Summable series and the Riemann rearrangement theorem

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1 Introduction

Let $\mathbb{N}$ be the set of positive integers. A function from $\mathbb{N}$ to a set is called a *sequence*. If X is a topological space and $x \in X$, a sequence $a : \mathbb{N} \to X$ is said to *converge to* x if for every open neighborhood U of x there is some N_U such that $n \geq N_U$ implies that $a_n \in U$. If there is no $x \in X$ for which a converges to x, we say that a *diverges*.

Let $a : \mathbb{N} \to \mathbb{R}$. We define $s(a) : \mathbb{N} \to \mathbb{R}$ by $s_n(a) = \sum_{k=1}^n a_k$. We call $s_n(a)$ the *nth partial sum of the sequence* a, and we call the sequence $s(a)$ a *series*. If there is some $\sigma \in \mathbb{R}$ such that $s(a)$ converges to σ, we write

$$\sum_{k=1}^{\infty} a_k = \sigma.$$

2 Goldbach

Euler [22, §110]: "If, as is commonly the case, we take the sum of a series to be the aggregate of all of its terms, actually taken together, then there is no doubt that only infinite series that converge continually closer to some value, the more terms we actually add, can have sums".

Euler Goldbach correspondence nos. 55, 161, 162.

3 Dirichlet

In 1837 Dirichlet proved that one can rearrange terms in an absolutely convergent series and not change the sum, and gave examples to show that this was not the case for conditionally convergent series.

If a is a sequence and the series $s(|a|)$ converges, we say that the series $s(a)$ is *absolutely convergent*. Because $\mathbb{R}$ is a complete metric space, a series being absolutely convergent implies that it is convergent.

Dirichlet [14] and [15, p. 176, §101]

Elstrodt [20] and [19]

In the following theorem we prove that if a series converges absolutely, then every rearrangement of it converges to the same value. Our proof follows Landau [45, p. 157, Theorem 216].

Theorem 1. *If a is a sequence for which $s(a)$ converges absolutely and*

$$\sum_{n=1}^{\infty} a_n = \sigma,$$

then for any bijection $\lambda : \mathbb{N} \to \mathbb{N}$, the series $s(a \circ \lambda)$ converges to σ.

Proof. Let $\epsilon > 0$, and let M be large enough so that

$$\sum_{n=M}^{\infty} |a_n| < \epsilon.$$

Let r be large enough so that

$$\{n : 1 \leq n < M\} \subseteq \{\lambda_n : 1 \leq n \leq r\}.$$

Fix $m \geq r$, and let $h : \mathbb{N} \to \mathbb{N}$ be the sequence whose terms are the elements of

$$\mathbb{N} \setminus \{\lambda_n : 1 \leq n \leq m\}$$

arranged in ascending order. If $t + m \geq \max_{1 \leq n \leq m} \lambda_n$ then

$$\{\lambda_n : 1 \leq n \leq m\} \cup \{h_n : 1 \leq n \leq t\} = \{n : 1 \leq n \leq t + m\},$$

and hence

$$\sum_{n=1}^{m} a_{\lambda_n} + \sum_{n=1}^{t} a_{h_n} = \sum_{n=1}^{t+m} a_n.$$

Taking $t \to \infty$, we get

$$\sum_{n=1}^{m} a_{\lambda_n} + \sum_{n=1}^{\infty} a_{h_n} = \sigma;$$

the series $s(a \circ h)$ converges because for sufficiently large n, $h_n = n$. Hence, for every $m \geq r$,

$$\left| \sum_{n=1}^{m} a_{\lambda_n} - \sigma \right| = \left| \sum_{n=1}^{\infty} a_{h_n} \right| \leq \sum_{n=1}^{\infty} |a_{h_n}| \leq \sum_{n=M}^{\infty} |a_n| < \delta,$$

which shows that $s(a \circ \lambda)$ converges to σ. □

4 Riemann rearrangement theorem

If $a : \mathbb{N} \to \mathbb{R}$ and $\lambda : \mathbb{N} \to \mathbb{N}$ is a bijection, we call the sequence $a \circ \lambda : \mathbb{N} \to \mathbb{R}$ a *rearrangement* of the sequence a.

