ and (ii) for $s, t \ge 0$, $Q(s + t) = Q(s) \circ Q(t)$. For a one-parameter semigroup to be strongly continuous, one proves that it is equivalent that $Q(t) \to \mathrm{id}_E$ in the strong operator topology as $t \downarrow 0$, i.e. for each $x \in E$, $Q(t)x \to x$.[8]

We now establish that the $\{T_t : t \ge 0\}$ is indeed a one-parameter semigroup and that it is strongly continuous.[9]

Theorem 13. Suppose μ is a Gaussian measure on $\mathbb{R}^n$ with mean 0 and let $p \ge 1$. Then $\{T_t : t \ge 0\}$ is a strongly continuous one-parameter semigroup in $\mathscr{B}(L^p(\gamma))$.

Proof. For $f \in L^p(\gamma)$, because γ is a probability measure,

$$T_0(f)(u) = \int_{\mathbb{R}^n} f(u) d\gamma(v) = f(u),$$

hence $T_0 = \mathrm{id}_{L^p(\mu)}$. For $s, t \ge 0$, define $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(u, v) = e^{-s} \frac{\sqrt{1 - e^{-2t}}}{\sqrt{1 - e^{-2t-2s}}} u + \frac{\sqrt{1 - e^{-2s}}}{\sqrt{1 - e^{-2t-2s}}} v.$$

[8]Walter Rudin, *Functional Analysis*, second ed., p. 376, Theorem 13.35.
[9]Vladimir I. Bogachev, *Gaussian Measures*, p. 10, Theorem 1.4.1.

By Theorem 6, $P_*(\gamma \times \gamma) = \gamma$, whence

$$\begin{aligned}
&(T_t(T_s f))(x) \\
=&\int_{\mathbb{R}^n} (T_s f)\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y) \\
=&\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f\left(e^{-s}\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) + \sqrt{1-e^{-2s}}w\right) d\gamma(w)\right) d\gamma(y) \\
=&\int_{\mathbb{R}^n \times \mathbb{R}^n} f\left(e^{-s-t}x + \sqrt{1-e^{-2t-2s}}P(y,w)\right) d(\gamma \times \gamma)(y,w) \\
=&\int_{\mathbb{R}^n \times \mathbb{R}^n} (f \circ M_{s+t})(x, P(y,w)) d(\gamma \times \gamma)(y,w) \\
=&\int_{\mathbb{R}^n} (f \circ M_{s+t})(x,z) d\gamma(z) \\
=&T_{s+t}(f)(x),
\end{aligned}$$

hence $T_t \circ T_s = T_{s+t}$. This establishes that $\{T_t : t \geq 0\}$ is a semigroup.

For $f \in C_b(\mathbb{R}^n)$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^n$, as $t \downarrow 0$ we have

$$f\left(e^{-t}u + \sqrt{1-e^{-2t}}v\right) - f(u) \to 0,$$

thus by the dominated convergence theorem, since

$$\left|f\left(e^{-t}u + \sqrt{1-e^{-2t}}v\right) - f(u)\right| \leq 2\|f\|_\infty$$

and γ is a probability measure, we have

$$\int_{\mathbb{R}^n} \left(f\left(e^{-t}u + \sqrt{1-e^{-2t}}v\right) - f(u)\right) d\gamma(v) \to 0,$$

and hence

$$\begin{aligned}
(T_t f - T_0 f)(u) &= \int_{\mathbb{R}^n} f\left(e^{-t}u + \sqrt{1-e^{-2t}}v\right) d\gamma(v) - \int_{\mathbb{R}^n} f(u) d\gamma(v) \\
&= \int_{\mathbb{R}^n} \left(f\left(e^{-t}u + \sqrt{1-e^{-2t}}v\right) - f(u)\right) d\gamma(v) \\
&\to 0.
\end{aligned}$$

