The Dunford-Pettis theorem

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1 Weak topology and weak-* topology

If (E, τ) is a topological vector space, we denote by E^* the set of continuous linear maps $E \to \mathbb{C}$, the **dual space of** E. The **weak topology on** E, denoted $\sigma(E, E^*)$, is the coarsest topology on E with which each function $x \mapsto \lambda x$, $\lambda \in E^*$, is continuous $E \to \mathbb{C}$. Thus, $\sigma(E, E^*) \subset \tau$. If (E, τ) is a locally convex space, it follows by the Hahn-Banach separation theorem that E^* separates X, and hence $|\lambda|, \lambda \in E^*$, is a separating family of seminorms on E that induce the topology $\sigma(E, E^*)$. Therefore, if (E, τ) is a locally convex space, then $(E, \sigma(E, E^*))$ is a locally convex space.

If (E, τ) is a topological vector space, the **weak-* topology on** E^*, denoted $\sigma(E^*, E)$, is the coarsest topology on E^* with which each function $\lambda \mapsto \lambda x$, $x \in E$, is continuous $E^* \to \mathbb{C}$. It is a fact that E^* with the topology $\sigma(E^*, E)$ is a locally convex space.

If E is a normed space, then $\|\lambda\|_{op} = \sup_{\|x\| \leq 1} |\lambda x|$ is a norm on the dual space E^*, and that E^* with this norm is a Banach space. The **Banach-Alaoglu theorem** states that $\{\lambda \in E^* : \|\lambda\|_{op} \leq 1\}$ is a compact subset of $(E^*, \sigma(E^*, E))$.

If (X, Σ, μ) is a σ-finite measure space, for $g \in L^\infty(\mu)$ define $\phi_g \in (L^1(\mu))^*$ by $\phi_g(f) = \int_X fg d\mu$. The map $g \mapsto \phi_g$ is an isometric isomorphism $L^\infty(\mu) \to (L^1(\mu))^*$.[1]

Let (X, Σ, μ) be a probability space. If $\Psi \in (L^\infty(\mu))^*$ and $A \mapsto \Psi(\chi_A)$ is countably additive on Σ, then there is some $f \in L^1(\mu)$ such that

$$\Psi(g) = \int_X gf d\mu, \qquad g \in L^\infty(\mu),$$

and $\|\Psi\|_{op} = \|f\|_1$.[2] Also, an additive function F on an algebra of sets $\mathscr{A}$ is countably additive if and only if whenever A_n is a decreasing sequence of elements of $\mathscr{A}$ with $\bigcap_{n=1}^\infty A_n = \emptyset$, we have $\lim_{n\to\infty} F(A_n) = 0$.[3] Using that μ is countably additive we get the following.

[1] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 190, Theorem 6.15.

[2] V. I. Bogachev, *Measure Theory*, volume I, p. 263, Proposition 4.2.2.

[3] V. I. Bogachev, *Measure Theory*, volume I, p. 9, Proposition 1.3.3.

Theorem 1. Suppose that (X, Σ, μ) be a probability space and that $\Psi \in (L^\infty(\mu))^*$, and suppose that for each $\epsilon > 0$ there is some $\delta > 0$ such that $E \in \Sigma$ and $\mu(E) \leq \delta$ imply that $|\Psi(\chi_A)| \leq \epsilon$. Then there is some $f \in L^1(\mu)$ such that

$$\Psi(g) = \int_X gf d\mu, \qquad g \in L^\infty(\mu).$$

2 Normed spaces

If E is a normed space, its dual space E^* with the operator norm is a Banach space, and $E^{**} = (X^*)^*$ with the operator norm is a Banach space. Define $i : E \to E^{**}$ by

$$i(x)(\lambda) = \lambda(x), \qquad x \in E, \quad \lambda \in E^*.$$

It follows from the Hahn-Banach extension theorem that $i : E \to E^{**}$ is an isometric linear map.

If E and F are normed spaces and $T : E \to F$ is a bounded linear map, we define the **transpose** $T^* : F^* \to E^*$ by $T^*\lambda = \lambda \circ T$ for $\lambda \in F^*$. If T is an isometric isomorphism, then $T^* : F^* \to E^*$ is an isometric isomorphism, where E^* and F^* are each Banach spaces with the operator norm. In particular, we have said that when (X, Σ, μ) is a σ-finite measure space, then the map $\phi : L^\infty(\mu) \to (L^1(\mu))^*$ defined for $g \in L^\infty(\mu)$ by

$$\phi_g(f) = \int_X fg d\mu, \qquad f \in L^1(\mu),$$

is an isometric isomorphism, and hence $\phi^* : (L^1(\mu))^{**} \to (L^\infty(\mu))^*$ is an isometric isomorphism. Therefore, for $E = L^1(\mu)$ we have that

$$\phi^* \circ i : L^1(\mu) \to (L^\infty(\mu))^* \tag{1}$$

is an isometric linear map. For $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$,

$$\begin{aligned}(\phi^* \circ i)(f)(g) &= (\phi^*(i(f)))(g) \\ &= (i(f) \circ \phi)(g) \\ &= i(f)(\phi_g) \\ &= \phi_g(f).\end{aligned}$$

The **Eberlein-Smulian theorem** states that if E is a normed space and A is a subset of E, then A is weakly compact if and only if A is weakly sequentially compact.[4]

[4]Robert E. Megginson, *An Introduction to Banach Space Theory*, p. 248, Theorem 2.8.6.

