$C[0,1]$: the Faber-Schauder basis, the Riesz representation theorem, and the Borel σ-algebra

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1 Introduction

In this note I work out some results about the space of continuous functions $[0,1] \to \mathbb{R}$. In some cases these are instances of general results about continuous functions on compact Hausdorff spaces or compact metrizable spaces.

2 $C(K)$

Let K be a compact topological space (not necessarily Hausdorff) and let $C(K)$ be the collection of continuous functions $K \to \mathbb{R}$. With the norm

$$\|f\| = \sup_{t\in K} |f(t)|, \qquad f \in C(K),$$

this is a real Banach algebra, with unity $1 : K \to \mathbb{R}$ defined by $1(t) = 1$ for all $t \in K$.[1] Generally, if X is a compact metrizable space and Y is a separable metrizable space then $C(X,Y)$ is separable.[2] Thus, when K is a compact metrizable space, $C(K)$ is a separable unital Banach algebra.

3 Schauder bases

If X is a real normed space, a **Schauder basis for** X is a sequence (h_k) in X such that for each $x \in X$ there is a unique sequence of real numbers $(c_k(x))$ such that $\sum_{k=1}^n c_k(x)h_k \to x$. A sequence (h_k) in X is called a **basic sequence** if it is a Schauder basis for the closure of its linear span.[3]

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[1] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 124, Lemma 3.97.

[2] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 125, Lemma 3.99.

[3] Because Schauder bases are not as familiar objects as orthonormal bases (Hilbert space bases) or vector space bases (Hamel bases), it is worth explicitly checking their properties to make ourselves familiar with how they work.

If (h_k) is a Schauder basis, then $c_k(0) = 0$ for all k. Suppose that for some real numbers $a_1, \ldots, a_n$ we have $\sum_{k=1}^n a_k h_k = 0$. Then $a_k = c_k(0) = 0$ for $1 \leq k \leq n$, which means that the set $\{h_1, \ldots, h_n\}$ is linearly independent. Therefore a Schauder basis is linearly independent. It is immediate that the linear span of a Schauder basis is a dense linear subspace of X. The following shows that a normed space with a Schauder basis is separable.[4]

Lemma 1. If (h_k) is a Schauder basis for a normed space X then

$$\left\{ \sum_{k=1}^n a_k h_k : n \geq 1, a_1, \ldots, a_n \in \mathbb{Q} \right\}$$

is dense in X.

Proof. Let $x \in X$ and let $\epsilon > 0$. There is some n for which

$$\left\| \sum_{k=1}^n c_k(x) h_k - x \right\| \leq \epsilon.$$

For each $1 \leq k \leq n$, let $a_k \in \mathbb{Q}$ satisfy $|a_k - c_k(x)| \leq \frac{\epsilon}{n\|h_k\|}$. Then

$$\begin{aligned} \left\| \sum_{k=1}^n a_k h_k - x \right\| &\leq \left\| \sum_{k=1}^n (a_k - c_k(x)) h_k \right\| + \epsilon \\ &\leq \sum_{k=1}^n |a_k - c_k(x)| \, \|h_k\| + \epsilon \\ &\leq 2\epsilon, \end{aligned}$$

which proves the claim. □

For each k we define $h_k^* : X \to \mathbb{R}$ by $h_k^*(x) = c_k(x)$. For $x, y \in X$, $\sum_{k=1}^n h_k^*(x) h_k \to x$ and $\sum_{k=1}^n h_k^*(y) h_k \to y$, so $\sum_{k=1}^n (h_k^*(x) + h_k^*(y)) h_k \to x + y$ and therefore $h_k^*(x + y) = h_k^*(x) + h_k^*(y)$, and for $\alpha \in \mathbb{R}$, $\sum_{k=1}^n \alpha h_k^*(x) h_k \to \alpha x$ and therefore $h_k^*(\alpha x) = \alpha h_k^*(x)$. This shows that $h_k^* : X \to \mathbb{R}$ is linear. It is apparent that $h_k^*(h_j) = \delta_{j,k}$.