Because this is true for each $u \in \mathbb{R}^n$ and

$$|(T_t f - T_0 f)(u)| \leq \int_{\mathbb{R}^n} 2\|f\|_\infty d\gamma(v) = 2\|f\|_\infty,$$

by the dominated convergence theorem we then have

$$\|T_t f - T_0 f\|_{L^p(\gamma)} \to 0. \tag{7}$$

Now let $f \in L^p(\gamma)$. There is a sequence $f_j \in C_b(\mathbb{R}^n)$ satisfying $\|f_j - f\|_{L^p(\gamma)} \to 0$, with $\|f_j\|_{L^p(\gamma)} \leq 2\|f\|_{L^p(\gamma)}$ for all j. For any $t \geq 0$,

$$\begin{aligned}
\|T_t f - T_0 f\|_{L^p(\gamma)} &\leq \|T_t f - T_t f_j\|_{L^p(\gamma)} + \|T_t f_j - T_0 f_j\|_{L^p(\gamma)} + \|T_0 f_j - T_0 f\|_{L^p(\gamma)} \\
&= \|T_t(f - f_j)\|_{L^p(\gamma)} + \|T_t f - T_0 f_j\|_{L^p(\gamma)} + \|f_j - f\|_{L^p(\gamma)} \\
&\leq \|f - f_j\|_{L^p(\gamma)} + \|T_t f - T_0 f_j\|_{L^p(\gamma)} + \|f_j - f\|_{L^p(\gamma)}.
\end{aligned}$$

Let $\epsilon > 0$ and let j be so large that $\|f - f_j\|_{L^p(\gamma)} < \epsilon$. Because $f_j \in C_b(\mathbb{R}^n)$, by (7) there is some $\delta > 0$ such that when $0 < t < \delta$, $\|T_t f_j - f_j\|_{L^p(\gamma)} < \epsilon$. Then when $0 < t < \delta$,

$$\|T_t f - T_0 f\|_{L^p(\gamma)} \leq \epsilon + \epsilon + \epsilon,$$

which shows that for each $f \in L^p(\gamma)$, $\|T_t f - T_0 f\|_{L^p(\gamma)}$ as $t \downarrow 0$, which suffices to establish that $\{T_t : t \geq 0\}$ is strongly continuous $[0, \infty) \to \mathscr{B}(L^p(\gamma))$. $\Box$

For $t > 0$, we define $L_t \in \mathscr{B}(L^p(\gamma))$ by

$$L_t f = \frac{1}{t}(T_t f - f), \qquad f \in L^p(\gamma).$$

We define $\mathscr{D}(L)$ to be the set of those $f \in L^p(\gamma)$ such that $L_t f$ converges to some element of $L^p(\gamma)$ as $t \downarrow 0$, and we define $L : \mathscr{D}(L) \to L^p(\gamma)$. This is the **infinitesimal generator** of the semigroup $\{T_t : t \geq 0\}$, and the infinitesimal generator L of the Ornstein-Uhlenbeck semigroup is called the **Ornstein-Uhlenbeck operator**. Because the Ornstein-Uhlenbeck semigroup is strongly continuous, we get the following.[10]

Theorem 14. Suppose μ is a Gaussian measure on $\mathbb{R}^n$ with mean 0, let $p \geq 1$, and let L be the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T_t : t \geq 0\}$. Then:

1. $\mathscr{D}(L)$ is a dense linear subspace of $L^p(\gamma)$ and $L : \mathscr{D}(L) \to L^p(\gamma)$ is a closed operator.

2. For each $f \in \mathscr{D}(L)$ and for each $t \geq 0$,

$$\frac{d}{dt}(T_t f) = (L \circ T_t) f = (T_t \circ L) f.$$

3. For $f \in L^p(\gamma)$ and K a compact subset of $[0, \infty)$, $(\exp(tL_\epsilon) f \to T_t f$ as $\epsilon \downarrow 0$ uniformly for $t \in K$.

4. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, $R(\lambda) : L^p(\gamma) \to L^p(\gamma)$ defined by

$$R(\lambda) f = \int_0^\infty e^{-\lambda t} T_t f dt, \qquad f \in L^p(\gamma),$$

[10] Walter Rudin, *Functional Analysis*, second ed., p. 376, Theorem 13.35.

belongs to $\mathscr{B}(L^p(\gamma))$, the range of $R(\lambda)$ is equal to $\mathscr{D}(L)$, and

$$((\lambda I - L) \circ R(\lambda))f = f, \quad f \in L^p(\gamma), \qquad (R(\lambda) \circ (\lambda I - L))f = \mathscr{D}(L),$$

where I is the identity operator on $L^p(\gamma)$.

We remind ourselves that if H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, an element A of $\mathscr{B}(H)$ is said to be a **positive operator** when $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We prove that each T_t is a positive operator on the Hilbert space $L^2(\gamma)$.[11]

Theorem 15. Suppose μ is a Gaussian measure on $\mathbb{R}^n$ with mean 0. For each $t \geq 0$, $T_t \in \mathscr{B}(L^2(\mu))$ is a positive operator.