3 Equi-integrability

Let (X, Σ, μ) be a probability space and let $\mathscr{F}$ be a subset of $L^1(\mu)$. We say that $\mathscr{F}$ is **equi-integrable** if for every $\epsilon > 0$ there is some $\delta > 0$ such that for any $A \in \Sigma$ with $\mu(A) \leq \delta$ and for all $f \in \mathscr{F}$,

$$\int_A |f| d\mu \leq \epsilon.$$

If $\mathscr{F}$ is a bounded subset of $L^1(\mu)$, it is a fact that $\mathscr{F}$ being equi-integrable is equivalent to

$$\lim_{C \to \infty} \sup_{f \in \mathscr{F}} \int_{\{|f| > C\}} |f| d\mu = 0. \tag{2}$$

The following theorem gives a condition under which a sequence of integrable functions is bounded and equi-integrable.[5]

Theorem 2. Let (X, Σ, μ) be a probability space and let f_n be a sequence in $L^1(\mu)$. If for each $A \in \Sigma$ the sequence $\int_A f_n d\mu$ has a finite limit, then $\{f_n\}$ is bounded in $L^1(\mu)$ and is equi-integrable.

4 The Dunford-Pettis theorem

A subset A of a topological space X is said to be **relatively compact** if A is contained in some compact subset of X. When X is a Hausdorff space, this is equivalent to the closure of A being a compact subset of X.

The following is the **Dunford-Pettis theorem**.[6]

Theorem 3 (Dunford-Pettis theorem)**.** Suppose that (X, Σ, μ) is a probability space and that $\mathscr{F}$ is a bounded subset of $L^1(\mu)$. $\mathscr{F}$ is equi-integrable if and only if $\mathscr{F}$ is a relatively compact subset of $L^1(\mu)$ with the weak topology.

Proof. Suppose that $\mathscr{F}$ is equi-integrable, and let $T = \phi^* \circ i : L^1(\mu) \to (L^\infty(\mu))^*$ be the isometric linear map in (1), for which

$$T(f)(g) = \int_X f g d\mu, \qquad f \in L^1(\mu), \quad g \in L^\infty(\mu).$$

Then $T(\mathscr{F})$ is a bounded subset of $(L^\infty(\mu))^*$, so is contained in some closed ball B in $(L^\infty(\mu))^*$. By the Banach-Alaoglu theorem, B is weak-* compact, and therefore the weak-* closure $\mathscr{H}$ of $T(\mathscr{F})$ is weak-* compact. Let $F \in \mathscr{H}$.

[5] V. I. Bogachev, *Measure Theory*, volume I, p. 269, Theorem 4.5.6.

[6] V. I. Bogachev, *Measure Theory*, volume I, p. 285, Theorem 4.7.18; Fernando Albiac and Nigel J. Kalton, *Topics in Banach Space Theory*, p. 109, Theorem 5.2.9; R. E. Edwards, *Functional Analysis: Theory and Applications*, p. 274, Theorem 4.21.2; P. Wojtaszczyk, *Banach Spaces for Analysts*, p. 137, Theorem 12; Joseph Diestel, *Sequences and Series in Banach Spaces*, p. 93; François Trèves, *Topological Vector Spaces, Distributions and Kernels*, p. 471, Theorem 46.1.

There is a net $F_\alpha = T(f_\alpha)$ in $T(\mathscr{F})$, $\alpha \in I$, such that for each $g \in L^\infty(\mu)$, $F_\alpha(g) \to F(g)$, i.e.,

$$\int_X f_\alpha g d\mu \to F(g), \qquad g \in L^\infty(\mu). \tag{3}$$

Let $\epsilon > 0$. Because $\mathscr{F}$ is equi-integrable, there is some $\delta > 0$ such that when $A \in \Sigma$ and $\mu(A) \le \delta$,

$$\sup_{\alpha \in I} \int_A |f_\alpha| d\mu \le \epsilon,$$

which gives

$$|F(\chi_A)| = \lim_\alpha \left| \int_X f_\alpha \chi_A d\mu \right| = \lim_\alpha \left| \int_A f_\alpha d\mu \right| \le \sup_{\alpha \in I} \int_A |f_\alpha| d\mu \le \epsilon.$$