We define $P_n : X \to X$ by

$$P_n x = \sum_{k=1}^n h_k^*(x) h_k,$$

and it is immediate that P_n is linear and that for each $x \in X$, $P_n x \to x$ as $n \to \infty$. It is also immediate that P_n is a **finite-rank operator**: $P_n(X)$ is a

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[4] In particular, a Banach space with a Schauder basis is separable. There is a celebrated counterexample found by Per Enflo of a separable Banach space for which there does not exist a Schauder basis.

finite-dimensional linear subspace of X. For $m, n \geq 1$,

$$\begin{aligned} P_n P_m x &= P_n \sum_{k=1}^{m} h_k^*(x) h_k \\ &= \sum_{k=1}^{m} h_k^*(x) P_n h_k \\ &= \sum_{k=1}^{m} h_k^*(x) \sum_{j=1}^{n} h_j^*(h_k) h_j \\ &= \sum_{k=1}^{m} h_k^*(x) \sum_{j=1}^{n} \delta_{j,k} h_j \\ &= P_{\min(m,n)} x, \end{aligned}$$

showing that

$$P_n P_m = P_{\min(m,n)}.$$

In particular, $P_n^2 = P_n$, namely, P_n is a **projection operator**. We prove that when X is a Banach space, P_n is continuous.[5] We indicate explicitly in the proof when we use that X is a Banach space rather than merely a normed space.

Theorem 2. If X is a Banach space and (h_n) is a Schauder basis for X, then each P_n is continuous, and $\sup_{n \geq 1} \|P_n\| < \infty$. Furthermore, each h_k^* is continuous.

Proof. For $x \in X$, $P_n x \to x$, so $\|P_n x\| \to \|x\|$, which implies that $\sup_{n\geq 1} \|P_n x\| < \infty$. It thus makes sense to define

$$p(x) = \sup_{n \geq 1} \|P_n x\|.$$

It is immediate that $p(\alpha x) = |\alpha| p(x)$ and that $p(x+y) \leq p(x) + p(y)$. If $p(x) = 0$, then $P_n x = 0$ for all n, which implies that $x = 0$. Therefore p is a norm on X.

For $n \geq 1$ and $x \in X$,

$$\|P_n x\| \leq p(x),$$

showing that $P_n : (X, p) \to (X, \|\cdot\|)$ is a bounded linear operator with operator norm ≤ 1. Suppose that (x_k) is a Cauchy sequence in the norm p. Then we have for each n that $P_n x_k$ is a Cauchy sequence in the norm $\|\cdot\|$, and because $(X, \|\cdot\|)$ is a Banach space, there is some $y_n \in X$ such that $P_n x_k \to y_n$ in the norm $\|\cdot\|$ as $k \to \infty$. For $\epsilon > 0$ there is some k_ϵ such that when $j, k \geq k_\epsilon$, $p(x_j - x_k) \leq \epsilon$, and thus for $k \geq k_\epsilon$ and for any n,

$$\|P_n x_k - y_n\| = \lim_{j \to \infty} \|P_n x_k - P_n x_j\| \leq \limsup_{j \to \infty} p(x_k - x_j) \leq \epsilon. \tag{1}$$

Now, for any m, n and any k,

$$\|y_n - y_m\| \leq \|y_n - P_n x_k\| + \|P_n x_k - x_k\| + \|P_m x_k - x_k\| + \|P_m x_k - y_m\|,$$

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[5] N. L. Carothers, *A Short Course in Banach Space Theory*, p. 26, Theorem 3.1.

so using the above, for $k \geq k_\epsilon$,

$$\|y_n - y_m\| \leq 2\epsilon + \|P_n x_k - x_k\| + \|P_m x_k - x_k\|.$$

Because $P_n x_k \to x_k$ as $n \to \infty$, there is some n_ϵ such that $\|P_n x_k - x_k\| \leq \epsilon$ when $n \geq n_\epsilon$, and in this case for $n, m \geq n_\epsilon$,

$$\|y_n - y_m\| \leq 4\epsilon,$$

which shows that y_n is a Cauchy sequence in the norm $\|\cdot\|$, and because $(X, \|\cdot\|)$ is a Banach space there is some $y \in X$ such that $y_n \to y$ in the norm $\|\cdot\|$.

For any n, m, the restriction of the linear map $P_n : X \to X$ to the finite-dimensional linear subspace $P_m(X)$ is continuous in the norm $\|\cdot\|$, thus

$$\begin{aligned} P_n y_m &= P_n \left(\lim_{k\to\infty} P_m x_k \right) \\ &= \lim_{k\to\infty} P_n P_m x_k \\ &= \lim_{k\to\infty} P_{\min(m,n)} x_k \\ &= y_{\min(m,n)}. \end{aligned}$$

Using this, we find by induction that for each n,

$$y_n = \sum_{k=1}^{n} h_k^*(y_k) h_k,$$

and because $y_n \to y$ in the norm $\|\cdot\|$ as $n \to \infty$ and (h_k) is a Schauder basis, this implies that $h_k^*(y) = h_k^*(y_k)$ for all k, and thus $P_n y = y_n$. Therefore

$$p(x_k - y) = \sup_{n\geq 1} \|P_n(x_k - y)\| = \sup_{n\geq 1} \|P_n x_k - y_n\|.$$

For $\epsilon > 0$ and $k \geq k_\epsilon$, by (1) we have $\|P_n x_k - y_n\| \leq \epsilon$ and therefore $p(x_k - y) \leq \epsilon$, which shows that $x_k \to y$ in the norm p as $k \to \infty$. Thus, (X, p) is a Banach space.