Because $\mathbb{N}$ is a well-ordered set, if there are at least n elements in the set $\{k \in \mathbb{N} : a_k \geq 0\}$ then it makes sense to talk about the nth nonnegative term in the sequence a. If a were not a function from $\mathbb{N}$ to $\mathbb{R}$ but merely a function from a countable set to $\mathbb{R}$, it would not make sense to talk about the nth nonnegative term in a or the nth negative term in a.

Riemann [60, pp. 96-97]

Our proof follows Landau [45, p. 158, Theorem 217].

Theorem 2 (Riemann rearrangement theorem)**.** *If $a : \mathbb{N} \to \mathbb{R}$ and $s(a)$ converges but $s(|a|)$ diverges, then for any nonnegative real number σ there is some rearrangement b of a such that $s(b) \to \sigma$.*

Proof. Define $p, q : \mathbb{N} \to \mathbb{R}$ by

$$p_n = \frac{|a_n| + a_n}{2}, \qquad q_n = \frac{|a_n| - a_n}{2}.$$

p_n and q_n are nonnegative, and satisfy $p_n - q_n = a_n$, $p_n + q_n = |a_n|$. If one of $s(p)$ or $s(q)$ converges and the other diverges, we obtain a contradiction from

$$s_n(a) = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (p_k - q_k) = \sum_{k=1}^{n} p_k - \sum_{k=1}^{n} q_k = s_n(p) - s_n(q)$$

and the fact that $s(a)$ converges. If both $s(p)$ and $s(q)$ converge, then we obtain a contradiction from

$$s_n(|a|) = \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} (p_k + q_k) = \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} q_k = s_n(p) + s_n(q)$$

and the fact that $s(|a|)$ diverges. Therefore, both $s(p)$ and $s(q)$ diverge.

Because $s(a)$ converges and $s(|a|)$ diverges, there are infinitely many n with $a_n > 0$ and there are infinitely many n with $a_n < 0$. Let P_n be the nth nonnegative term in the sequence a, and let Q_n be the absolute value of the nth negative term in the sequence a. The fact that $s(p)$ diverges implies that $s(P)$ diverges, and the fact that $s(q)$ diverges implies that $s(Q)$ diverges.

Let $\sigma \geq 0$. We define sequences $\mu, \nu : \mathbb{N} \to \mathbb{N}$ by induction as follows. Let μ_1 be the least element of $\mathbb{N}$ such that

$$s_{\mu_1}(P) > \sigma,$$

and with μ_1 chosen, let ν_1 be the least element of $\mathbb{N}$ such that

$$s_{\mu_1}(P) - s_{\nu_1}(Q) < \sigma.$$

Let m_2 be the least element of $\mathbb{N}$ such that

$$s_{\mu_2}(P) - s_{\nu_1}(Q) > \sigma,$$

and with μ_2 chosen, let ν_2 be the least element of $\mathbb{N}$ such that

$$s_{\mu_2}(P) - s_{\nu_2}(Q) < \sigma.$$

It is straightforward to check that $\mu_2 > \mu_1$ and $\nu_2 > \nu_1$.

Suppose that $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$ have been chosen, that μ_n is the least element of $\mathbb{N}$ such that

$$s_{\mu_n}(P) - s_{\nu_{n-1}}(Q) > \sigma,$$

that ν_n it the least element of $\mathbb{N}$ such that

$$s_{\mu_n}(P) - s_{\nu_n}(Q) < \sigma$$

and that $\mu_n > \mu_{n-1}$ and $\nu_n > \nu_{n-1}$. Let μ_{n+1} be the least element of $\mathbb{N}$ such that

$$s_{\mu_{n+1}}(P) - s_{\nu_n}(Q) > \sigma,$$

and with μ_{n+1} chosen, let ν_{n+1} be the least element of $\mathbb{N}$ such that

$$s_{\mu_{n+1}}(P) - s_{\nu_{n+1}}(Q) < \sigma.$$

It is straightforward to check that $\mu_{n+1} > \mu_n$ and $\nu_{n+1} > \nu_n$.

Define $b : \mathbb{N} \to \mathbb{R}$ by taking b_n to be the nth term in

$$P_1, \ldots, P_{\mu_1}, -Q_1, \ldots, -Q_{\nu_1}, P_{\mu_1+1}, \ldots, P_{\mu_2}, -Q_{\nu_1+1}, \ldots, -Q_{\nu_2}, \ldots,$$

which, because the sequences μ and ν are strictly increasing, is a rearrangement of the sequence a.

