Book I of Euclid's *Elements* and application of areas

Jordan Bell
jordan.bell@gmail.com

February 22, 2023

Taisbak [32, pp. 28–29]:

> It may be appropriate to introduce the *The Helping Hand*, a well-known factotum in Greek geometry, who takes care that lines are drawn, points are taken, circles described, perpendiculars dropped, etc. The perfect imperative passive is its verbal mask: 'Let a circle have been described with centre A and radius AB'; 'let it lie given' *keistho dedomenon*. No one who has done the *Elements* in Greek will have missed it; never is there any of the commands or exhortations so familiar from our own class-rooms: 'Draw the median from vertex A', or 'If we cut the circle by that secant', or 'Let us add those squares together'. Always *The Helping Hand* is there first to see that things are done, and to keep the operations free from contamination by our mortal fingers.

On the perfect passive imperative, cf. Priscian, *Institutiones Grammaticae*: Book 8, Keil, 1870, II: 406, 15–27; 407, 1–9; Book 18, Keil, 1870, III: 238, 12–26.

Netz [29, p. xvii]:

> As is explained in chapter 1, most of the diagrams in Greek mathematical works have not yet been edited from manuscripts. The figures in modern editions are reconstructions made by modern editors, based on their modern understanding of what a diagram should look like. However, as will be argued below, such an understanding is culturally variable. It is therefore better to keep, as far as possible, to the diagrams as they are found in Greek manuscripts (that is, generally speaking, in Byzantine manuscripts).

Netz [29, p. 16]:

> Diagrams, as a rule, were not drawn on site. The limitations of the media available suggest, rather, the preparation of the diagram prior to the communicative act – a consequence of the inability to erase.

Netz [29, p. 25]:

> What we see, in short is that while the text is being worked through, the diagram is assumed to exist. The text takes the diagram for granted. This reflects the material implementation discussed above. This, in fact, is the simple explanation for the use of *perfect* imperatives in the references to the setting out – 'let the point A have been taken'. It reflects nothing more than the fact that, by the time one comes to discuss the diagram, it has already been drawn.

Netz [29, pp. 94–95]:

> That numbers are absent from the original is not just an accident, the absence of a tool we find useful but the Greeks did not require. The absence signifies a different approach to definitions. The text of the definitions appears as a continuous piece of prose, not as a discrete juxtaposition of so many definitions. . . . . So the principle is this. Mathematical texts start, most commonly, with some piece of prose preceding the sequence of proved results. Often, this is developed into a full 'introduction', usually in the form of a letter (prime examples: Archimedes or Apollonius). Elsewhere, the prose is very terse, and supplies no more than some reflections on the mathematical objects (prime example: Euclid).
>
> I suggest that we see the shorter, Euclid-type introduction as an extremely abbreviated, impersonal variation upon the theme offered more richly in Archimedes or Apollonius. Then it becomes possible to understand such baffling 'definitions' as, e.g., *Elements* I.3: 'and the limits of a line are points'. This 'definition' is not a definition of any of the three nouns it contains (lines and points are defined elsewhere, and no definition of limits is required here). It is a brief second-order commentary, following the definitions of 'line' and 'point'. Greek mathematical works do not start with definitions. They start with second-order statements, in which the goals and the means of the work are settled. Often, this includes material we identify as 'definitions'. In counting definitions, snatches of text must be taken out of context, and the decision concerning where they start is somewhat arbitrary. (Bear in mind of course that the text was written – even in late manuscripts – as a continuous, practically unparagraphed whole.)
>
> . . . .
>
> Most definitions do not prescribe equivalences between expressions (which can then serve to abbeviate, no more). They specify the situations under which properties are considered to belong to objects. Now that we see that most definitions are simply part of the introductory prose, this makes sense. There is no metamathematical

> theory of definition at work here. Before getting down to work, the mathematician describes what he is doing – that's all.

Netz [29, p. 238]:

> The first floor of Greek mathematics is the general tool-box, Euclid's results. To master it, even superficially, is to become a passive mathematician, an initiate.
>
> The second floor is made up of such works as the first four books of Apollonius' *Conics*; other, comparable works are, e.g. works on trigonometry. Such results are understood passively even by the passive mathematician, the one who knows no more than Euclid's *Elements*. They are mastered by a creative mathematician in the same way in which the first floor is mastered by the passive mathematician.
>
> The third floor relies on the results of the second floor. This is where the professionals converse. They have little incentive and occasion to master this level in the way in which the second floor is mastered. What is more important, the cognitive situation excludes such mastery, for oral storing and retrieval have their limitations. Already the second floor is invoked only through some very specific results. When Archimedes entered the scene, the tool-box was already full. Mathematics would explode exponentially only when storing and retrieval became much more written, and when the construction of the tool-box was done methodically rather than through sheer exposure.

Proclus 83–84 [26, pp. 68–69]:

> The book is divided into three major parts. The first reveals the construction of triangles and the special properties of their angles and their sides, comparing triangles with one another as well as studying each by itself. Thus it takes a single triangle and examines now the angles from the standpoint of the sides and now the sids from the standpoint of the angles, with respect to their equality or inequality; and then, assuming two triangles, it investigates the same properties in various ways. The second part develops the theory of parallelograms, beginning with the special characteristics of parallel lines and the method of constructing the parallelograms and then demonstrating the properties of parallelograms. The third part reveals the kinship between triangles and parallelograms both in their properties and in their relations to one another. Thus it proves that triangles or parallelograms on the same or equal bases have identical properties; it shows [what is the relation between] a triangle and a parallelogram on the same base, how to construct a parallelogram equal to a triangle, and finally, with respect to the squares on the sides of a right-angled triangle, what is the relation of the square on

> the side that subtends the right angle to the squares on the two sides that contain it. Something like this may be said to be the purpose of the first book of the *Elements* and the division of its contents.

First part: 1–26, second part: 27–34, third part: 35–48; cf. Proclus 352–353 [26, p. 275] and 395 [26, p. 311].

Proclus 203 [26, p. 159]:

> Every problem and every theorem that is furnished with all its parts should contain the following elements: an enunciation, an exposition, a specification, a construction, a proof, and a conclusion. Of these the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.

Netz [29, pp. 10–11]:

> enunciation=*protasis*
>
> exposition=setting out=*ekthesis*
>
> specification=definition of goal=*diorismos*
>
> construction=*kataskeuē*
>
> proof=*apodeixis*
>
> conclusion=*sumperasma*

Mueller [27, p. 11]: *protasis*, *ekthesis*, *diorismos*, *kataskeuē*, *apodeixis*, *sumperasma*

Al-Nayrizi [24, p. 102]:

> The figures, all of them, theorems and constructions, have been named with a common name, and each one of them, namely, theorem and construction (and locating too, if it is something else apart from the two of them), is divided into six divisions, namely, proposition, exemplification, separation, construction, proof, and conclusion.

Al-Nayrizi [24, p. 102]:

> The separation is what separates what is requested in the proposition, what is set down in the exemplification, from its common genus and requests that it be constructed and proved.

Al-Nayrizi [24, p. 103]:

> The conclusion is what teaches the proposition, as when you say, "We have now proven that in every triangle, the three angles are truly equal to two right angles." We say it with confidence since it has been proven, and for that reason we do not add anything at all to it except "therefore."

Proclus 207 [26, p. 162]:

> Furthermore, mathematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of the given figure, they infer that it is true in general, going from the particular to the universal conclusion. Because they do not make use of the particular qualities of the subjects but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort. It is not as having such-and-such a size that the angle before me is bisected, but as being rectilineal and nothing more. Its particular size is a character of the given angle, but its having rectilineal sides is a common feature of all rectilineal angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilineal angles; but if I make no use of its rightness and consider only its rectilineal character, the proposition will apply equally to all angles with rectilineal sides.