The identity map $\mathrm{id}_X : X \to X$ is a linear isomorphism, and for $x \in X$,

$$\|\mathrm{id}_X x\| = \|x\| = \lim_{n\to\infty} \|P_n x\| \leq p(x),$$

showing that id_X is continuous $(X, p) \to (X, \|\cdot\|)$. Because (X, p) and $(X, \|\cdot\|)$ are Banach spaces and id_X is a continuous linear isomorphism, by the open mapping theorem[6] there is some $a > 0$ such that

$$\|x\| = \|\mathrm{id}_X\| \geq a p(x), \qquad x \in X.$$

Then

$$\|P_n x\| \leq p(x) \leq \frac{1}{a} \|x\|,$$

which means that $P_n : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is a bounded linear operator with operator norm $\leq \frac{1}{a}$. $\square$

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[6] Walter Rudin, *Functional Analysis*, second ed., p. 49, Corollary 2.12c.

Each P_n is a bounded linear operator on X, and for each $x \in X$, $P_n x \to x$, which means that $P_n \to \mathrm{id}_X$ in the **strong operator topology**. The fact that the functions $h_k^* : X \to \mathbb{R}$ are linear and continuous means that they belong to the dual space X^*.

In Theorem 2, if $\sup_{n \geq 1} \|P_n\| = 1$, we call the Schauder basis (x_k) **monotone**.

The following theorem gives sufficient and necessary conditions under which a sequence in a Banach space X is a basic sequence.[7] Thus if a sequence satisfies this condition and its linear span is dense in X space, then it is a Schauder basis.

Theorem 3. Suppose that X is a Banach space and that (x_k) is a sequence of nonzero elements of X. There is some K such that for any sequence of real numbers (a_k) and any $n < N$,

$$\left\| \sum_{k=1}^{n} a_k x_k \right\| \leq K \left\| \sum_{k=1}^{N} a_k x_k \right\| \tag{2}$$

if and only if (x_k) is a basic sequence.

Proof. It is apparent that $K \geq 1$. Let S be the linear span of (x_k), and let S_n be the linear span of $\{x_1, \ldots, x_n\}$. For $n < N$,

$$|a_n| \, \|x_n\| \leq \left\| \sum_{k=1}^{n-1} a_k x_k \right\| + \left\| \sum_{k=1}^{n} a_k x_k \right\| \leq 2K \left\| \sum_{k=1}^{N} a_k x_k \right\|. \tag{3}$$

Thus if $\sum_{k=1}^{N} a_k x_k = 0$ then using the above with $a_{N+1} = 0$, for $1 \leq n \leq N$ we get $a_n = 0$, showing that (x_k) is linearly independent. Because (x_k) is linearly independent, it makes sense to define a linear map $Q_n : S \to S_n$ by

$$Q_n x_k = \begin{cases} x_k & k \leq n \\ 0 & k > n. \end{cases}$$

For $x = \sum_{k=1}^{N} a_k x_k \in S$, if $n < N$ then by (2), $\|Q_n x\| \leq K \|x\|$, and if $n \geq N$ then $Q_n x = x$. Thus for each n, $Q_n : S \to S_n$ is a bounded linear operator with operator norm $\leq K$, and because X is a Banach space and $\overline{S}_n = S_n$, there is a unique bounded linear operator $P_n : \overline{S} \to S_n$ whose restriction to S is equal to Q_n, which satisfies $\|P_n\| = \|Q_n\| \leq K$.[8] For $s \in S$, $Q_n Q_m s = Q_{\min(m,n)} s$, and for $x \in \overline{S}$, there is a sequence $s_k \in S$ that tends to x, thus

$$P_n P_m x = \lim_{k \to \infty} P_n P_m s_k = \lim_{k \to \infty} P_{\min(m,n)} s_k = P_{\min(m,n)} x. \tag{4}$$

For $x \in \overline{S}$ and $\epsilon > 0$, there is some $s \in S$ with $\|s - x\| \leq \epsilon$. Then there are

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[7] Joseph Diestel, *Sequences and Series in Banach Spaces*, p. 36, Chapter V, Theorem 1.

