The inclusion map from the integers to the reals and universal properties of the floor and ceiling functions

Jordan Bell

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1 Categories

If X is a set, by a partial order on X we mean a binary relation $\leq$ on X that is reflexive, antisymmetric, and transitive, and we call $(X, \leq)$ a **poset**. If $(X, \leq)$ is a poset, we define it to be a category whose objects are the elements of X, and for $x, y \in X$,

$$\mathrm{Hom}(x, y) = \begin{cases} \{(x, y)\} & x \leq y \\ \emptyset & \neg(x \leq y). \end{cases}$$

In particular, $\mathrm{id}_x = (x, x)$.

Let $U : \mathbb{Z} \to \mathbb{R}$ be the inclusion map. If $(j, k) \in \mathrm{Hom}(j, k)$, define $U(j, k) = (Uj, Uk) \in \mathrm{Hom}(Uj, Uk)$.

$$U\mathrm{id}_j = U(j, j) = (Uj, Uj) = \mathrm{id}_{Uj}.$$

If $(j, k) \in \mathrm{Hom}(j, k)$ and $(k, l) \in \mathrm{Hom}(k, l)$, then $(k, l) \circ (j, k) = (j, l)$ and

$$U(k, l) \circ U(j, k) = (Uk, Ul) \circ (Uj, Uk) = (Uj, Ul) = U(j, l) = U((j, l) \circ (j, k)).$$

This shows that $U : (\mathbb{Z}, \leq) \to (\mathbb{R}, \leq)$ is a functor.

2 Galois connections

If $(A, \leq)$ and $(B, \leq)$ are posets, a function $G : A \to B$ is said to be **order-preserving** if $a \leq a'$ implies $G(a) \leq G(a')$. A **Galois connection from A to B** is an order-preserving function $G : A \to B$ and an order-preserving function $H : B \to A$ such that

$$G(a) \leq b \text{ if and only if } a \leq H(b), \qquad a \in A, \quad b \in B.$$

We say that G is the **left-adjoint of H** and that H is the **right-adjoint of G**.

Let $I : \mathbb{Z} \to \mathbb{R}$ be the inclusion map. Define $F : \mathbb{R} \to \mathbb{Z}$ by $F(x) = \lfloor x \rfloor$. For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, suppose $I(n) \leq x$. Then $F(I(n)) \leq F(x)$. But $F(I(n)) = n$, so $n \leq F(x)$. Suppose $n \leq F(x)$. Then $I(n) \leq I(F(x)) \leq x$. Therefore $F : \mathbb{R} \to \mathbb{Z}$, $F(x) = \lfloor x \rfloor$ is the right-adjoint of $I : \mathbb{Z} \to \mathbb{R}$:[1]

$$I(n) \leq x \iff n \leq F(x), \qquad n \in \mathbb{Z}, \quad x \in \mathbb{R}.$$

Define $C : \mathbb{R} \to \mathbb{Z}$ by $C(x) = \lceil x \rceil$. For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, suppose $C(x) \leq n$. Then $I(C(x)) \leq I(n)$. But $I(C(x)) \geq x$, so $x \leq I(n)$. Suppose $x \leq I(n)$. Then $C(x) \leq C(I(n))$. But $C(I(n)) = n$, so $C(x) \leq n$. Therefore $C : \mathbb{R} \to \mathbb{Z}$, $C(x) = \lceil x \rceil$ is the left-adjoint of $I : \mathbb{Z} \to \mathbb{R}$:

$$C(x) \leq n \iff x \leq I(n), \qquad x \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

Lemma 1. *For $x \geq 0$,*

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor.$$

Proof. For $k \in \mathbb{Z}_{\geq 0}$ and $y \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} k \leq \lfloor \sqrt{\lfloor y \rfloor} \rfloor &\iff I(k) \leq \sqrt{\lfloor y \rfloor} \\ &\iff k^2 \leq \lfloor y \rfloor \\ &\iff k^2 \leq y \\ &\iff k \leq \sqrt{y} \\ &\iff k \leq \lfloor \sqrt{y} \rfloor. \end{aligned}$$

□

Lemma 2. *If $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 1}$, then*

$$\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor.$$

Proof. For $k \in \mathbb{Z}$,

$$\begin{aligned} k \leq F(I(F(x))/I(n)) &\iff I(k) \leq I(F(x))/I(n) \\ &\iff I(k)I(n) \leq I(F(x)) \\ &\iff I(kn) \leq I(F(x)) \\ &\iff kn \leq F(x) \\ &\iff I(kn) \leq x \\ &\iff I(k) \leq x/I(n) \\ &\iff k \leq F(x/I(n)). \end{aligned}$$

This means that $F(I(F(x))/I(n)) = F(x/I(n))$. □

[1] See Roland Backhouse, *Galois Connections and Fixed Point Calculus*, `http://www.cs.nott.ac.uk/~psarb2/G53PAL/FPandGC.pdf`, p. 14; Samson Abramsky and Nikos Tzevelekos, *Introduction to Categories and Categorical Logic*, `http://arxiv.org/abs/1102.1313`, p. 44, §1.5.1.