Proclus 208 [26, pp. 162–163]:

> Let us view the things that have been said by applying them to this our first problem. Clearly it is a problem, for it bids us devise a way of constructing an equilateral triangle. In this case the enunciation consists of both what is given and what is sought. What is given is a finite straight line, and what is sought is how to construct an equilateral triangle on it. The statement of the given precedes and the statement of what is sought follows, so that we may weave them together as "If there is a finite straight line, it is possible to construct an equilateral triangle on it."

Proclus 208 [26, p. 163]:

> Next after the enunciation is the exposition: "Let this be the given finite line." You see that the exposition itself mentions only the

given, without reference to what is sought. Upon this follows the specification: "It is required to construct an equilateral triangle on the designated finite straight line." In a sense the purpose of the specification is to fix our attention; it makes us more attentive to the proof by announcing what is to be proved, just as the exposition puts us in a better position for learning by producing the given element before our eyes.

Book I, Definitions [18, pp. 153–154]:

8 A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9 And when the lines containing the angle are straight, the angle is called **rectilineal**.

10 When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.

13 A **boundary** is that which is an extremity of anything.

14 A **figure** is that which is contained by any boundary or boundaries.

15 A **circle** is a plane figure contained by one line such that all the straight lines falling upon it it from one point among those lying within the figure are equal to one another;

16 And the point is called the **centre** of the circle.

22 Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia**.

23 **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Book I, Postulates [18, pp. 154–155]:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

If A, B, C, D are angles, the statement that A and B are less than C and D means that $A + B < C + D$, not that $A < C$ and $B < D$. The statement that A and B are less than C and D respectively means that $A < C$ and $B < D$. Likewise, the statement that A and B are equal to C and D means that $A + B = C + D$, and the statement that A and B are equal to C and D respectively means that $A = C$ and $B = D$.

Plural predication in Plato, *Hippias Major* 300d–e [35, p. 259]:

> Hippias: You'll have finer knowledge than anyone whether or not I'm playing games, Socrates, when you try to describe these notions of yours and are shown to be talking nonsense. It's quite impossible – you'll never find an attribute which neither you nor I have, but which both of us have.
>
> Socrates: Are you sure, Hippias? I suppose you've got a point, but *I* don't understand. Let me explain more clearly what I'm getting at: it seems to me that both of us together may possess as an attribute something which neither I have as an attribute nor am (and neither are you); and, to put it the other way round, that neither of us, as individuals, may be something which both of us together have as an attribute.

Socrates ironically says the following, 301d–e [35, p. 260]:

> You see, before you spoke, my friend, we were so inane as to believe that *each* of us – you and I – is one, but that both of us together, being two not one, are not what each individual is. See how stupid we were! But now we know better: you've explained that if both together are two, then each individual must be two as well; and if each individual is one, both must be one as well.

Heath [18, p. 201]:

> As to the *raison d'être* and the place of Post. 4 one thing is quite certain. It was essential from Euclid's point of view that it should come before Post. 5, since the condition in the latter that a certain pair of angles are together less than two right angles would be useless unless it were first made clear that right angles are angles of determinate and invariable magnitude.

Common Notions [18, p. 155]:

1. Things which are equal to the same thing are also equal to one another.

2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Conspectus siglorum:

P BAV, Vat. gr. 190

F Florence, Biblioteca Medicea Laurenziana, Plut. 28.03

B Bodleian, MS. D'Orville 301

V Österreichische Nationalbibliothek, Cod. Phil. gr. 31 Han

I.1: "On a given finite straight line to construct an equilateral triangle."

I.2: "To place at a given point (as an extremity) a straight line equal to a given straight line."

Proof. *ekthesis*: Let the point A be given and let the straight line BC be given.

diorismos: Thus it is required to place at the point A as an extremity a straight line equal to the given straight line BC.

kataskeuē: Let the straight line AB be joined from A to B (Postulate 1). On this straight let the equilateral triangle DAB be constructed (I.1). Let the straight line AE be produced in a straight line with DA and let the straight line BF be produced in a straight line with DB (Postulate 2). Let the circle CGH be described with center B and distance BC (Postulate 3), intersecting the line BF at G; and let the circle GKL be described with center D and distance DG (Postulate 3), intersecting the line AE at L.

apodeixis: Because B is the center of the circle CGH, BC is equal to BG. Because D is the center of the circle GKL, DL is equal to DG. And DA is equal to DB, so when DA is subtracted from DL and DB is subtracted from DG, the remainders AL and BG are equal (Common Notion 3). But BC is equal to BG, so each of the straight lines AL, BC is equal to BG. Therefore AL is equal to BC (Common Notion 1).

sumperasma: Therefore the straight line AB, equal to the given straight line BC, has been placed at the given point A. □

Proclus 222–223 [26, p. 174]:

> Some problems have no cases, while others have many; and the same is true of theorems. A proposition is said to have cases when it has the same force in a variety of diagrams, that is, can be demonstrated in the same way despite changes in position, whereas one that succeeds only with a single position and a single construction is without cases.

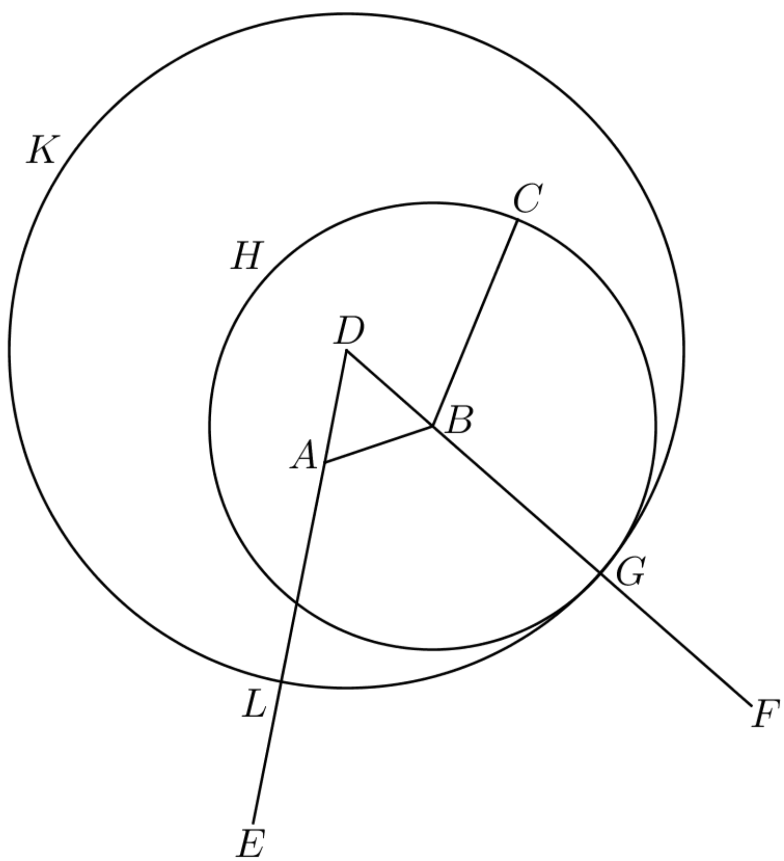

Figure 1: I.2: P 15v, F 2r, B 8v, V 10

1.3: "Given two unequal straight lines, to cut off from the greater a straight line equal to the less."

I.4: "If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend."

Proclus 235–236 [26, pp. 183–184]:

> There are three things proved and two things given about these triangles. One of the given elements is the equality of two sides (really two given sides, but obviously given in ratio to one another) and the equality of the angles contained by the equal sides. And the things to be proved are three: the equality of base to base, the equality of triangle to triangle, and the equality of the other angles to the other angles. Since it would be possible for the triangles to have two sides equal to two sides and yet the theorem to be false because the sides are not equal one to another but one pair to the other pair, he did not simply say, in his statement of the given, that the lines are equal, but that they are equal "respectively." For if it should happen that one of the triangles had one side of three units and the other of four, while the other triangle had one side of five units and another of two (the angle included between them being a right angle), the two sides of the one would be equal to the two sides of the other, since their sum is seven in each case. But this would not show the one triangle equal to the other; for the area of the former is six, of the latter five.