[8] Gert K. Pedersen, *Analysis Now*, revised printing, p. 47, Proposition 2.1.11.

some $a_1, \dots, a_m \in \mathbb{R}$ for which $s = \sum_{i=1}^m a_i x_i$, and for $n > m$,

$$\begin{aligned} \|x - P_n x\| &\leq \|x - s\| + \|s - P_n s\| + \|P_n s - P_n x\| \\ &= \|x - s\| + \|P_n s - P_n x\| \\ &\leq \epsilon + K \|s - x\| \\ &\leq (K+1)\epsilon, \end{aligned}$$

showing that $P_n x \to x$. It follows from this and (4) that for each $x \in \overline{S}$ there is a sequence of real numbers (c_k) such that $\sum_{k=1}^n c_k x_k \to x$. If (b_k) is another such sequence, let $a_k = c_k - b_k$, and then we obtain from (3) that $a_k = 0$ for each k. Therefore, (c_k) is the unique sequence of real numbers such that $\sum_{k=1}^n c_k x_k \to x$, which establishes that (x_k) is a Schauder basis for $\overline{S}$. $\square$

4 Haar system and Faber-Schauder system

For $k \geq 0$, for $1 \leq i \leq 2^k$, and for $n = 2^k + i$, write

$$\Delta_n = \Delta_k^i = \left(\frac{i-1}{2^k}, \frac{i}{2^k} \right),$$

and we write $\Delta_1 = \Delta_0^0 = (0,1)$.[9] Thus

$$\Delta_1 = \Delta_0^0 = (0,1), \ \Delta_2 = \Delta_0^1 = (0,1), \ \Delta_3 = \Delta_1^1 = \left(0, \frac{1}{2}\right), \ \Delta_4 = \Delta_1^2 = \left(\frac{1}{2}, 1\right).$$

If $(a,b) \subset [0,1]$, let $(a,b)^- = \left(a, \frac{a+b}{2}\right)$ and $(a,b)^+ = \left(\frac{a+b}{2}, b\right)$. Thus

$$\Delta_n^- = (\Delta_k^i)^- = \left(\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}} \right) = \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}} \right) = \Delta_{k+1}^{2i-1}$$

and

$$\Delta_n^+ = (\Delta_k^i)^+ = \left(\frac{2i-1}{2^{k+1}}, \frac{i}{2^k} \right) = \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}} \right) = \Delta_{k+1}^{2i}.$$

Lemma 4. If $n \geq m$ then either $\Delta_n \subset \Delta_m$ or $\Delta_n \cap \Delta_m = \emptyset$.

Proof. Let $n = 2^k + i$ and $m = 2^l + j$, with $k \geq l$. Then

$$\Delta_l^j = \left(\frac{j-1}{2^l}, \frac{j}{2^l} \right) = \left(\frac{2^{k-l}(j-1)}{2^k}, \frac{2^{k-l}j}{2^k} \right).$$

There are three cases: (i) $i \leq 2^{k-l}(j-1)$, (ii) $2^{k-l}(j-1) < i \leq 2^{k-l}j$, (iii) $i > 2^{k-l}j$. In the first case, $\Delta_k^i \cap \Delta_l^j = \emptyset$, i.e. $\Delta_n \cap \Delta_m = \emptyset$. In the second case, $i - 1 \geq 2^{k-l}(j-1)$ and $i \leq 2^{k-l}$, so $\Delta_k^i \subset \Delta_l^j$, i.e. $\Delta_n \subset \Delta_m$. In the third case, $i - 1 \geq 2^{k-l}j$ so $\Delta_k^i \cap \Delta_l^j = \emptyset$, i.e. $\Delta_n \cap \Delta_m = \emptyset$. $\square$

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[9] We are partly following the presentation in B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, p. 61, Chapter III.

We define $\chi_1 = 1$, and for $k \geq 0$, $1 \leq i \leq 2^k$, and $n = 2^k + i$, we define

$$\chi_n(t) = \begin{cases} 1 & t \in \Delta_n^- = \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right) \\ -1 & t \in \Delta_n^+ = \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right) \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$\chi_2(t) = \begin{cases} 1 & t \in \left(0, \frac{1}{2}\right) \\ -1 & t \in \left(\frac{1}{2}, 1\right) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_3(t) = \begin{cases} 1 & t \in \left(0, \frac{1}{4}\right) \\ -1 & t \in \left(\frac{1}{4}, \frac{1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_4(t) = \begin{cases} 1 & t \in \left(\frac{1}{2}, \frac{3}{4}\right) \\ -1 & t \in \left(\frac{3}{4}, 1\right) \\ 0 & \text{otherwise.} \end{cases}$$