By Theorem 1, this tells us that there is some $f \in L^1(\mu)$ for which

$$F(g) = \int_X g f d\mu, \qquad g \in L^\infty(\mu),$$

and hence $F = T(f)$. This shows that $\mathscr{H} \subset T(L^1(\mu))$, and

$$\int_X f_\alpha g d\mu \to \int_X f g d\mu, \qquad g \in L^\infty(\mu)$$

tells us that $f_\alpha \to f$ in $\sigma(L^1(\mu), (L^1(\mu))^*)$, in other words $T^{-1}(F_\alpha)$ converges weakly to $T(F)$. Thus $T^{-1} : \mathscr{H} \to L^1(\mu)$ is continuous, where $\mathscr{H}$ has the subspace topology $\tau_{\mathscr{H}}$ inherited from $(L^\infty(\mu))^*$ with the weak-* topology and $L^1(\mu)$ has the weak topology. $(\mathscr{H}, \tau_{\mathscr{H}})$ is a compact topological space, so $T^{-1}(\mathscr{H})$ is a weakly compact subset of $L^1(\mu)$. But $\mathscr{F} \subset T^{-1}(\mathscr{H})$, which establishes that $\mathscr{F}$ is a relatively weakly compact subset of $L^1(\mu)$.

Suppose that $\mathscr{F}$ is a relatively compact subset of $L^1(\mu)$ with the weak topology and suppose by contradiction that $\mathscr{F}$ is not equi-integrable. Then by (2), there is some $\eta > 0$ such that for all C_0 there is some $C \ge C_0$ such that

$$\sup_{f \in \mathscr{F}} \int_{\{|f| > C\}} |f| d\mu > \eta,$$

whence for each n there is some $f_n \in \mathscr{F}$ with

$$\int_{\{|f_n| > n\}} |f_n| d\mu \ge \eta, \tag{4}$$

On the other hand, because $\mathscr{F}$ is relatively weakly compact, the Eberlein-Smulian theorem tells us that $\mathscr{F}$ is relatively weakly sequentially compact, and so there is a subsequence $f_{a(n)}$ of f_n and some $f \in L^1(\mu)$ such that $f_{a(n)}$ converges weakly to f. For $A \in \Sigma$, as $\chi_A \in L^\infty(\mu)$ we have

$$\lim_{n \to \infty} \int_A f_{a(n)} d\mu = \int_A f d\mu,$$

and thus Theorem 2 tells us that the collection $\{f_{a(n)}\}$ is equi-integrable, contradicting (4). Therefore, $\mathscr{F}$ is equi-integrable. $\square$

Corollary 4. Suppose that (X, Σ, μ) is a probability space. If $\{f_n\} \subset L^1(\mu)$ is bounded and equi-integrable, then there is a subsequence $f_{a(n)}$ of f_n and some $f \in L^1(\mu)$ such that

$$\int_X f_{a(n)} g d\mu \to \int_X f g d\mu, \qquad g \in L^\infty(\mu).$$

Proof. The Dunford-Pettis theorem tells us that $\{f_n\}$ is relatively weakly compact, so by the Eberlein-Smulian theorem, $\{f_n\}$ is relatively weakly sequentially compact, which yields the claim. □

5 Separable topological spaces

It is a fact that if E is a separable topological vector space and K is a compact subset of $(E^*, \sigma(E^*, E))$, then K with the subspace topology inherited from $(E^*, \sigma(E^*, E))$ is metrizable. Using this and the Banach-Alaoglu theorem, if E is a separable normed space it follows that $\{\lambda \in E^* : \|\lambda\|_{op} \leq 1\}$ with the subspace topology inherited from $(E, \sigma(E^*, E))$ is compact and metrizable, and hence is sequentially compact.[7] In particular, when E is a separable normed space, a bounded sequence in E^* has a weak-* convergent subsequence.

If X is a separable metrizable space and μ is a σ-finite Borel measure on X, then the Banach space $L^p(\mu)$ is separable for each $1 \leq p < \infty$.[8]

Theorem 5. Suppose that X is a separable metrizable space and μ is a σ-finite Borel measure on X. If $\{g_n\}$ is a bounded subset of $L^\infty(\mu)$, then there is a subsequence $g_{a(n)}$ of g_n and some $g \in L^\infty(\mu)$ such that

$$\int_X f g_{a(n)} d\mu \to \int_X f g d\mu, \qquad f \in L^1(\mu).$$

[7] A second-countable T_1 space is compact if and only if it is sequentially compact: Stephen Willard, *General Topology*, p. 125, 17G.

[8] René L. Schilling, *Measures, Integrals and Martingales*, p. 270, Corollary 23.20.