Proclus 236 [26, p. 184]:

> As to the "base" of a triangle, when no side has previously been named, we must suppose it to denote the side towards the observer, but when two sides have already been mentioned, it must mean the remaining side.

Proclus 236 [26, p. 185]:

> Two triangles are said to be equal when their areas are equal. It can happen that two triangles with equal perimeters have unequal areas because of the inequality of their angles. "Area" I call the space itself which is cut off by the sides of the triangle, and "perimeter" the line composed of the three sides of the triangle.

Proclus 237–238 [26, p. 185]:

> Base is said to be equal to base and generally a straight line to another straight line when the congruence of their extremities makes

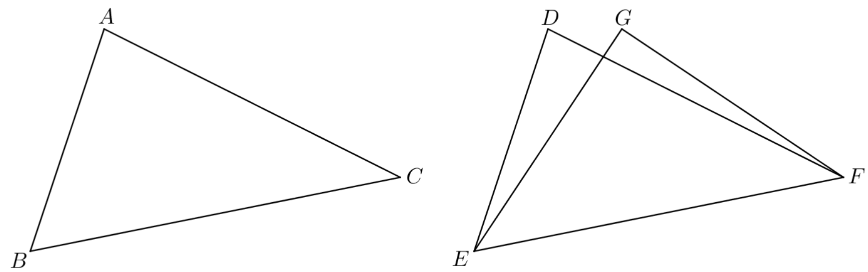

Figure 2: I.8: P 21r, F 3v, B 12r, V 17

> the whole of the one line coincide with the whole of the other. Every straight line coincides with every other, and in the case of equal lines their extremities also coincide. A rectilineal angle is said to be equal to a rectilineal angle when, if one of the sides containing it is placed upon one of the sides containing the other, the second side of the first coincides with the second side of the second.

Proclus 238 [26, p. 186]:

> This also must be understood in advance, that the side that lies opposite an angle is said to subtend it. Every angle in a triangle is contained by two sides of the triangle and subtended by the other. When the other sides fail to coincide, that angle is greater whose side falls outside, and that angle less whose side falls inside. For in the one case the one angle includes the other, in the other case it is included by the other.

I.7: "Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it."

I.8: "If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines."

Proof. ekthesis: Let ABC, DEF be two triangles with the two sides AB, AC equal to the two sides DE, DF respectively and with the base BC equal to the base EF.

diorismos: I say that the angle BAC is equal to the angle EDF.

kataskeuē: Let the triangle ABC be applied to the triangle DEF with the point B placed on the point E and the straight line BC on EF.

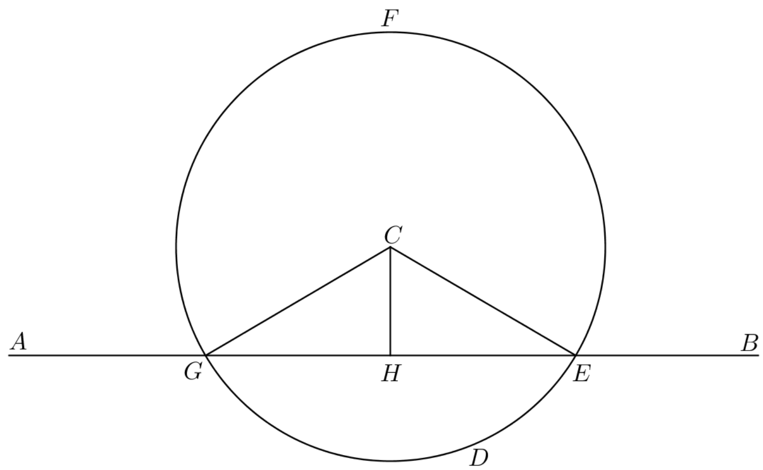

Figure 3: I.12: P 23r, F 4v, B 13v, V 19

apodeixis: Then the point C will coincide with the point F because BC is equal to EF. If the base BC coincides with the base EF and the sides BA, AC do not coincide with ED, DF but as EG, EF miss them, then there have been constructed on the straight line EF the straight lines ED, FD meeting at D, and on the same side the straight lines EG, FG meeting at G. AB, AC are equal respectively to DE, DF and are also equal respectively to GE, GF, so DE, DF are equal respectively to GE, GF (Common Notion 1). But such straight lines cannot be so constructed (I.7). Therefore, with the base BC having been applied to the base ED, it is impossible for the sides BA, AC not to coincide with the sides ED, DF; therefore the sides BA, AC will coincide with the sides ED, DF, and then the angle BAC will coincide with the angle EDF, and therefore will be equal to it (Common Notion 4).

sumperasma: Therefore, etc. □

I.9: "To bisect a given rectilineal angle."

I.10: "To bisect a given finite straight line."

I.11: "To draw a straight line at right angles to a given straight line from a given point on it."

I.12: "To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line."

Proof. Let AB be the given infinite straight line and let C be the given point not on this line. Let a point D be taken on the other side of AB and let the circle EFG be described with center C and distance CD (Postulate 3). Let the

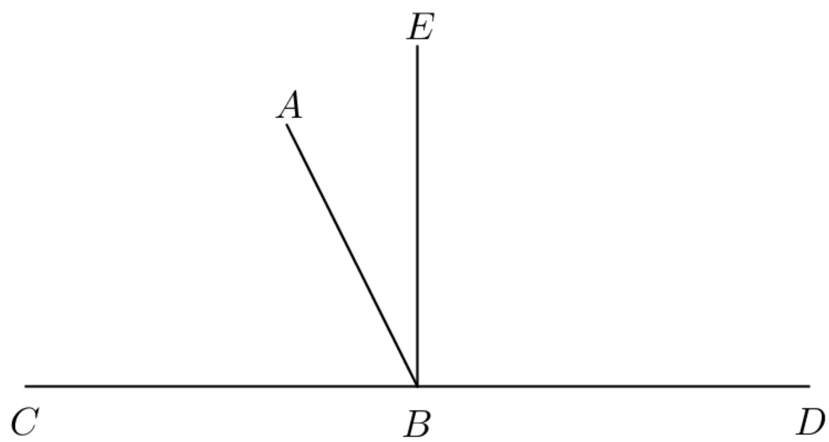

Figure 4: I.13: P 23v, F 4v, B 14r, V 20

straight line EG be bisected at H (I.10). Let the straight lines CG, CH, CE be joined (Postulate 1).

GHC and EHC are triangles with a common side HC. As GH is equal to EH, the two sides GH, HC are equal to the two sides EH, HC respectively; and because EFG is a circle with center C, CG is equal to CE; therefore the respective bases CG and CE of GHC and EHC are equal and the two sides GH, HC are equal to the two sides EH, HC respectively; therefore the angle GHC is equal to the angle EHC (I.8).

The straight line CH set up on the straight line AB has adjacent angles GHC, EHC, which are equal; therefore GHC, EHC are right angles, and CH is perpendicular to AB (Definition 10). □

I.13: "If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles."