We call (χ_n) the **Haar system**. It is a fact that (χ_n) is a monotone Schauder basis for the Banach space $L^1[0,1]$ with the norm $\|f\|_{L^1} = \int_0^1 |f(t)|dt$.[10]

Now we define $\phi_1 = 1$ and for $n > 1$ we define $\phi_n : [0,1] \to \mathbb{R}$ by

$$\phi_n(t) = \int_0^t \chi_{n-1}(u)du.$$

Each ϕ_n belongs to $C[0,1]$, and we call (ϕ_n) the **Faber-Schauder system**. For example,

$$\phi_2(t) = \int_0^t \chi_1(u)du = t,$$

and

$$\phi_3(t) = \begin{cases} \int_0^t 1du & t \in \left[0, \frac{1}{2}\right] \\ \int_0^{1/2} 1du + \int_{1/2}^t -1du & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} t & t \in \left[0, \frac{1}{2}\right] \\ -t + 1 & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$\phi_4(t) = \begin{cases} \int_0^t 1du & t \in \left[0, \frac{1}{4}\right] \\ \int_0^{1/4} 1du + \int_{1/4}^t -1du & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ 0 & t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} t & t \in \left[0, \frac{1}{4}\right] \\ -t + \frac{1}{2} & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ 0 & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

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[10] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach Spaces I and II*, p. 3.

and

$$\phi_5(t) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right] \\ \int_{1/2}^{t} 1 du & t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ \int_{1/2}^{3/4} 1 du + \int_{3/4}^{t} -1 du & t \in \left[\frac{3}{4}, 1\right] \end{cases} = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right] \\ t - \frac{1}{2} & t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ -t + 1 & t \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Generally for $k \geq 0$, for $1 \leq i \leq 2^k$, and for $n = 2^k + i$,

$$\phi_{n+1}(t) = \begin{cases} t - \frac{i-1}{2^k} & t \in \Delta_n^- = \left(\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}\right) \\ -t + \frac{i}{2^k} & t \in \Delta_n^+ = \left(\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}\right) \\ 0 & \text{otherwise.} \end{cases}$$

We remark that

$$\|\phi_{n+1}\| = \frac{2i-1}{2^{k+1}} - \frac{i-1}{2^k} = 2^{-k-1}.$$

5 Riesz representation theorem

Let $(X, \mathfrak{M})$ be a measurable space. A **signed measure** is a function $\mu : \mathfrak{M} \to [-\infty, \infty]$ such that (i) $\mu(\emptyset) = 0$, (ii) μ assumes at most one of the values $-\infty, \infty$, and (iii) if (E_j) is a sequence of disjoint elements of $\mathfrak{M}$ then $\mu\left(\bigcup E_j\right) = \sum \mu(E_j)$. A **finite signed measure** is a signed measure whose image is contained in $\mathbb{R}$. We denote by $\mathrm{ca}(\mathfrak{M})$ the collection of all finite signed measures on $\mathfrak{M}$. For $\mu, \lambda \in \mathrm{ca}(\mathfrak{M})$ and for $c \in \mathbb{R}$, define

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E), \qquad (c\mu)(E) = c\mu(E)$$

for $E \in \mathfrak{M}$, and we check that with addition and scalar multiplication thus defined $\mathrm{ca}(\mathfrak{M})$ is a real vector space. A **positive measure** is a signed measure whose imaged is contained in $[0, \infty]$. For $\mu, \lambda \in \mathrm{ca}(\mathfrak{M})$, we write

$$\mu \geq \lambda$$

if $\mu - \lambda$ is a positive measure; in any case $\mu - \lambda \in \mathrm{ca}(\mathfrak{M})$. We check that $\leq$ is a partial order on $\mathrm{ca}(\mathfrak{M})$ with which $\mathrm{ca}(\mathfrak{M})$ is an **ordered vector space**. Finally, a **probability measure** is a positive measure satisfying $\mu(X) = 1$, which in particular belongs to $\mathrm{ca}(\mathfrak{M})$.

For $E \in \mathfrak{M}$, a **partition of** E is a countable subset $\{E_i\}$ of $\mathfrak{M}$ whose members are pairwise disjoint. For $\mu \in \mathrm{ca}(\mathfrak{M})$ and $E \in \mathfrak{M}$ we define

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : \{E_i\} \text{ is a partition of } E \right\}.$$

It is immediate that $|\mu(E)| \leq |\mu|(E)$. It is proved that $|\mu|$ is a finite positive measure on $\mathfrak{M}$, called the **total variation measure of** μ.[11] For $\mu \in \mathrm{ca}(\mathfrak{M})$, define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \qquad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

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[11] Walter Rudin, *Real and Complex Analysis*, third ed., pp. 117–118, Theorem 6.2 and Theorem 6.4.