□

5 Symmetry

Don't use order where it is accidental.

6 Nets

A *directed set* is a set D and a binary relation $\preceq$ satisfying

- if $m, n, p \in D$, $m \preceq n$, and $n \preceq p$, then $m \preceq p$
- if $m \in D$, then $m \preceq m$
- if $m, n \in D$, then there is some $p \in D$ such that $m \preceq p$ and $n \preceq p$.

For example, let A be a set, let D be the set of all subsets of A, and say that $F \preceq G$ when $F \subseteq G$. Check that $(D, \preceq)$ is a directed set: for $F, G \in D$, we have $F \cup G \in D$, and $F \cup G$ is an upper bound for both F and G.

A *net* is a function from a directed set $(D, \preceq)$ to a set X. Let (X, τ) be a topological space, let $S : (D, \preceq) \to (X, \tau)$ be a net, and let $x \in X$. We say that S *converges to* x if for every $U \in \tau$ with $x \in U$ there is some $N_U \in D$ such that $N_U \preceq i$ implies that $S(i) \in U$. One proves that a topological space is Hausdorff if and only if every net in this space converges to at most one point [41, p. 67, Theorem 3].

A net $S : (D, \preceq) \to \mathbb{R}$ is said to be *increasing* if $m \preceq n$ implies that $S(m) \leq S(n)$.

Lemma 3. *If $S : (D, \preceq) \to \mathbb{R}$ is an increasing net and the range R of S has an upper bound, then S converges to the supremum of R.*

Proof. Because R is a subset of $\mathbb{R}$ that has an upper bound, it has a supremum, call it σ. To say that σ is the supremum of R means that for all $r \in R$ we have $r \leq \sigma$ (σ is an upper bound) and that for all $\epsilon > 0$ there is some $r_\epsilon \in R$ with $\sigma - \epsilon < r_\epsilon$ (nothing less than σ is an upper bound). Take $\epsilon > 0$. There is some $r_\epsilon \in R$ with $\sigma - \epsilon < r_\epsilon$. As $r_\epsilon \in R$, there is some $n_\epsilon \in D$ with $S(n_\epsilon) = r_\epsilon$. If $n_\epsilon \preceq n$, then because S is increasing, $S(n_\epsilon) \leq S(n)$, and hence

$$\sigma - \epsilon < r_\epsilon = S(n_\epsilon) \leq S(n).$$

But $S(n) \in R$, so $S(n) \leq \sigma$. Hence $n_\epsilon \preceq n$ implies that $|S(n) - \sigma| < \epsilon$, showing that S converges to σ. □

7 Unordered sums

Let A be a set, and let $\mathscr{P}_0(A)$ be the set of all finite subsets of A. Check that $(\mathscr{P}_0(A), \subseteq)$ is a directed set: if $F, G \in \mathscr{P}_0(A)$ then $F \cup G \in \mathscr{P}_0(A)$ and $F \cup G$ is an upper bound for both F and G. Let $f : A \to \mathbb{R}$ be a function, and define $S_f : \mathscr{P}_0(A) \to \mathbb{R}$ by

$$S_f(F) = \sum_{a \in F} f(a), \qquad F \in \mathscr{P}_0(A).$$

If the net S_f converges, we say that the function f is *summable*, and we call the element of $\mathbb{R}$ to which S_f converges the *unordered sum of* f, denoted by

$$\sum_{a \in A} f(a).$$

If B is a subset of A, we say that f is *summable over* B if the restriction of f to B is summable. If f_B is the restriction of f to B and f is summable over B (i.e. f_B is summable), by

$$\sum_{a \in B} f(a)$$

we mean

$$\sum_{a\in B} f_B(a).$$

Lemma 4. *Suppose that $f, g : A \to \mathbb{R}$ are functions and $\alpha, \beta \in \mathbb{R}$. If f and g are summable, then $\alpha f + \beta g$ is summable and*

$$\sum_{a\in A}(\alpha f(a) + g(a)) = \alpha \sum_{a\in A} f(a) + \beta \sum_{a\in A} g(a).$$