Proof. Let a straight line AB be set up on a straight line CD make angles CBA, ABD. If CBA is equal to ABD then CBA, ABD are right angles (Definition 10). If not, let BE be drawn from B at right angles to CD (I.11); therefore the angles CBE, EBD are two right angles. The angle CBE is equal to the two angles CBA, ABE. Then the angles CBE, EBD are equal to the three angles CBA, ABE, EBD (Common Notion 2). The angle DBA is equal to the two angles DBE, EBA; therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC (Common Notion 2). But the angles CBE, EBD are also equal to these three angles; therefore the angles DBA, ABC are equal to the angles CBE, EBD (Common Notion 1). But the angles CBE, EBD are two right angles; therefore the angles DBA, ABC are equal to two right angles. □

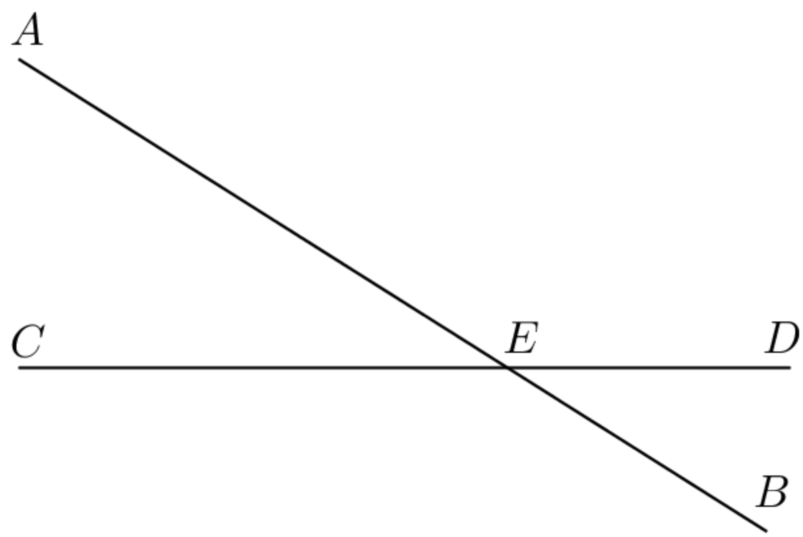

Figure 5: I.15: F 5r, B 15r, V 21

Proclus 292–293 [26, pp. 228–229]:

> But what does he intend when he adds that it makes "either two right angles or angles equal to two right angles?" For when it makes two right angles, it makes angles equal to two right angles, since all right angles are equal to one another. Is it not that the one expression denotes an attribute common to both equal and unequal angles, the other a property of equal angles only? Whenever both a general and a special attribute can be affirmed truly of something, we are accustomed to indicate its character by the special attribute; but whenever we cannot hit upon this, we are satisfied with the general character for the clarification of the things under consideration.

I.14: "If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another."

I.15: "If two straight lines cut one another, they make the vertical angles equal to one another."

Proof. The straight line AE stands on the straight line CD and makes the angles CEA, AED; thus the angles CEA, AED are equal to two right angles (I.13). And the straight line DE stands on the straight line AB and makes the angles AED, DEB; thus the angles AED, DEB are equal to two right angles (I.13).

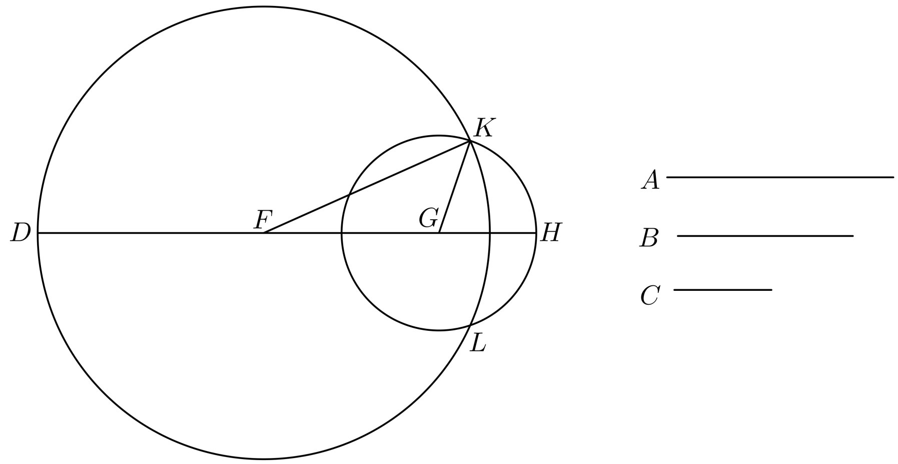

Figure 6: I.22: P 25r, F 6v, B 18r, V 26

Therefore the angles CEA, AED are equal to the angles AED, DEB (Postulate 4 and Common Notion 1). Let the angle AED be subtracted from CEA, AED and AED, DEB; then the remainders CEA is equal to DEB (Common Notion 3). It can be proved in the same way that CEB is equal to DEA. □

Proclus 298 [26, p. 233]:

> Vertical angles are different from adjacent angles, we say, in that they arise from the intersection of two straight lines, whereas adjacent angles are produced when one only of the two straight lines is divided by the other. That is, if a straight line, itself undivided, cuts the other with its extremity and makes two angles, we call these angles adjacent; but if two straight lines cut each other, they make vertical angles. We call them so because their vertices come together at the same point; and their vertices are the points at which the lines converging make the angles.

I.16: "In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles."

I.20: "In any triangle two sides taken together in any manner are greater than the remaining one."

I.22: "Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one."

Proof. Let the three given straight lines be A, B, C, with A, B greater than C, and A, C greater than B, and B, C greater than A. Let DE be a straight line that terminates at D and is infinite in the direction E. Let DF be cut off from

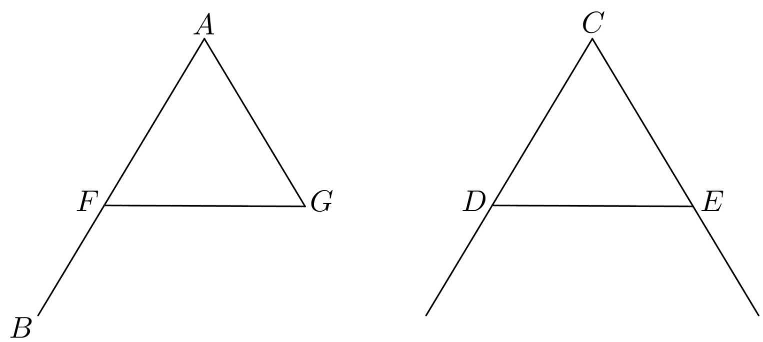

Figure 7: I.23: P 25v, F 7r, B 18v, V 27

DE equal to A, let FG be cut off from FE equal to B, and let GH be cut off from GE equal to C (I.3). With center F and distance FD let the circle DKL be described and with center G and distance GH let the circle KLH be described (Postulate 3), with K and L the points at which the two circles intersect. Let KF, KG be joined (Postulate 1).

Because F is the center of the circle DKL, FD is equal to FK; but FD is equal to A, so KF is equal to A (Common Notion 1). Because G is the center of the circle LKH, GH is equal to GK; but GH is equal to C, so KG is equal to C (Common Notion 1). But FG is equal to B, so the three straight lines KF, FG, GK are equal respectively to the three straight lines A, B, C. □

I.23: "On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle."

Proof. ekthesis: Let the given straight line be AB, let the point on the line be A, and let the given angle be DCE.

diorismos: Thus it is required to construct on the line AB at the point A an angle equal to the angle DCE.

kataskeuē: On the straight lines CD, CE respectively let points taken by chance be D, E. Let DE be joined. From the three straight lines CD, DE, CE let the triangle AFG be constructed such that CD is equal to AF, CE to AG, and DE to FG (I.22).[1]

apodeixis: Since the two sides DC, CE are respectively equal to the two sides FA, AG and the base DE is equal to the base FG, the angle contained by

[1]What is used here is more than what I.22 provides.

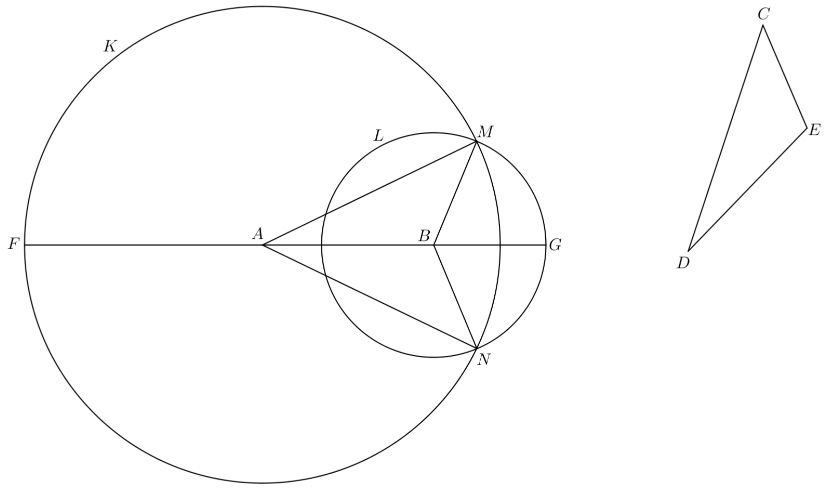

Figure 8: Proclus, I.23

the straight lines DC, CE is equal to the angle contained by the straight lines FA, AG (I.8). That is, the angle DCE is equal to the angle FAG.

sumperasma: On the given straight line AB at the point A, the angle FAG has been constructed that is equal to the given angle DCE. □

Scholia for I.23 [19, pp. 161–162].

