Diophantine vectors

Jordan Bell

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1 Dirichlet's approximation theorem

Let $m \geq 1$, and for $v \in \mathbb{R}^m$ write

$$|v|_\infty = \max\{|v_j| : 1 \leq j \leq m\}.$$

For a positive integer r, let

$$V_r = \{k \in \mathbb{Z}^m : 0 < |k|_\infty \leq r\},$$

which has $N_r = (2r+1)^m - 1$ elements. For any $k \in V_r$,

$$|\langle v, k\rangle| \leq m|k|_\infty |v|_\infty \leq mr|v|_\infty,$$

Let $I_1, \ldots, I_{N_r-1}$ be consecutive closed intervals with

$$[0, mr|v|_\infty] = \bigcup_{j=1}^{N_r-1} I_j.$$

Then there is some j and some $k', k'' \in V_r$, $k' \neq k''$, with $|\langle v, k'\rangle|, |\langle v, k''\rangle| \in I_j$. If $\langle v, k'\rangle, \langle v, k''\rangle$ have the same sign, then $k = k' - k''$ satisfies $|\langle v, k\rangle| \leq |I_j|$, and if $\langle v, k'\rangle, \langle v, k''\rangle$ have different signs then $k = k' + k''$ satisfies $|\langle v, k\rangle| \leq |I_j|$. In either case, $k \in V_{2r}$, and k satisfies

$$|\langle v, k\rangle| \leq |I_j| = \frac{mr|v|_\infty}{N_r - 1} = \frac{mr|v|_\infty}{(2r+1)^m - 2}.$$

2 Diophantine vectors

For real $\tau, \gamma > 0$, let $D(\tau, \gamma)$ be the set of those $v \in \mathbb{R}^m$ such that for any nonzero $k \in \mathbb{Z}^m$,

$$|\langle v, k\rangle| \geq \gamma |k|_\infty^{-\tau}.$$

In other words,

$$D(\tau, \gamma) = \bigcap_{k \in \mathbb{Z}^m \setminus \{0\}} \{v \in \mathbb{R}^m : |\langle v, k\rangle| \geq \gamma |k|_\infty^{-\tau}\} = \bigcap_{k \in \mathbb{Z}^m \setminus \{0\}} D(\tau, \gamma, k).$$

Each $D(\tau, \gamma, k)$ is closed, so $D(\tau, \gamma)$ is closed. Let

$$D(\tau) = \bigcup_{\gamma > 0} D(\tau, \gamma).$$

If $\gamma_1 \geq \gamma_2$ and $v \in D(\tau, \gamma_1)$, let $k \in \mathbb{Z}^m \setminus \{0\}$. Then $| \langle v, k \rangle | \geq \gamma_1 |k|_\infty^{-\tau} \geq \gamma_2 |k|_\infty^{-\tau}$, so $v \in D(\tau, \gamma_2)$, i.e.

$$D(\tau, \gamma_1) \subset D(\tau, \gamma_2), \qquad \gamma_1 \geq \gamma_2.$$

Therefore

$$D(\tau, N_1^{-1}) \subset D(\tau, N_2^{-1}) \qquad N_1 \leq N_2,$$

and

$$D(\tau) = \bigcup_{N \geq 1} D(\tau, N^{-1}),$$

showing that $D(\tau)$ is an F_σ set.

If $0 \leq \tau < m - 1$ and $\gamma > 0$, suppose by contradiction that there is some $v \in D(\tau, \gamma)$. Now, by Dirichlet's theorem, for each positive integer r there is some $k_r \in V_{2r}$ satisfying $| \langle v, k_r \rangle | \leq m|v|_\infty 2^{-m} r^{-m+1}$. Then, as $|k_r|_\infty \leq 2r$,

$$m|v|_\infty 2^{-m} r^{-m+1} \geq | \langle v, k_r \rangle | \geq \gamma |k_r|_\infty^{-\tau} \geq \gamma (2r)^{-\tau} = \gamma (2r)^{\tau - m + 1} (2r)^{m-1},$$