Because $|\mu|$ is a finite positive measure and $|\mu(E)| \leq |\mu(E)|$, μ^+ and μ^- are finite positive measures, called the **positive and negative variations of** μ. Then $\mu = \mu^+ - \mu^-$, called the **Jordan decomposition of** μ.

For $\mu \in \mathrm{ca}(\mathfrak{M})$, define

$$\|\mu\| = |\mu|(X) = \mu^+(X) + \mu^-(X).$$

One checks that $\|\cdot\|$ is a norm on $\mathrm{ca}(\mathfrak{M})$.

For a compact Hausdorff space K and for $f, g \in C(K)$, write $g \geq f$ when $(g-f)(t) \geq 0$ for all $t \in K$. A **positive linear functional** is a linear map $\phi : C(K) \to \mathbb{R}$ such that $\phi(f) \geq 0$ when $f \geq 0$. In this case, because $\|f\| \cdot 1 + f \geq 0$,

$$\|f\| \, \phi(1) + \phi(f) = \phi(\|f\| \cdot 1 + f) \geq 0$$

i.e. $-\phi(f) \leq \phi(1) \, \|f\|$, and because $\|f\| \cdot 1 - f \geq 0$,

$$\|f\| \, \phi(1) - \phi(f) = \phi(\|f\| \cdot 1 - f) \geq 0,$$

i.e. $\phi(f) \leq \phi(1) \, \|f\|$, showing because $\|1\| = 1$ that the operator norm of ϕ is $\|\phi\| = \phi(1)$, and in particular that $\phi \in C(K)^*$.

For normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, an **isometric isomorphism** from V to W is a linear isomorphism $T : V \to W$ satisfying $\|Tv\|_W = \|v\|_V$ for all $v \in V$. The simplest version of the Riesz representation theorem is for compact metrizable spaces, for which we do not need to speak about regular Borel measures or continuous functions vanishing at infinity.[12]

Theorem 5 (Riesz representation theorem for compact metrizable spaces)**.** Let K be a compact metrizable space and define $\Lambda : \mathrm{ca}(\mathscr{B}_K) \to C(K)^*$ by

$$\Lambda(\mu)(f) = \int_K f d\mu, \qquad \mu \in \mathrm{ca}(\mathscr{B}_K), \quad f \in C(K).$$

Λ is an isometric isomorphism, and is order preserving: if $\mu \geq 0$ then $\Lambda(\mu) \in C(K)^*$ is a positive linear functional, and thus if $\mu \geq \lambda$ then $\Lambda(\mu) \geq \Lambda(\lambda)$.

6 The Borel σ-algebra of $C[0,1]$

Let $I = [0,1]$, with the relative topology inherited from $\mathbb{R}$, with which I is a compact metric space. For $t_1, \ldots, t_n \in I$, we define $\pi_{t_1,\ldots,t_n} : C(I) \to \mathbb{R}^n$ by

$$\pi_{t_1,\ldots,t_n}(x) = (x(t_1), \ldots, x(t_n)), \qquad x \in C(I),$$

which is continuous.

For a set X and a collection of functions $f_t : X \to \mathbb{R}$, the coarsest σ-algebra on X such that each f_t is measurable $X \to \mathbb{R}$, where $\mathbb{R}$ has the Borel σ-algebra, is called the **σ-algebra generated by** $\{f_t : t \in I\}$, and is denoted by $\sigma(\{f_t : t \in I\})$. We show that the Borel σ-algebra of $C(I)$ is equal to the σ-algebra generated by the family of projection maps $C(I) \to \mathbb{R}$.[13]

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[12] Walter Rudin, *Real and Complex Analysis*, third ed., p. 130, Theorem 6.19.

[13] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, p. 212, Theorem 2.1.

Theorem 6. Let $\mathscr{A}$ be the σ-algebra generated by the family $\{\pi_t : t \in [0,1]\}$. Then $\mathscr{B}_{C[0,1]} = \mathscr{A}$.