The figures for I.23 in Al-Nayrizi [24, pp. 146–147] and Adelard of Bath [6, pp. 49–50] are the same as I.23 in Heiberg.

Proclus 334–335 [26, pp. 261–262] gives the following proof of I.23.

Proof. Let AB be a given straight line, A the given point on it, and CDE the given rectilineal angle, with DE equal to AB. Let CE be joined and let AB be produced in both directions to F and G. Let FA be cut off equal to CD and let BG be cut off equal to CE (I.3). With A as center and distance FA let the circle K be described and with B as center and distanc BG let the circle L be described (Postulate 3). These circles intersect at the points M, N. Let MA, MB be joined and let NA, NB be joined (Postulate 1). Because FA is equal to each of AM, AN and FA is equal CD, each of AM, AN is equal to CD, and likewise each of BM, BN is equal to CE (Common Notion 1). But AB is equal to DE, so the two lines AB, AM are equal respectively to DE, DC and the base BM is equal to CE, hence the angle MAB is equal to the angle CDE (I.8); it can likewise be proved that the angle NAB is equal to the angle CDE. □

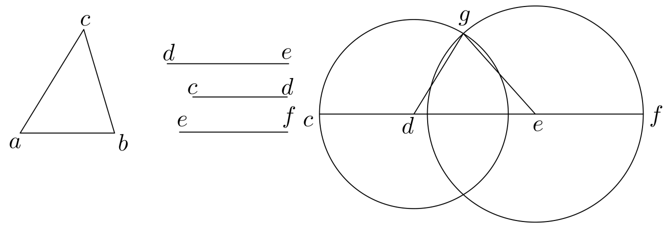

Figure 9: Johannes de Tinemue, I.23

Johannes de Tinemue [7, pp. 52–53] gives a proof of I.23 that does not invoke I.22.

> Data recta linea supra terminum eius cuilibet angulo proposito equum angulum designare.
>
> Dispositio. Clausis itaque b, c terminis interpositione bc linee adequetur de ad ac. Duabus lineis altrinsecus eidem scilicet de directe protractis cd equali ab et ef equali bc, deinde secundum premissam ex tribus lineis ipsis cd, de, ef equalibus designetur deg triangulus.
>
> Ratiocinatio. Age. Si memineris priorum, istorum triangulorum abc, deg, ab, dg mediante cd et ac, de et bc, eg mediante ef sunt equalia. Ergo secundum 8^{am} d et a anguli sunt equales. Sicque super terminum d designatus est angulus equalis a. Quod proposuimus.

Albert the Great [25, pp. 82–84] gives a proof of I.23 that does not invoke I.22.

Proof. Let the given straight line be AB and let the given angle be FGH. If AB is longer than GH then apply I.3 to cut off AB equal to GH. If AB is shorter than GH then apply I.2 to extend AB to be equal to GH. Extend AB on the side of A to C so that AC is equal to FG. Extend AB on the side of B to E so that BE is equal to FH. Draw the circle with center A and radius AC and the circle with center B and radius BE. These circles either intersect or they do not.

If they intersect let D be one of the points of intersection. AC is equal to FG and AC is equal to AD so AD is equal to FG. Likewise, BE is equal to FH and BE is equal to BD so BD is equal to FH. Finally, AB was made equal to GH. Thus the two triangles DAB, FGH have the two sides DA, AB are respectively equal to the two sides FG, GH and the base DB is equal to

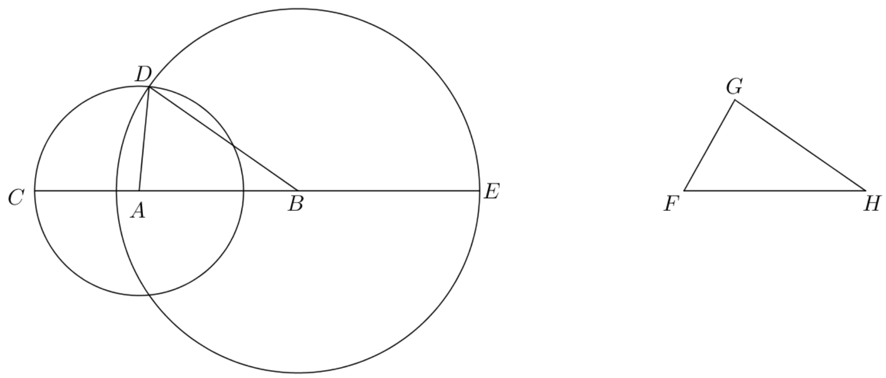

Figure 10: Albert the Great, I.23, intersecting circles

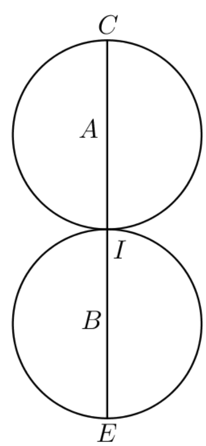

Figure 11: Albert the Great, I.23, touching circles

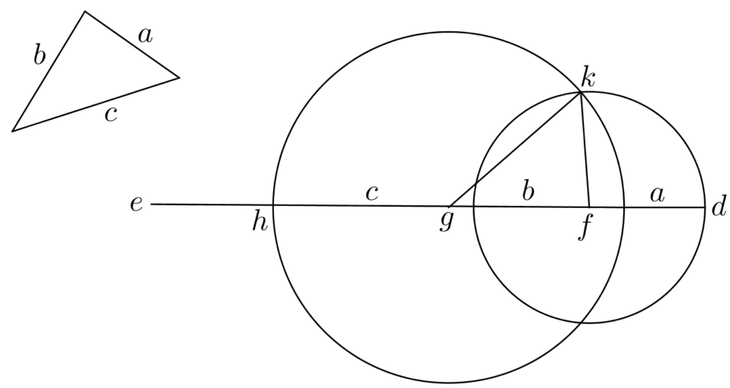

Figure 12: Campanus, I.23

the base FH, so by I.8 the angle contained by the sides DA, AB is equal to the angle contained by FG, GH, that is, the angle DAB is equal to the given angle FGH.

If the circles do not intersect then either they touch each other on line AB or they do not touch each other. If they touch each other, let I be the point of contact. Because I lies on the circle AC, AC is equal to AI. Because I lies on the circle BE, EB is equal to BI. Because the circles touch each other at I, AIB is a straight line, so AB is equal to AC, EB. But AB is equal to GH, AC is equal to FG, and EB is equal to FH. So GH is equal to FG, FH, which according to I.20 is absurd.

If the circles do not touch each other, then by the above reasoning we get that GH is greater than FG, FH, which according to I.20 is absurd. $\square$

Campanus [8, p. 74]:

> Data recta linea super terminum eius cuilibet angulo proposito equum angulum designare.