hence

$$(2r)^{-\tau + m - 1} \geq \frac{2}{cm|v|_\infty}.$$

As $\tau < m - 1$, taking $r \to \infty$ yields a contradiction. Therefore

$$D(\tau) = \emptyset, \qquad 0 \leq \tau < m - 1.$$

3 Measures of sets

Denote by μ Lebesgue measure on $\mathbb{R}^m$. Let $e_1, \ldots, e_m$ be the standard basis for $\mathbb{R}^m$, so

$$|v|_1 = \sum_{j=1}^m |v_j| = \sum_{j=1}^m | \langle v, e_j \rangle |.$$

Let $C = \{v \in \mathbb{R}^m : |v|_\infty \leq 1\}$. Let A_m be the supremum of the $(m-1)$-dimensional Hausdorff measure of the intersection of an $(n-1)$-dimensional affine subspace of $\mathbb{R}^m$ and C.

We calculate the following.[1]

Theorem 1. *For $\tau > m - 1$ and $\gamma > 0$,*

$$\mu(C \setminus D(\tau, \gamma)) \leq 4\gamma m A_m 3^{m-1} \zeta(\tau + 2 - m).$$

[1] Dmitry Treschev and Oleg Zubelevich, *Introduction to the Perturbation Theory of Hamiltonian Systems*, p. 166, Theorem 9.3.

Proof. Let $k \in \mathbb{Z}^m \setminus \{0\}$, and for $t \in \mathbb{R}$, let

$$P_{k,t} = \{x \in \mathbb{R}^m : \langle x, k \rangle = t\},$$

and let

$$U_k = \left\{x \in \mathbb{R}^m : |\langle x, k \rangle| < \gamma |k|_\infty^{-\tau}\right\}.$$

U is the set of points between the hyperplanes $P_{k,-\gamma|k|_\infty^{-\tau}}$ and $P_{k,\gamma|k|_\infty^{-\tau}}$. The distance between the hyperplanes $P_{k,s}$ and $P_{k,s}$ is $\frac{|s-t|}{|k|_2}$, so the distance between the hyperplanes $P_{k,-\gamma|k|_\infty^{-\tau}}$ and $P_{k,\gamma|k|_\infty^{-\tau}}$ is $d_k = \frac{2\gamma|k|_\infty^{-\tau}}{|k|_2}$. And $|x|_2 \geq |x|_\infty$, so $d_k \leq 2\gamma|k|_\infty^{-\tau-1}$. But $\mu(C \cap U_k) \leq dA_m$, so

$$\mu(C \cap U) \leq 2\gamma |k|_\infty^{-\tau-1} A_m.$$

Now, $U_k = \mathbb{R}^m \setminus D(\tau, \gamma, k)$, so

$$C \setminus D(\tau, \gamma) = C \setminus \bigcap_{k \in \mathbb{Z}^m \setminus \{0\}} D(\tau, \gamma, k) = \bigcup_{k \in \mathbb{Z}^m \setminus \{0\}} (C \cap U_k).$$

We remind ourselves that for r a positive integer, the set $V_r = \{k \in \mathbb{Z}^m : 0 < |k|_\infty \leq r\}$ has $N_r = (2r+1)^m - 1$ elements. Therefore

$$\{k \in \mathbb{Z}^m \setminus \{0\} : |k|_\infty = r\} = V_r \setminus V_{r-1}$$

has

$$N_r - N_{r-1} = (2r+1)^m - (2r-1)^m \leq 2m(2r+3)^{m-1}$$

elements, using $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1})$. Therefore

$$\begin{aligned}
\mu(C \setminus D(\tau, \gamma)) &\leq \sum_{k \in \mathbb{Z}^m \setminus \{0\}} \mu(C \cap U_k) \\
&\leq \sum_{k \in \mathbb{Z}^m \setminus \{0\}} 2\gamma |k|_\infty^{-\tau-1} A_m \\
&= 2\gamma A_m \sum_{r=1}^{\infty} \sum_{\{k \in \mathbb{Z}^m \setminus \{0\} : |k|_\infty = r\}} r^{-\tau-1} \\
&\leq 2\gamma A_m \sum_{r=1}^{\infty} 2m(2r+1)^{m-1} \cdot r^{-\tau-1} \\
&= 4\gamma m A_m \sum_{r=1}^{\infty} (2r+1)^{m-1} r^{-\tau-1}.
\end{aligned}$$