Proof. Let $\sigma_1 = \sum_{a\in A} f(a)$ and $\sigma_2 = \sum_{a\in A} g(a)$, and set $h = \alpha f + \beta g$. For $\epsilon > 0$, there is some $F_\epsilon \in \mathscr{P}_0(A)$ such that $F_\epsilon \subseteq F \in \mathscr{P}_0(A)$ implies that $|S_f(F) - \sigma_1| < \epsilon$, and there is some $G_\epsilon \in \mathscr{P}_0(A)$ such that $G_\epsilon \subseteq G \in \mathscr{P}_0(A)$ implies that $|S_g(G) - \sigma_2| < \epsilon$. Let $H_\epsilon = F_\epsilon \cup G_\epsilon \in \mathscr{P}_0(A)$. If $H_\epsilon \subseteq H \in \mathscr{P}_0(A)$, then, as $F_\epsilon \subseteq H$ and $G_\epsilon \subseteq H$,

$$\begin{aligned}
|S_h(H) - (\alpha\sigma_1 + \beta\sigma_2)| &= \left|\sum_{a\in H}(\alpha f(a) + \beta g(a)) - \alpha\sigma_1 - \beta\sigma_2\right| \\
&= |\alpha S_f(H) + \beta S_g(H) - \alpha\sigma_1 - \beta\sigma_2| \\
&\leq |\alpha||S_f(H) - \sigma_1| + |\beta||S_g(H) - \sigma_2| \\
&\leq |\alpha|\epsilon + |\beta|\epsilon;
\end{aligned}$$

we write $\leq$ rather than $<$ in the last inequality to cover the case where $\alpha = \beta = 0$. It follows that S_h converges to $\alpha\sigma_1 + \beta\sigma_2$. □

The following lemma is simple to prove and ought to be true, but should not to be called obvious. For example, the Cesàro sum of the sequence $1, -1, 1, -1, \dots$ is $\frac{1}{2}$, while the Cesàro sum of the sequence $1, -1, 0, 1, -1, 0, \dots$ is $\frac{1}{3}$.

Lemma 5. *If $f : A \to \mathbb{R}$ is summable, then for any set C that contains A, the function $g : C \to \mathbb{R}$ defined by*

$$g(c) = \begin{cases} f(c) & c \in A \\ 0 & \textit{otherwise} \end{cases}$$

is summable, and

$$\sum_{a\in A} f(a) = \sum_{c\in C} g(c).$$

Proof. Let $\sigma = \sum_{a\in A} f(a)$. For $\epsilon > 0$, there is some $F_\epsilon \in \mathscr{P}_0(A)$ such that $F_\epsilon \subseteq F \in \mathscr{P}_0(A)$ implies that $|S_f(F) - \sigma| < \epsilon$. If $F_\epsilon \subseteq H \in \mathscr{P}_0(C)$, then, as

$F_\epsilon \subseteq H \cap A \in \mathscr{P}_0(A)$,

$$\begin{aligned}
|S_g(H) - \sigma| &= \left| \sum_{c \in H} g(c) - \sigma \right| \\
&= \left| \sum_{c \in H \cap A} g(c) + \sum_{c \in H \setminus A} g(c) - \sigma \right| \\
&= \left| \sum_{a \in H \cap A} f(a) + \sum_{c \in H \setminus A} 0 - \sigma \right| \\
&= |S_f(H \cap A) - \sigma| \\
&< \epsilon.
\end{aligned}$$

This shows that S_g converges to σ. □

The previous two lemmas are useful, and also convince us that unordered summation works similarly to finite sums. We now establish conditions under which a function is summable.

Lemma 6. *If $f : A \to \mathbb{R}$ is nonnegative and there is some $M \in \mathbb{R}$ such that $F \in \mathscr{P}_0(A)$ for all $S_f(F) \leq M$, then f is summable. If $f : A \to \mathbb{R}$ is nonnegative and summable, then $S_f(F) \leq \sum_{a \in A} f(a)$ for all $F \in \mathscr{P}_0(A)$.*

Proof. Suppose there is some $M \in \mathbb{R}$ such that if $F \in \mathscr{P}_0(A)$ then $S_f(F) \leq M$. That is, M is an upper bound for the range of S_f. Because f is nonnegative, the net S_f is increasing. We apply Lemma 3, which tells us that S_f converges to the supremum of its range. That S_f converges means that f is summable.