> Sit data linea fe que est in superiori figura et sint linee b, a continentes angulum datum cui subtendam basim c. Supra punctum f

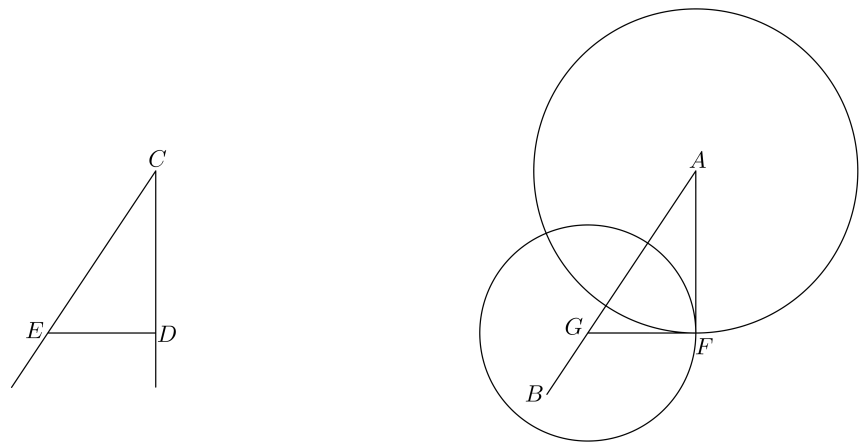

Figure 13: Commandinus, I.23

> linee ef iubemur facere equalem angulum angulo dato ad lineam ef. Adiungo fd equalem linee a et ex fe sumo fg equalem b et ex ge sumo gh equalem c et super puncta f et g describo duos circulos dk et kh secundum quantitatem duarum linearum fd et gh intersecantes se in puncto k sicut docuit precedens. Ductisque lineis kf et kg erunt duo latera kf et fg trianguli kfg equalia duobus lateribus a et b trianguli abc et basis gk equalis basi c, ergo per 8 angulus kfg equalis erit angulo contento ab a et a b. Quod est propositum.

Proof. Let the given line be fe and let the given angle be contained by the sides b, a and subtended by the base c. Extend fd to be equal to a; from fe take fg equal to b; from ge take gh equal to c; and at the points f and g describe two circles dk and kh. It follows from I.22 that these circles intersect, and let one of the points of intersection be k. Join the lines kf and kg. Then the two sides kf and fg of the triangle kfg are equal to the two sides a and b of the triangle abc respectively, and the base gk is equal to the base c, so by I.8 the angle kfg is equal to the angle contained by a and b. □

Peletarius, *Euclidis elementa geometrica demonstrationum libri sex*, 1557, p. 42.

Commandinus, *Euclidis Elementorum libri XV*, 1572, f. 17r, in his commentary on I.23, gives the following construction.

Proof. Let AB be the given line and let A be the given point. Let DCE be the given angle and let DE be joined. Cut AG from AB equal to CE. With center A, describe a circle with radius CD and with center G describe a circle

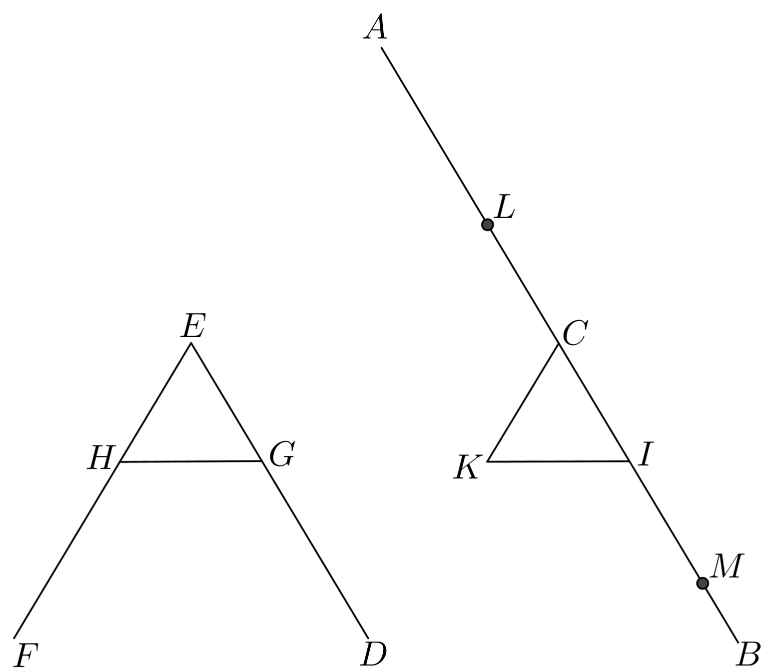

Figure 14: Clavius, I.23

with radius ED. The circles intersect at a point F. Join AF, FG. Then FAG is equal to DCE. □

Clavius, *Euclidis Elementorum libri XV*, *Opera mathematica*, pp. 44–45, I.23.

Proof. Let the given line be AB, let the given point on the line be C, and let the given angle be DEF.

Take CI equal to EG, take CL equal to EH, and take IM equal to GH. Describe a circle with center C and radius CL and describe a circle with center I and radius IM. These circles intersect at K. □

I.26: "If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle."

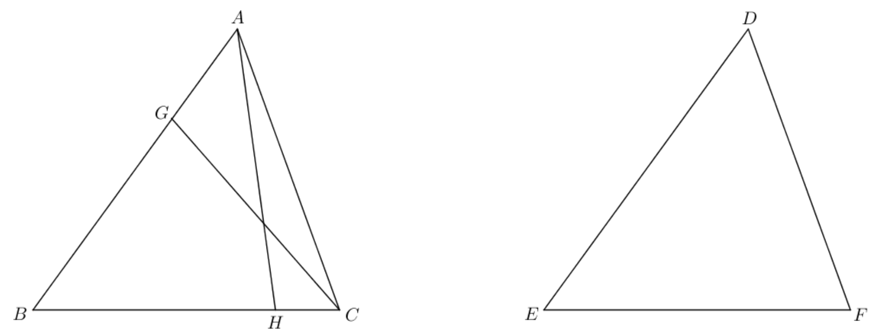

Figure 15: I.26: P 27r, F 8r, B 20r, V 30

Proof. Let ABC, DEF be two triangles having the two angles ABC, BCA equal respectively to the two angles DEF, EFD. First, let the sides adjoining the equal angles be equal, namely let BC be equal to EF.

If AB is unequal to DE, one is greater and let AB be greater than DE. Let BG be made equal to DE (I.3), and let GC be joined (Postulate 1). Because BG is equal to DE and BC is equal to EF, and the angle GBC is equal to the angle DEF, the base GC is equal to the base DF and the angles GCB, BGC are equal respectively to the angles DFE, EDF (I.4). But by hypothesis the angle DFE is equal to the angle BCA; thus the angle BCG is equal to the angle BCA (Common Notion 1), namely the less is equal to the greater, which is impossible. Therefore AB is not unequal to DE, so AB is equal to DE.

But by hypothesis BC is equal to EF, so the sides AB, BC are equal respectively to the sides DE, EF; and the angle ABC is equal to the angle DEF, thus the base AC is equal to the base DF and the remaining angle BAC is equal to the remaining angle EDF (I.4).

Second, let sides subtending equal angles be equal, with AB equal to DE. If BC is unequal to EF, one is greater and let BC be greater than EF. Let BH be made equal to EF (I.3), and let AH be joined (Postulate 1). Then the two sides AB, BH are equal respectively to DE, EF, and by hypothesis the angle ABH is equal to the angle DEF; thus the base AH is equal to the base DF, the angle BHA is equal to the angle EFD, and the angle BAH is equal to the angle EDF (I.4). But the angle EFD is equal by hypothesis to the angle BCA, so the angle BHA is equal to the angle BCA (Common Notion 1). Therefore for the triangle AHC, the exterior angle BHA is equal to the interior and opposite angle BCA, which is impossible (I.16). Therefore BC is not unequal to EF, so BC is equal to EF.

But by hypothesis AB is equal to DE. Thus the two sides AB, BC are equal respectively to the two sides DE, EF, and the angle ABC is equal to the angle

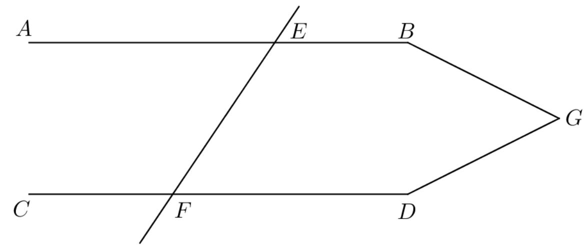

Figure 16: I.27: P 27v, F 8v, B 21v, V 31

DEF; thus the base AC is equal to the base DF and the remaining angle BAC is equal to the remaining angle EDF (I.4). □

I.27: "If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another."