We estimate

$$\sum_{r=1}^{\infty} (2r+1)^{m-1} r^{-\tau-1} \leq \sum_{r=1}^{\infty} (3r)^{m-1} r^{-\tau-1} = 3^{m-1} \sum_{r=1}^{\infty} r^{m-\tau-2},$$

and therefore

$$\mu(C \setminus D(\tau, \gamma)) \leq 4\gamma m A_m \cdot 3^{m-1} \zeta(\tau + 2 - m).$$

□

But

$$C \setminus D(\tau) = \bigcap_{N \geq 1} (C \setminus D(\tau, N^{-1})),$$

and by Theorem 1, if $\tau > m - 1$ then $\mu(C \setminus D(\tau, N^{-1})) \to 0$ as $N \to \infty$. Therefore

$$\mu(C \cap D(\tau)) = 0, \qquad \tau > m - 1.$$

4 Cohomological equation

Let $\mathbb{T}^m = \{z \in \mathbb{C}^m : |z_1| = 1, \dots, |z_m| = 1\}$ and write ν for the Haar measure on $\mathbb{T}^m$ for which $\nu(\mathbb{T}^m) = 1$. For $k \in \mathbb{Z}^m$ let $\chi_k(z) = \prod_{j=1}^m z_j^{k_j}$. Let $\Delta(\tau, \gamma)$ be the set of those $z \in \mathbb{T}^m$ such that

$$|\chi_k(z) - 1| \geq \gamma |k|_1^{-\tau}, \qquad k \in \mathbb{Z}^m \setminus \{0\}.$$

Let

$$\Delta(\tau) = \bigcup_{\gamma > 0} \Delta(\tau, \gamma),$$

and then

$$\Delta = \bigcup_{\tau > 0} \Delta(\tau).$$

For $\lambda \in \mathbb{T}^m$ define $R_\lambda : \mathbb{T}^m \to \mathbb{T}^m$ by

$$R_\lambda(z) = \lambda \cdot z = (\lambda_1 z_1, \dots, \lambda_m z_m).$$

In the following theorem, (1) is called a **cohomological equation**.[2]

Theorem 2. *For $\lambda \in \mathbb{T}^m$, $\lambda \in \Delta$ if and only if for any $h \in C^\infty(\mathbb{T}^m)$ there is some $\psi \in C^\infty(\mathbb{T}^m)$ such that*

$$h(z) - \int_{\mathbb{T}^m} h d\nu = \psi(R_\lambda z) - \psi(z), \qquad z \in \mathbb{T}^m. \tag{1}$$

Proof. It is a fact that $\chi_k \in \widehat{\mathbb{T}}^m$ and that $k \mapsto \chi_k$ is an isomorphism of topological groups $\mathbb{Z}^d \to \widehat{\mathbb{T}}^m$. For $f \in L^1(\nu)$, $\widehat{f} : \mathbb{Z}^m \to \mathbb{C}$ is defined by

$$\widehat{f}(k) = \int_{\mathbb{T}^m} f(z) \overline{\chi_k(z)} d\nu(z) = \int_{\mathbb{T}^m} f(z) \chi_k(z)^{-1} d\nu(z).$$

If the Fourier series of f converges pointwise,

$$f(z) = \sum_{k \in \mathbb{Z}^m} \widehat{f}(k) \chi_k(z), \qquad z \in \mathbb{T}^m.$$

[2] Anatole Katok, *Combinatorial Constructions in Ergodic Theory and Dynamics*, p. 71, Theorem 11.5.