Suppose that f is summable, and let $\sigma = \sum_{a \in A} f(a)$. Suppose by contradiction that there is some $F_0 \in \mathscr{P}_0(A)$ such that $S_f(F_0) > \sigma$, and let $\epsilon = S_f(F_0) - \sigma$. Then there is some $F_\epsilon \in \mathscr{P}_0(A)$ such that $F_\epsilon \subseteq F \in \mathscr{P}_0(A)$ implies that $|S_f(F) - \sigma| < \epsilon$. As $F_\epsilon \subseteq F_0 \cup F_\epsilon \in \mathscr{P}_0(A)$, we have $|S_f(F_0 \cup F_\epsilon) - \sigma| < \epsilon$, and hence

$$S_f(F_0 \cup F_\epsilon) < \sigma + \epsilon = S_f(F_0).$$

But F_0 is contained in $F_0 \cup F_\epsilon$ and f is nonnegative, so

$$S_f(F_0) \leq S_f(F_0 \cup F_\epsilon),$$

which gives $S_f(F_0) < S_f(F_0)$, a contradiction. Therefore, there is no $F \in \mathscr{P}_0(A)$ for which $S_f(F_0) > \sigma$. □

Lemma 7. *Suppose that $f : A \to \mathbb{R}$ is a function and that $A_+ = \{a \in A : f(a) \geq 0\}$ and $A_- = \{a \in A : f(a) \leq 0\}$. Then, f is summable if and only if f is summable over both A_+ and A_-. If f is summable, then*

$$\sum_{a \in A} f(a) = \sum_{a \in A_+} f(a) + \sum_{a \in A_-} f(a).$$

Proof. Suppose that f is summable. Because f is summable, there is some $E \in \mathscr{P}_0(A)$ such that $E \subseteq F \in \mathscr{P}_0(A)$ implies that $|S_f(F) - \sigma| < 1$. Define

$$E_+ = \{a \in E : f(a) \geq 0\} \in \mathscr{P}_0(A_+), \qquad E_- = \{a \in E : f(a) \leq 0\} \in \mathscr{P}_0(A_-).$$

If $G \in \mathscr{P}_0(A_+)$ then $E \subseteq G \cup E \in \mathscr{P}_0(A)$, and hence $|S_f(G \cup E) - \sigma| < 1$. We have

$$S_{f_+}(G) = \sum_{a \in G} f(a) \leq \sum_{a \in G \cup E_+} f(a) = \sum_{a \in G \cup E} f(a) - \sum_{a \in E_-} f(a),$$

and hence

$$S_{f_+}(G) \leq S_f(G \cup E) - S_f(E_-) < \sigma + 1 - S_f(E_-).$$

That is, $\sigma + 1 - S_f(E_-)$ is an upper bound for the range of S_{f_+}. The net S_{f_+} is increasing, hence applying Lemma 3 we get that S_{f_+} converges. That is, f_+ is summable. If $H \in \mathscr{P}_0(A_-)$, then $E \subseteq H \cup E \in \mathscr{P}_0(A)$, and hence $|S_f(H \cup E) - \sigma| < 1$. We have

$$S_{f_-}(H) = \sum_{a \in H} f(a) \geq \sum_{a \in H \cup E_-} f(a) = \sum_{a \in H \cup E} f(a) - \sum_{a \in E_+} f(a),$$

and then

$$S_{f_-}(H) \geq S_f(H \cup E) - S_f(E_+) > \sigma - 1 - S_f(E_+),$$

showing that $-\sigma + 1 + S_f(E_+)$ is an upper bound for the net $-S_{f_-}$. As $-S_{f_-}$ is increasing, by Lemma 3 it converges, and it follows that S_{f_-} converges. That is, f_- is summable.

Suppose that f is summable over both A_+ and A_-. Let f_+ be the restriction of f to A_+ and let f_+ be the restriction of f to A_+, and define $g_+, g_- : A \to \mathbb{R}$ by

$$g_+(a) = \begin{cases} f(a) & a \in A_+ \\ 0 & a \in A_-, \end{cases} \qquad g_-(a) = \begin{cases} 0 & a \in A_+ \\ f(a) & a \in A_-. \end{cases}$$

By Lemma 5, f_+ being summable implies that g_+ is summable, with

$$\sum_{a \in A_+} f_+(a) = \sum_{a \in A} g_+(a),$$

and f_- being summable implies that g_- is summable, with

$$\sum_{a \in A_-} f_-(a) = \sum_{a \in A} g_-(a).$$

But $f = g_+ + g_-$, so by Lemma 4 we get that f is summable, with

$$\sum_{a \in A} f(a) = \sum_{a \in A} g_+(a) + \sum_{a \in A} g_-(a) = \sum_{a \in A_+} f_+(a) + \sum_{a \in A_-} f_-(a).$$

□

If $f : A \to \mathbb{R}$ is a function, we define $|f| : A \to \mathbb{R}$ by $|f|(a) = |f(a)|$.