Proof. Because $C(I)$ is a separable metric space it is second-countable, and so if U is an open subset of $C(I)$ then U is equal to the union of countably many open balls. Each open ball is equal to the union of countably many closed balls: $B(x,r) = \bigcup \overline{B(x, r-1/n)}$. Therefore each open subset of $C(I)$ is equal to the union of countably many closed balls, and to prove that $\mathscr{B}_{C(I)} \subset \mathscr{A}$ it suffices to prove that all closed balls belong to $\mathscr{A}$. To this end, let $q_1, q_2, \ldots$ be an enumeration of $[0,1] \cap \mathbb{Q}$, let $x \in C(I)$, and let $r > 0$. Suppose that $|y(q_n) - x(q_n)| \leq r$ for all n, and take $t \in [0,1]$. Then there is a subsequence q_{a_n} of q_n that tends to t, and because $y - x : [0,1] \to \mathbb{R}$ is continuous, $|y(q_{a_n}) - x(q_{a_n})| \to |y(t) - x(t)|$, and then because for each n we have $|y(q_{a_n}) - x(q_{a_n})| \leq r$ it follows that $|y(t) - x(t)| \leq r$. This establishes

$$\{y \in C(I) : \|y - x\| \leq r\} = \bigcap_{n=1}^{\infty} \{y \in C(I) : |y(q_n) - x(q_n)| \leq r\}.$$

But

$$\{y \in C(I) : |y(q_n) - x(q_n)| \leq r\} = \pi_{q_n}^{-1}([\pi_{q_n}(x) - r, \pi_{q_n}(x) + r]),$$

which belongs to $\mathscr{A}$. Thus $\overline{B(x,r)}$ is a countable intersection of elements of $\mathscr{A}$ and so belongs to $\mathscr{A}$, which shows that $\mathscr{B}_{C(I)} \subset \mathscr{A}$.

On the other hand, for each $t \in I$ the map $\pi_t : C(I) \to \mathbb{R}$ is continuous and hence is measurable $\mathscr{B}_{C(I)} \to \mathscr{B}_{\mathbb{R}}$.[14] Therefore $\mathscr{A} \subset \mathscr{B}_{C(I)}$. ☐

7 Relatively compact subsets of $C[0,1]$

If (M,d) is a metric space, a subset A of M is called **totally bounded** if for each $\epsilon > 0$ there are finitely many points $x_1, \ldots, x_n \in M$ such that for any point $x \in M$ there is some i for which $d(x, x_i) < \epsilon$. It is immediate that a compact metric space is totally bounded, and the **Heine-Borel theorem** states that a metric space is compact if and only if it is complete and totally bounded.[15] On the other hand, one checks that if (M,d) is a metric space and A is a totally bounded subset of M, then the closure $\overline{A}$ is a totally bounded subset of M. Thus, if A is a totally bounded subset of a complete metric space (M,d), then the closure $\overline{A}$ is itself a complete metric space (because (M,d) is a complete metric space), and because $\overline{A}$ is complete and totally bounded, by the Heine-Borel theorem it is compact.

If X is a topological space and $\mathscr{F}$ is a subset of $C(X)$, we say that $\mathscr{F}$ is **equicontinuous at** $x \in X$ if for each $\epsilon > 0$ there is a neighborhood $U_{x,\epsilon}$ of

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[14] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 140, Corollary 4.26.

[15] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 86, Theorem 3.28.

x such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathscr{F}$ and for all $y \in U_{x,\epsilon}$, and we call $\mathscr{F}$ **equicontinuous** if it is equicontinuous at each $x \in X$. We call $\mathscr{F}$ **pointwise bounded** if for each $x \in X$, $\{f(x) : f \in \mathscr{F}\}$ is a bounded subset of $\mathbb{R}$. The **Arzelà-Ascoli theorem** states that for a compact Hausdorff space X and for $\mathscr{F} \subset C(X)$, $\mathscr{F}$ is equicontinuous and pointwise bounded if and only if $\mathscr{F}$ is totally bounded.[16] Then the closure $\overline{\mathscr{F}}$ is totally bounded and is itself a complete metric space, and hence is a compact metric space. That is, for a compact Hausdorff space X, $\mathscr{F}$ is equicontinuous and pointwise bounded if and only if $\mathscr{F}$ is relatively compact in $C(X)$.

For $x \in C[0,1]$ and $\delta > 0$, define

$$\omega_x(\delta) = \sup_{s,t \in I, |s-t| \leq \delta} |x(s) - x(t)|.$$

For $x \in C[0,1]$ because $x : I \to \mathbb{R}$ is continuous and I is compact, x is uniformly continuous on I. Thus for $\epsilon > 0$ there is some $\delta_\epsilon > 0$ such that when $|s-t| \leq \delta_\epsilon$, $|x(s) - x(t)| \leq \epsilon$, hence if $\delta \leq \delta_\epsilon$ then $\omega_x(\delta) \leq \epsilon$, i.e. $\omega_x(\delta) \to 0$ as $\delta \to 0$. We give sufficient and necessary conditions for a subset of $C[0,1]$ to be relatively compact.