Proof. For $t \geq 0$, define $N_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$O_t(x, y) = \left(e^{-t}x + \sqrt{1 - e^{-2t}}y, -\sqrt{1 - e^{-2t}}x + e^{-t}y\right), \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

whose transpose is the linear operator $N_t^* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$O_t^*(u, v) = \left(e^{-t}u - \sqrt{1 - e^{-2t}}v, \sqrt{1 - e^{-2t}}u + e^{-t}v\right), \qquad (u, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle (x,y),(u,v)\rangle} d(O_{t*}(\gamma \times \gamma))(u, v) \\
=&\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle (x,y),O_t(u,v)\rangle} d(\gamma \times \gamma)(u, v) \\
=&\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle O_t^*(x,y),(u,v)\rangle} d(\gamma \times \gamma)(u, v) \\
=&\widetilde{\gamma \times \gamma}(O_t^*(x, y)) \\
=&\widetilde{\gamma \times \gamma}(e^{-t}x - \sqrt{1 - e^{-2t}}y, \sqrt{1 - e^{-2t}}x + e^{-t}y) \\
=&\widetilde{\gamma}(e^{-t}x - \sqrt{1 - e^{-2t}}y)\widetilde{\gamma}(\sqrt{1 - e^{-2t}}x + e^{-t}y) \\
=&\exp\left(-\frac{1}{2}\left\langle K(e^{-t}x - \sqrt{1 - e^{-2t}}y), e^{-t}x - \sqrt{1 - e^{-2t}}y\right\rangle\right) \\
&\cdot \exp\left(-\frac{1}{2}\left\langle K(\sqrt{1 - e^{-2t}}x + e^{-t}y), \sqrt{1 - e^{-2t}}x + e^{-t}y\right\rangle\right) \\
=&\exp\left(-\frac{1}{2}\langle Kx, x\rangle - \frac{1}{2}\langle Ky, y\rangle\right) \\
=&\widetilde{\gamma}(x)\widetilde{\gamma}(y) \\
=&\widetilde{\gamma \times \gamma}(x, y),
\end{aligned}$$

[11] Vladimir I. Bogachev, *Gaussian Measures*, p. 10, Theorem 1.4.1.

which shows that $O_{t*}(\gamma \times \gamma)$ and $\gamma \times \gamma$ have equal characteristic functions and hence are themselves equal.

For $f, g \in L^2(\gamma)$ and $t \geq 0$,

$$\begin{aligned}
\langle T_t f, g\rangle_{L^2(\gamma)} &= \int_{\mathbb{R}^n} (T_t f)(x) g(x) d\mu(x) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) g(x) d(\gamma \times \gamma)(x,y) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} (f \circ \pi_1 \circ O_t)(x,y)(g \circ \pi_1 \circ O_t^{-1} \circ O_t)(x,y) d(\gamma \times \gamma)(x,y) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} (f \circ \pi_1)(u,v)(g \circ \pi_1 \circ O_t^{-1})(u,v) d(O_{t*}(\gamma \times \gamma))(u,v) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} (f \circ \pi_1)(u,v)(g \circ \pi_1 \circ O_t^{-1})(u,v) d(\gamma \times \gamma)(u,v) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(u) g\left(e^{-t}u - \sqrt{1-e^{-2t}}v\right) d(\gamma \times \gamma)(u,v) \\
&= \int_{\mathbb{R}^n} f(u) \left(\int_{\mathbb{R}^n} g(M_t(u,-v)) d\gamma(v)\right) d\gamma(u) \\
&= \int_{\mathbb{R}^n} f(u) \left(\int_{\mathbb{R}^n} g(M_t(u,v)) d\gamma(v)\right) d\gamma(u) \\
&= \int_{\mathbb{R}^n} f(u)(T_t g)(u) d\gamma(u) \\
&= \langle f, T_t g\rangle_{L^2(\gamma)},
\end{aligned}$$

which establishes that T_t is a self-adjoint operator on $L^2(\gamma)$.