Lemma 3. *If $n \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}$, then*

$$\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor .$$

Proof. For $k \in \mathbb{Z}$,

$$\begin{aligned} k \leq F(I(m+n-1)/I(n)) &\iff I(k) \leq I(m+n-1)/I(n) \\ &\iff I(k)I(n) \leq I(m+n-1) \\ &\iff kn \leq m+n-1 \\ &\iff kn-n+1 \leq m \\ &\iff kn-n < m \\ &\iff I(k-1) < I(m)/I(n) \\ &\iff k-1 < C(I(m)/I(n)) \\ &\iff k \leq C(I(m)/I(n)). \end{aligned}$$

This means

$$F(I(m+n-1)/I(n)) = C(I(m)/I(n)).$$

□

3 The Euclidean algorithm and continued fractions

Let $a, b \in \mathbb{Z}_{\geq 1}$, $a > b$. Let

$$v_0 = a, \quad v_1 = b.$$

Let

$$a_1 = \lfloor v_0/v_1 \rfloor, \quad v_2 = v_0 - a_1 v_1.$$

For $m \geq 2$, if $v_m \neq 0$ then let

$$a_m = \lfloor v_{m-1}/v_m \rfloor, \quad v_{m+1} = v_{m-1} - a_m v_m.$$

Then $0 \leq v_{m+1} < v_m$.[2]

For example, let $a = 83$, $b = 14$. Then

$$v_0 = 83, \quad v_1 = 14.$$

Then

$$a_1 = \lfloor 83/14 \rfloor = 5, \quad v_2 = 83 - 5 \cdot 14 = 13.$$

Then

$$a_2 = \lfloor v_1/v_2 \rfloor = 14/13 \rfloor = 1, \quad v_3 = v_1 - a_2 v_2 = 14 - 1 \cdot 13 = 1.$$

[2]See Marius Iosifescu and Cor Kraaikamp, *Metrical Theory of Continued Fractions*, p. 1, Chapter 1.

Then

$$a_3 = \lfloor v_2/v_3 \rfloor = \lfloor 13/1 \rfloor = 13, \quad v_4 = v_2 - a_3 v_3 = 13 - 13 \cdot 1 = 0.$$

As $v_3 = 1$ and $v_4 = 0$,

$$\gcd(83, 14) = 1.$$

Written as a continued fraction, we get

$$\frac{14}{83} = [0; 5, 1, 13].$$

For example, let $a = 168$, $b = 43$. Then

$$v_0 = 168, \quad v_1 = 43.$$

Then

$$a_1 = \lfloor 168/43 \rfloor = 3, \quad v_2 = v_0 - a_1 v_1 = 168 - 3 \cdot 43 = 39.$$

Then

$$a_2 = \lfloor 43/39 \rfloor = 1, \quad v_3 = v_1 - a_2 v_2 = 43 - 1 \cdot 39 = 4.$$

Then

$$a_3 = \lfloor v_2/v_3 \rfloor = \lfloor 39/4 \rfloor = 9, \quad v_4 = v_2 - a_3 v_3 = 39 - 9 \cdot 4 = 3.$$

Then

$$a_4 = \lfloor v_3/v_4 \rfloor = \lfloor 4/3 \rfloor = 1, \quad v_5 = v_3 - a_4 v_4 = 4 - 1 \cdot 3 = 1.$$

Then

$$a_5 = \lfloor v_4/v_5 \rfloor = \lfloor 3/1 \rfloor = 3, \quad v_6 = v_4 - a_5 v_5 = 3 - 3 \cdot 1 = 0.$$

As $v_5 = 1$ and $v_6 = 0$,

$$\gcd(168, 43) = 1.$$

Written as a continued fraction, we get

$$\frac{43}{168} = [0; 3, 1, 9, 1, 3].$$

For example, let $a = 1463$ and $b = 84$. Then

$$v_0 = 1463, \quad v_1 = 84.$$

Then

$$a_1 = \lfloor 1463/84 \rfloor = 17, \quad v_2 = 1463 - 17 \cdot 84 = 35.$$

Then

$$a_2 = \lfloor 84/35 \rfloor = 2, \quad v_3 = 84 - 2 \cdot 35 = 14.$$

Then

$$a_3 = \lfloor 35/14 \rfloor = 2, \quad v_4 = 35 - 2 \cdot 14 = 7.$$

Then

$$a_4 = \lfloor 14/7 \rfloor 2, \quad v_5 = 14 - 2 \cdot 7 = 0.$$

As $v_4 = 7$ and $v_5 = 0$,

$$\gcd(1463, 84) = 7.$$

Written as a continued fraction, we get

$$\frac{84}{1463} = [0; 17, 2, 2, 2].$$