It is a fact that $f \in C^\infty(\mathbb{T}^m)$ if and only if for any $R > 0$ there is some C_R such that

$$|\widehat{f}(k)| \leq C_R |k|_1^{-R}, \qquad k \in \mathbb{Z}^m \setminus \{0\}.$$

For $\psi \in L^1(\mathbb{T}^m)$ and $k \in \mathbb{T}^m$, because ν is invariant under multiplication in $\mathbb{T}^m$,

$$\begin{aligned}
\widehat{\psi \circ R_\lambda}(k) &= \int_{\mathbb{T}^m} \psi(\lambda z)\overline{\chi_k(z)} d\nu(z) \\
&= \int_{\mathbb{T}^m} \psi(z)\overline{\chi_k(\lambda^{-1}x)} d\nu(z) \\
&= \chi_k(\lambda)\widehat{\psi}(k).
\end{aligned}$$

Suppose that for every h is C^∞ that there is some $\psi \in C^\infty(\mathbb{T}^m)$ satisfying (1). Taking the Fourier transform of (1),

$$\widehat{h}(k) - \delta_0(k) \cdot \int_{\mathbb{T}^m} h d\nu = \chi_k(\lambda)\widehat{\psi}(k) - \widehat{\psi}(k), \qquad k \in \mathbb{Z}^m,$$

then, if $\chi_k(\lambda) \neq 1$,

$$\widehat{\psi}(k) = \frac{\widehat{h}(k)}{\chi_k(\lambda) - 1}.$$

Now suppose by contradiction that $\lambda \notin \Delta$. This means that there are $\tau_N \to \infty$ such that for each N, there is some $\gamma_N > 0$ and some $k_N \in \mathbb{Z}^m \setminus \{0\}$ such that $|\chi_{k_N}(\lambda) - 1| < \gamma_N |k_N|_1^{-\tau_N}$. Define

$$\widehat{h}(k) = |\chi_k(\lambda) - 1|^{1/2} \cdot 1_{\{k_N\}}(k).$$

For $R > 0$ let $\tau_N \geq 2R$. Then for $k \in \mathbb{Z}^m$, either $\widehat{h}(k) = 0$ or if $k = k_N$ then

$$|\widehat{h}(k)| = |\chi_{k_N}(\lambda) - 1|^{1/2} < \gamma_N^{1/2} |k_N|_1^{-\tau_N/2} \leq \gamma_N^{1/2} |k_N|_1^{-R} = \gamma_N^{1/2} |k|_1^{-R},$$

which shows that h is C^∞. There is some $\psi \in C^\infty(\mathbb{T}^m)$ satisfying (1), according to which, for $\chi_k(\lambda) \neq 1$,

$$\widehat{\psi}(k) = \frac{\widehat{h}(k)}{\chi_k(\lambda) - 1}.$$

But $|\widehat{\psi}(k)|$ is either 0 or if $k = k_N$ then $|\chi_{k_N}(\lambda) - 1|^{-1/2} > \gamma_N^{-1/2} |k_N|_1^{\tau_N/2}$. Thus the Fourier coefficients of ψ are unbounded, which contradicts that ψ is C^∞. Therefore $\lambda \in \Delta$.

Now suppose that $\lambda \in D$ and let $h \in C^\infty(\mathbb{T}^m)$. Define ψ by

$$\widehat{\psi}(k) = \begin{cases} \frac{\widehat{h}(k)}{\chi_k(\lambda) - 1} & \chi_k(\lambda) \neq 1 \\ 0 & \chi_k(\lambda) = 1. \end{cases}$$

The facts that $\lambda \in D$ and that h is C^∞ yield that ψ is C^∞. It is straightforward from the definition of $\widehat{\psi}(k)$ that ψ satisfies (1). □