Proof. Let the straight line EF falling on the two straight lines AB, CD make the alternate angles AEF, EFD equal to one another. If AB is not parallel to CD, then AB, CD when produced will meet either in the direction of B, D or in the direction of A, C (Definition 23); let them be produced and meet in the direction of B, D at G. For the triangle GEF, by hypothesis the exterior angle AEF is equal to the interior and opposite angle EFG, which is impossible (I.16). Therefore AB, CD when produced will not meet in the direction of B, D. It can likewise be proved that AB, CD when produced will not meet in the direction of A, C. Therefore AB is parallel to CD. □

Proclus 357 [26, p. 278]:

> Angles that are produced in different directions and are not adjacent to one another, but separated by the intersecting line, both of them within the parallels but differing in that one lies above and the other below, he calls "alternate" angles.

I.28: "If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another."

Proof. First, let the straight line EF falling on the two straight lines AB, CD make the exterior angle EGB equal to the interior and opposite angle GHD. Since the straight lines EF, AB cut one another, the vertical angles EGB, AGH are equal (I.15). Then since by hypothesis the angle EGB is equal to the angle

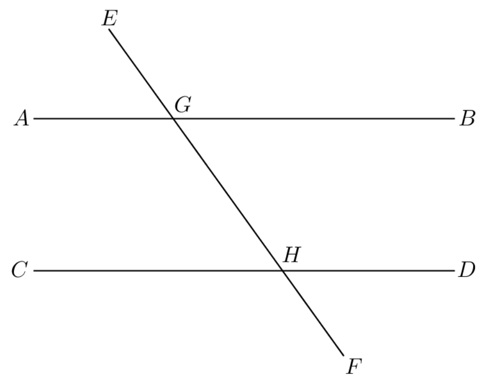

Figure 17: I.28: P 28r, F 9r, B 21v, V 32

GHD, the angle AGH is equal to the angle GHD (Common Notion 1); the angles AGH, GHD are alternate, so AB is parallel to CD (I.27).

Second, let the straight line EF falling on the two straight lines AB, CD make the interior angles BGH, GHD equal to two right angles. The straight line GH set up on the straight line AB makes the angles AGH, BGH, which thus are equal to two right angles (I.13). Since the angles BGH, GHD are equal to two right angles and the angles AGH, BGH are equal to two right angles, the angles AGH, BGH are equal to the angles BGH, GHD (Postulate 4 and Common Notion 1). Let the angle BGH be subtracted from each; then the angle AGH is equal to the angle GHD (Common Notion 3). The alternate angles AGH, GHD are equal, so the straight line AB is parallel to the straight line CD (I.27). □

I.29: "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles."

Proof. Let the straight line EF fall on the parallel straight lines AB, CD. If the alternate angles AGH, GHD are unequal then one is greater; let the angle AGH be greater than the angle GHD. Let the angle BGH be added to each; then the angles AGH, BGH are greater than the angles BGH, GHD. But the angles AGH, BGH are equal to two right angles (I.13); therefore the angles BGH, GHD are less than two right angles. Thus the straight line EF falling on the two straight lines AB, CD makes the interior angles BGH, GHD on the same side less than two right angles, so the straight lines AB, CD if produced indefinitely meet on the same side as are the interior angles BGH, GHD (Postulate 5). But the straight lines AB, CD are by hypothesis parallel so do not meet; therefore the angle AGH is not unequal to the angle GHD, so the alternate angles AGH, GHD are equal.

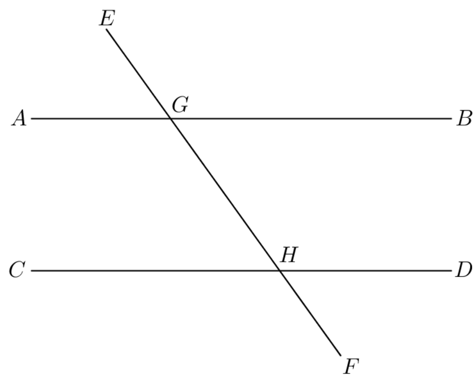

Figure 18: I.29: P 28r, F 9r, B 22r, V 33

Because the straight line EF cuts the straight line AB, the vertical angles AGH, EGB are equal (I.15). But the alternate angles AGH, GHD are equal, so the exterior angle EGB is equal to the interior and opposite angle GHD (Common Notion 1).

Let the angle BGH be added to each of EGB, GHD; then the angles EGB, BGH are BGH, GHD (Common Notion 2). But the angles EGB, BGH are equal to two right angles (I.13). Therefore the interior angles on the same side BGH, GHD are equal to two right angles (Common Notion 1). $\square$

I.30: "Straight lines parallel to the same straight line are also parallel to one another."

Proof. Let each of the straight lines AB, CD be parallel to EF. Let the straight line GK fall upon these straight lines. Since the straight line GK has fallen on the parallel straight lines AB, EF, the alternate angles AGK, GHF are equal (I.29). Likewise, since the straight line GK has fallen on the parallel straight lines EF, CD, the exterior angle GHF is equal to the interior and opposite angle GKD (I.29). But the angle AGK was proved to be equal to the angle GHF, so the angle AGK is equal to the angle GKD (Common Notion 1). The line GK falling on the lines AB, CD makes the alternate angles AGK, GKD, and because these are equal, the lines AB, CD are parallel (I.27). $\square$

I.31: "Through a given point to draw a straight line parallel to a given straight line."

Proof. Let A be the given point and let BC be the given straight line. Let D be a point on BC taken by chance and let AD be joined (Postulate 1). On the straight line DA at the point A let the angle DAE be constructed equal to the angle ADC (I.23). Then let the straight line AF be produced in a straight line with EA (Postulate 2). The straight line AD falling on the two

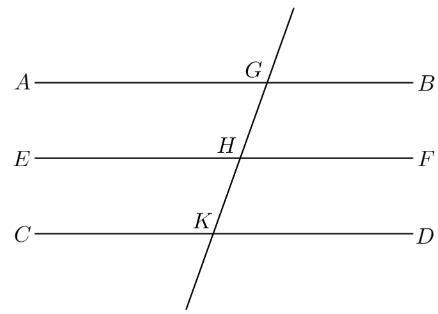

Figure 19: I.30: P 28v, F 9v, B 23r, V 34

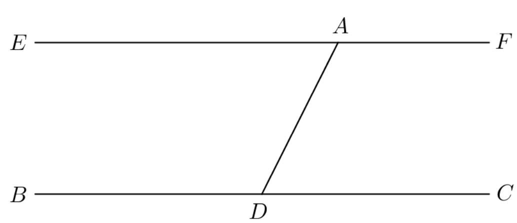

Figure 20: I.31: P 28v, F 9v, B 23r, V 34

straight lines BC, EF makes the alternate angles EAD, ADC; but the angles EAD, ADC have been proved to be equal; therefore the lines BC, EAF are parallel (I.27). □

I.32: "In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles."

Proclus 381–382 [26, pp. 300–301], on *Elements* I.32 (Proclus refers to *Timaeus* 53c):

> We can now say that in every triangle the three angles are equal to two right angles. But we must find a method of discovering for all the other rectilineal polygonal figures – for four-angled, five-angled, and all the succeeding many-sided figures – how many right angles their angles are equal to. First of all, we should know that every rectilineal figure may be divided into triangles, for the triangle is the source from which all things are constructed, as Plato teaches us when he says, "Every rectilineal plane face is composed of triangles." Each rectilineal figure is divisible into triangles two less in number than the number of its sides: if it is a four-sided figure, it is divisible into two triangles; if five-sided, into three; and if six-sided, into four. For two triangles put together make at once a four-sided figure, and this difference between the number of the constituent triangles and the sides of the first figure composed of triangles is characteristic of all succeeding figures. Every many-sided figure, therefore, will have two more sides than the triangles into which it can be resolved. Now every triangle has been proved to have its angles equal to two right angles. Therefore the number which is double the number of the constituent triangles will give the number of right angles to which the angles of a many-sided figure are equal. Hence every four-sided figure has angles equal to four right angles, for it is composed of two triangles; and every five-sided figure, six right angles; and similarly for the rest.

