The nonlinear Schrödinger equation is Hamiltonian

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Let

$$H(u)=\int \frac{1}{2}|\nabla u|^2+\frac{2\mu}{p+1}|u|^{p+1}dx.$$

We have

$$H(u+\epsilon v)=\int \frac{1}{2}\nabla(u+\epsilon v)\nabla(\overline{u}+\epsilon\overline{v})+\frac{2\mu}{p+1}(u+\epsilon v)^{\frac{p+1}{2}}(\overline{u}+\epsilon\overline{v})^{\frac{p+1}{2}}.$$

We have, for $v\in T_u$,

$$\begin{aligned}
dH(u)v &= \frac{d}{d\epsilon}|_{\epsilon=0}H(u+\epsilon v)\\
&= \int \frac{1}{2}\left(\nabla v\nabla\overline{u}+\nabla u\nabla\overline{v}\right)\\
&\quad +\frac{2\mu}{p+1}\left(\frac{p+1}{2}u^{\frac{p-1}{2}}v\overline{u}^{\frac{p+1}{2}}+\frac{p+1}{2}u^{\frac{p+1}{2}}\overline{u}^{\frac{p-1}{2}}\overline{v}\right)dx\\
&= \Re\int \nabla\overline{v}\nabla u+2\mu|u|^{p-1}u\overline{v}dx\\
&= \Re\int\left(-\Delta u+2\mu|u|^{p-1}u\right)\overline{v}dx.
\end{aligned}$$

Define

$$\omega(v_1,v_2)=-2\Im\int v_1\overline{v_2}dx.$$

The Hamiltonian vector field X_H of H is defined, for $v\in T_u$, by

$$\omega(X_H(u),v)=dH(u)v.$$

Therefore we have for $v\in T_u$ that

$$\begin{aligned}
-2\Im\int X_H(u)\overline{v}dx &= \Re\int\left(-\Delta u+2\mu|u|^{p-1}u\right)\overline{v}dx\\
&= \Im\int\left(-i\Delta u+2i\mu|u|^{p-1}u\right)\overline{v}dx.
\end{aligned}$$

It follows that

$$-2X_H(u) = -i\Delta u + 2i\mu|u|^{p-1}u,$$

i.e.,

$$X_H(u) = \frac{i}{2}\Delta u - i\mu|u|^{p-1}u.$$

The flow $S(t)$ of the Hamiltonian vector field X_H satisfies, for any u_0 and for $u(t) = S(t)u_0$,

$$u_t(t) = X_H(u(t)) = \frac{i}{2}\Delta u(t) - i\mu|u(t)|^{p-1}u(t).$$

This equation can be written as

$$iu_t(t) + \frac{1}{2}\Delta u(t) = \mu|u(t)|^{p-1}u(t).$$