Furthermore, using that $T_t = T_{t/2} \circ T_{t/2}$ and that $T_{t/2}$ is self-adjoint,

$$\langle T_t f, f\rangle_{L^2(\gamma)} = \left\langle T_{t/2}T_{t/2}f, f\right\rangle_{L^2(\gamma)} = \left\langle T_{t/2}f, T_{t/2}^* f\right\rangle_{L^2(\gamma)} = \left\langle T_{t/2}f, T_{t/2}f\right\rangle_{L^2(\gamma)},$$

which is ≥ 0, which establishes that T_t is a positive operator on $L^2(\gamma)$. □

We now write the Ornstein-Uhlenbeck semigroup using the orthogonal projections $I_k : L^2(\gamma_n) \to \mathcal{X}_k$, where γ_n is the standard Gaussian measure on $\mathbb{R}^n$.[12]

Theorem 16. For each $t \geq 0$ and $f \in L^2(\gamma_n)$,

$$T_t f = \sum_{k=0}^{\infty} e^{-kt} I_k(f).$$

Proof. Define $S_t : L^2(\gamma_n) \to L^2(\gamma_n)$ by $S_t f = \sum_{k=0}^{\infty} e^{-kt} I_k(f)$, which satisfies, using that the subspaces $\mathcal{X}_k$ are pairwise orthogonal,

$$\|S_t f\|^2_{L^2(\gamma_n)} = \sum_{k=0}^{\infty} e^{-kt} \|I_k(f)\|^2_{L^2(\gamma_n)} \leq \sum_{k=0}^{\infty} \|I_k(f)\|^2_{L^2(\gamma_n)} = \|f\|^2_{L^2(\gamma_n)},$$

[12] Vladimir I. Bogachev, *Gaussian Measures*, p. 11, Theorem 1.4.4.

so $S_t \in \mathscr{B}(L^2(\gamma_n))$. To prove that $T_t = S_t$, it suffices to prove that $T_t H_\alpha = S_t H_\alpha$ for each Hermite polynomial, which are an orthonormal basis for $L^2(\gamma_n)$. For $\alpha = (k_1, \dots, k_n)$ with $k = |\alpha| = k_1 + \cdots + k_n$,

$$S_t H_\alpha = e^{-kt} H_\alpha,$$

and

$$\begin{aligned}(T_t H_\alpha)(x) &= \int_{\mathbb{R}^n} H_\alpha\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma_n(y) \\ &= \int_{\mathbb{R}^n} \prod_{j=1}^n H_{k_j}\left(e^{-t}x_j + \sqrt{1-e^{-2t}}y_j\right) d\gamma_n(y) \\ &= \prod_{j=1}^n \int_{\mathbb{R}} H_{k_j}\left(e^{-t}x_j + \sqrt{1-e^{-2t}}y_j\right) d\gamma_1(y_j).\end{aligned}$$

To prove that $T_t H_\alpha = e^{-kt} H_\alpha$, it thus suffices to prove that for any t, for any k_j, and for any x_j,

$$\int_{\mathbb{R}} H_{k_j}\left(e^{-t}x_j + \sqrt{1-e^{-2t}}y_j\right) d\gamma_1(y_j) = e^{-k_j t} H_{k_j}(x_j). \tag{8}$$

For $k_j = 0$, as $H_0 = 1$ and γ_1 is a probability measure, (8) is true. Suppose that (8) is true for $\leq k_j$. That is, for each $0 \leq h \leq k_j$, $T_t H_h = e^{-ht} H_h$. For any l, because the Hermite polynomial H_l is a polynomial of degree l, one checks that $T_t H_l(x_j)$ is a polynomial of degree l: using the binomial formula,

$$\int_{\mathbb{R}} (e^{-t}x_j + \sqrt{1-e^{-2t}}y_j)^l \exp\left(-\frac{y_j^2}{2}\right) d\gamma_1(y_j)$$

is a polynomial in x_j of degree l. Hence $T_t H_l$ a linear combination of $H_0, H_1, \dots, H_l$. For $0 \leq h \leq k_j$,

$$\left\langle T_t H_{k_j+1}, H_h \right\rangle_{L^2(\gamma_1)} = \left\langle H_{k_j+1}, T_t H_h \right\rangle_{L^2(\gamma_1)} = \left\langle H_{k_j+1}, e^{-ht} H_h \right\rangle_{L^2(\gamma_1)} = 0.$$

Therefore there is some $c \in \mathbb{R}$ such that $T_t H_{k_j+1} = c H_{k_j+1}$. Then check that $c = e^{-(k_j+1)t}$. □

We now give an explicit expression for the domain $\mathscr{D}(L)$ of the Ornstein-Uhlenbeck operator L and for L applied to an element of its domain.[13]

Theorem 17.

$$\mathscr{D}(L) = \left\{ f \in L^2(\gamma_n) : \sum_{k=0}^\infty k^2 \left\| I_k(f) \right\|^2_{L^2(\gamma_n)} < \infty \right\}.$$

For $f \in \mathscr{D}(L)$,

$$Lf = -\sum_{k=0}^\infty k I_f(f).$$

[13] Vladimir I. Bogachev, *Gaussian Measures*, p. 12, Proposition 1.4.5.