Lemma 7. *Let $A \subset C[0,1]$. A is relatively compact if and only if*

$$\sup_{x \in A} |x(0)| < \infty \tag{5}$$

and

$$\lim_{\delta \downarrow 0} \sup_{x \in A} \omega_x(\delta) = 0. \tag{6}$$

Proof. For $t \in I$ and for $r > 0$, let

$$B_r(t) = \{s \in I : |s - t| < r\},$$

which is an open neighborhood of t. The Arzelà-Ascoli theorem tells us that A is relatively compact if and only if A is equicontinuous and pointwise bounded. If A is relatively compact, then being pointwise bounded yields (5). Let $\epsilon > 0$. Because A is equicontinuous, for each $t \in I$ there is some $\delta_t > 0$ such that for all $x \in A$ and for all $s \in B_{\delta_t}(t)$ we have $|x(s) - x(t)| < \epsilon/2$. Then because I is compact, there are $t_1, \ldots, t_n \in I$ such that $I = \bigcup_{i=1}^n B_{\delta_{t_i}}(t_i)$. Let $\delta_\epsilon > 0$ be the **Lebesgue number** for this open cover: for each $t \in I$ there is some i for which $B_{\delta_\epsilon}(t) \subset B_{\delta_{t_i}}(t_i)$. For $0 < \delta \leq \delta_\epsilon$, for $x \in A$, and for $|s - t| \leq \delta/2$, there is some i for which $B_\delta(t) \subset B_{\delta_{t_i}}(t_i)$, so $s, t \in B_{\delta_{t_i}}(t_i)$, which means that $|x(s) - x(t_i)| < \epsilon/2$ and $|x(t) - x(t_i)| < \epsilon/2$, and thus $|x(s) - x(t)| \leq |x(s) - x(t_i)| + |x(t_i) - x(t)| < \epsilon$. Therefore $\omega_{\delta/2}(x) \leq \epsilon$, and this is true for all $x \in A$ which proves (6).

Suppose now that (5) and (6) are true. By (6), there is some $m \geq 1$ such that $\sup_{x \in A} \omega_x(1/m) \leq 1$, and therefore for $x \in A$, $\|x\| \leq |x(0)| + m$. With (5)

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[16] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 137, Theorem 4.43.

this yields $\sup_{x\in A} \|x\| < \infty$, whence A is pointwise bounded. Let $t \in I$ and let $\epsilon > 0$. By (6) there is some $\delta > 0$ such that $\sup_{x\in A} \omega_x(\delta) < \epsilon$. Thus for $x \in A$ and for $|s - t| \leq \delta$, $|x(s) - x(t)| < \epsilon$, which shows that A is equicontinuous at t. This is true for all $t \in I$, so A is equicontinuous and by the Arzelà-Ascoli theorem we get that A is relatively compact in $C[0, 1]$. $\square$

8 Continuously differentiable functions

Let $f : [0, 1] \to \mathbb{R}$ be a function. We say that f is **differentiable at** $t \in [0, 1]$ if there is some $f'(t) \in \mathbb{R}$ such that[17]

$$\lim_{s\to t, s\in[0,1]\setminus\{t\}} \frac{f(s) - f(t)}{s - t} = f'(t).$$

If f is differentiable at each $t \in [0, 1]$, we say that f is **differentiable on** $[0, 1]$ and define $f' : [0, 1] \to \mathbb{R}$ by $t \mapsto f'(t)$. We define $C^1[0, 1]$ to be the collection of those $f : [0, 1] \to \mathbb{R}$ such that f is differentiable on $[0, 1]$ and $f' : [0, 1] \to \mathbb{R}$ is continuous. $C^1[0, 1]$ is contained in $C[0, 1]$, and it turns out that $C^1[0, 1]$ is a Borel set in $C[0, 1]$.[18] On the other hand, the collection differentiable functions $[0, 1] \to \mathbb{R}$, which is a subset of $C[0, 1]$, is not a Borel set in $C[0, 1]$.[19]

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[17] cf. Nicolas Bourbaki, *Elements of Mathematics. Functions of a Real Variable: Elementary Theory*, p. 3, Chapter I, §1, no. 1, Definition 1.

[18] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 70, §11.B, Example 2.

[19] S. M. Srivastava, *A Course on Borel Sets*, p. 139, Proposition 4.2.7.