Theorem 8. *If $f : A \to \mathbb{R}$ is a function, then f is summable if and only if $|f|$ is summable.*

Proof. Let $A_+ = \{a \in A : f(a) \geq 0\}$ and $A_- = \{a \in A : f(a) \leq 0\}$, and let f_+ and f_- be the restrictions of f to A_+ and A_- respectively. Suppose that f is summable. Then by Lemma 7 we get that f_+ is summable and f_- is summable. Let $F \in \mathscr{P}_0(A)$ and write $F_+ = \{a \in F : f(a) \geq 0\}$, $F_- = \{a \in F : f(a) \leq 0\}$. We have

$$S_{|f|}(F) = \sum_{a \in F} |f(a)| = \sum_{a \in F_+} f(a) - \sum_{a \in F_-} f(a) = S_{f_+}(F_+) - S_{f_-}(F_-).$$

But by Lemma 6, because the net S_{f_+} is increasing we have $S_{f_+}(F_+) \leq \sum_{a \in A_+} f_+(a)$, and because the net $-S_{f_-}$ is increasing we have $-S_{f_-}(F_-) \leq -\sum_{a \in A_-} f_-(a)$. Therefore, $\sum_{a \in A_+} f_+(a) - \sum_{a \in A_-} f_-(a)$ is an upper bound for the range of $S_{|f|}$. Moreover, $S_{|f|}$ is increasing, so by Lemma 6 it follows that $S_{|f|}$ converges, i.e. that $|f|$ is summable.

Suppose that $|f|$ is summable. By Lemma 6, for any $F \in \mathscr{P}_0(A_+)$ we have

$$S_{f_+}(F) = S_{|f|}(F) \leq \sum_{a \in A} |f|(a),$$

i.e., $\sum_{a \in A} |f|(a)$ is an upper bound for the range of S_{f_+}. As S_{f_+} is increasing, by Lemma 6 it follows that S_{f_+} converges, i.e., that f_+ is summable. Because $-S_{f_-}$ is increasing, we likewise get that $-S_{f_-}$ converges and hence that S_{f_-} converges, i.e. that f_- is summable. Now applying Lemma 7, we get that f is summable. $\square$

Theorem 9. *If $f : A \to \mathbb{R}$ is summable, then $\{a \in A : f(a) \neq 0\}$ is countable.*

Proof. Suppose by contradiction that $\{a \in A : f(a) \neq 0\}$ is uncountable. We have

$$\{a \in A : f(a) \neq 0\} = \{a \in A : |f(a)| > 0\} = \bigcup_{n \in \mathbb{N}} \left\{a \in A : |f(a)| \geq \frac{1}{n}\right\}.$$

Since this is a countable union, there is some $n \in \mathbb{N}$ such that $\left\{a \in A : |f(a)| \geq \frac{1}{n}\right\}$ is uncountable; in particular, this set is infinite. Because f is summable, by Theorem 8 we have that $|f|$ is summable, with unordered sum σ. Hence, there is some $F_1 \in \mathscr{P}_0(A)$ such that $F_1 \subseteq F \in \mathscr{P}_0(A)$ implies that $|S_{|f|}(F) - \sigma| < 1$. Let F be a finite subset of $\left\{a \in A : |f(a)| \geq \frac{1}{n}\right\}$ with at least $n(\sigma+1)$ elements. Then

$$S_{|f|}(F \cup F_1) = \sum_{a \in F \cup F_1} |f(a)| \geq \sum_{a \in F} |f(a)| \geq n(\sigma + 1) \cdot \frac{1}{n} = \sigma + 1.$$

But $F_1 \subseteq F \cup F_1 \in \mathscr{P}_0(A)$, so $S_{|f|}(F \cup F_1) < \sigma + 1$, a contradiction. Therefore, $\{a \in A : f(a) \neq 0\}$ is countable. $\square$

8 References

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