Proof. Let $f \in \mathscr{D}(L)$, i.e. $\frac{T_t f - f}{t} \to Lf$ in $L^2(\gamma_n)$ as $t \downarrow 0$. For any $k \geq 0$, using Theorem 16,

$$\begin{aligned}
I_k L f &= I_k \left(\lim_{t \downarrow 0} \frac{T_t f - f}{t} \right) \\
&= \lim_{t \downarrow 0} \frac{I_k T_t f - I_k f}{t} \\
&= \lim_{t \downarrow 0} \frac{T_t I_k f - I_k f}{t} \\
&= \lim_{t \downarrow 0} \frac{e^{-kt} I_k f - I_k f}{t} \\
&= \left(\lim_{t \downarrow 0} \frac{e^{-kt} - 1}{t} \right) I_k f \\
&= \left(e^{-kt}\right)' \big|_{t=0} I_k f \\
&= -k I_k f.
\end{aligned}$$

Using this,

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 \left\| I_k f \right\|_{L^2(\gamma_n)}^2 &= \sum_{k=0}^{\infty} \left\| I_k L f \right\|_{L^2(\gamma_n)}^2 \\
&= \left\| \sum_{k=0}^{\infty} I_k L f \right\|_{L^2(\gamma_n)}^2 \\
&= \left\| L f \right\|_{L^2(\gamma_n)}^2 \\
&< \infty.
\end{aligned}$$

Moreover,

$$Lf = L \left(\sum_{k=0}^{\infty} I_k f \right) = \sum_{k=0}^{\infty} L I_k f = \sum_{k=0}^{\infty} I_k L f = \sum_{k=0}^{\infty} -k I_f.$$

Let $f \in L^2(\gamma_n)$ satisfy

$$\sum_{k=0}^{\infty} k^2 \left\| I_k f \right\|_{L^2(\gamma_n)}^2 < \infty.$$

For $t > 0$,

$$\begin{aligned}
\left\| \frac{T_t f - f}{t} + \sum_{k=0}^{\infty} k I_k f \right\|_{L^2(\gamma_n)}^2 &= \left\| \sum_{k=0}^{\infty} \left(\frac{e^{-kt} I_k f - I_k f}{t} + k I_k f \right) \right\|_{L^2(\gamma_n)}^2 \\
&= \sum_{k=0}^{\infty} \left| \frac{e^{-kt} - 1}{t} + k \right|^2 \left\| I_k f \right\|_{L^2(\gamma_n)}^2 .
\end{aligned}$$

For $t > 0$ and $k \geq 0$,

$$|t^{-1}(e^{-kt} - 1)| \leq k,$$

and thus

$$\sum_{k=0}^{\infty} \left| \frac{e^{-kt} - 1}{t} + k \right|^2 \|I_k f\|^2_{L^2(\gamma_n)} \leq \sum_{k=0}^{\infty} (2k)^2 \|I_k f\|^2_{L^2(\gamma_n)} < \infty.$$

For each $k \geq 0$, as $t \downarrow 0$,

$$\frac{e^{-kt} - 1}{t} + k \to 0,$$

thus as $t \downarrow 0$,

$$\sum_{k=0}^{\infty} \left| \frac{e^{-kt} - 1}{t} + k \right|^2 \|I_k f\|^2_{L^2(\gamma_n)} \to 0$$

and hence

$$\left\| \frac{T_t f - f}{t} + \sum_{k=0}^{\infty} k I_k f \right\|^2_{L^2(\gamma_n)} \to 0.$$

This means that $\frac{T_t f - f}{t}$ converges in $L^2(\gamma_n)$ to $-\sum_{k=0}^{\infty} k I_k f$ as $t \downarrow 0$, and since $\frac{T_t f - f}{t}$ converges, $f \in \mathscr{D}(L)$. $\square$