I.33: "The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel."

Proof. Let AB, CD be equal and parallel and let the straight lines AC, BD joins the points on the same sides. Let BC be joined (Postulate 1). Since BC falls on the parallel lines AB, CD, the alternate angles ABC, BCD are equal (I.29). By hypothesis, AB is equal to CD, and BC is a common side of the triangles ABC, DCB, so in the triangles ABC, DCB, the two sides AB, BC are equal to the two sides CD, BC, and the angles ABC, BCD contained by the equal sides are equal; thus the base AC of the triangle ABC is equal to the base BD of the triangle DCB, the angle ACB is equal to the angle CDB, and the angle BAC is equal to the angle BDC (I.4). Then the straight line BC falling on the

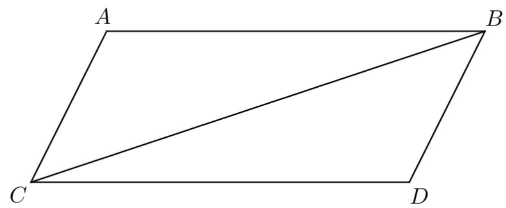

Figure 21: I.33: P 30v, F 10r, B 24r, V 36

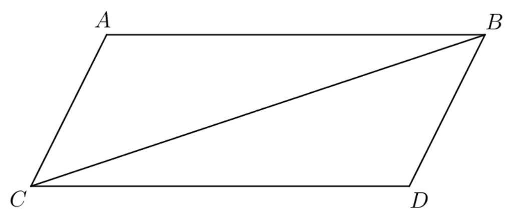

Figure 22: I.34: P 32r, F 10r, B 24v, V 37

straight lines AB, CD makes the alternate angles ACB, CBD equal; therefore AC is parallel to BD (I.27). And AC was proved equal to BD, so the lines AC, BD are equal and parallel. □

I.34: "In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas."

Proof. Let $ACDB$ be a parallelogrammic area and let BC be its diameter. Since the straight line BC falls on the parallel straight lines AB, CD, the alternate angles ABC, BCD are equal (I.29). And since the straight line BC falls on the parallel straight lines AC, BD, the alternate angles ACB, CBD are equal (I.29). The two angles ABC, BCA of the triangle ABC are equal respectively to the two angles DCB, CBD of the triangle DCB, and the side BC adjoining the equal angles is common to both triangles; therefore the side AB of ABC is equal to the side CD of DCB, the side AC is equal to the side BD of DCB, and the angle BAC of ABC is equal to the angle CDB of DCB (I.26). The angles ABC, CBD are equal respectively to the angles BCD, ACB; and the whole angle ABD is equal to the angles ABC, CBD, and the whole angle ACD is equal to the angles ACB, BCD; therefore the angle ABD is equal to the

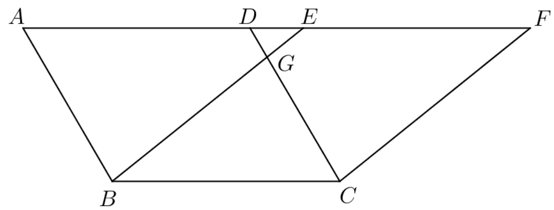

Figure 23: I.35: P 32v, F 10v, B 25r, V 37

angle ACD (Common Notion 2). But the angle BAC was proved equal to the angle CDB, the side AB was proved equal to the side CD, and the side AC was proved equal to the side BD; therefore in the parallelogrammic area $ACDB$ the opposite sides are equal to the opposite sides and the opposite angles are equal to the opposite angles.

For the triangles ABC, DCB, the side AB is equal to the side CD and the side BC is common, the two sides AB, BC are equal respectively to the two sides DC, CB; and the angle ABC is equal to the angle BCD; therefore the base AC of ABC is equal to the base DB of DCB and the triangle ABC is equal to the triangle DCB (I.4). Since the triangle ABC is equal to the triangle DCB, the diameter BC bisects the parallelogram $ACDB$. □

Proclus 393–394 [26, p. 309]:

> It seems also that this very term "parallelogram" was coined by the author of the *Elements* and that it was suggested by the preceding theorem. For when he had shown that the straight lines connecting equal and parallel lines in the same directions are themselves equal and parallel he had clearly shown that both pairs of opposite sides, the connecting and the connected lines, are parallel; and he rightly called the figure enclosed by parallel lines a "parallelogram," just as he had designated as "rectilinear" the figure enclosed by straight lines.

I.35: "Parallelograms which are on the same base and in the same parallels are equal to one another."

Proof. Let $ABCD, EBCF$ be parallelograms on the same base BC and in the same parallels AF, BC. Because $ABCD$ is a parallelogram, the opposite sides AD, BC are equal, and because $EBCF$ is a parallelogram, the opposite sides EF, BC are equal (I.34); therefore AD, EF are equal (Common Notion 1). Let DE be added to each of AD, EF; thus AE is equal to DF (Common Notion 2). And because $ABCD$ is a parallelogram, the opposite sides AB, DC are equal

(I.34). Therefore the sides EA, AB of the triangle EAB are equal respectively to the sides FD, DC of the triangle FDC, and the angle; and the straight line FA falling on the parallel straight lines DC, AB makes the exterior angle FDC equal to the interior and opposite angle EAB (I.29); therefore the base EB of the triangle EAB is equal to the base FC of the triangle FDC and the triangle EAB is equal to the triangle FDC (I.4).

Let the triangle DGE be subtracted from each of the triangles EAB, FDC; then the remaining trapezium $ABGD$ is equal to the remaining trapezium $EGCF$ (Common Notion 3). Let the triangle GBC be added to each of the trapezia $ABGD, EGCF$; then the parallelogram $ABCD$ is equal to the parallelogram $EBCF$ (Common Notion 2). □

Heath [18, p. 327]:

> It is important to observe that we are in this proposition introduced for the first time to a new conception of equality between figures. Hitherto we have had equality in the sense of *congruence* only, as applied to straight lines, angles, and even triangles (cf. I.4). Now, without any explicit reference to any change in the meaning of the term, figures are inferred to be *equal* which are equal in *area* or in *content* but need not be of the same *form*. No *definition* of equality is anywhere given by Euclid; we are left to infer its meaning from the few *axioms* about "equal things."

Proclus 396–397 [26, pp. 312–313]:

> It may seem a great puzzle to those inexperienced in this science that the parallelograms constructed on the same base [and between the same parallels] should be equal to one another. For when the sides of the areas constructed on the same base can be extended indefinitely – and we can increase the length of these sides of the parallelograms as far as we can extend the parallel lines – we may well ask how the areas can remain equal when this happens. For if the breadth is the same (since the base is identical) while the side becomes greater, how could the area fail to become greater? This theorem, then, and the following one about triangles belong among what are called the "paradoxical" theorems in mathematics. The mathematicians have worked out what they call the "locus of paradoxes," as the Stoics have done in their dogmas, and this theorem is included among them. Most people at least are immediately startled to learn that multiplying the length of the side does not destroy the equality of the areas when the base remains the same. The truth is, nevertheless, that the equality or inequality of the angles is the factor of greatest weight in determining the increase or decrease of the area. For the more unequal we make the angles, the more we decrease the area, if the side and base remain the same; hence if we are to preserve equality, we must increase the side.

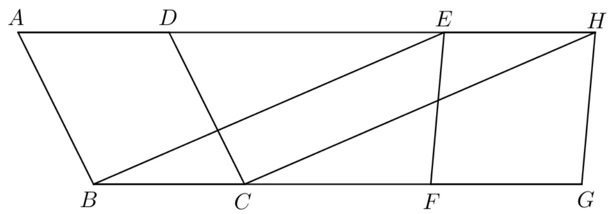

Figure 24: I.36: P 32v, F 10v, B 25v, V 38

Proclus 398 [26, p. 314]:

> With regard to the theorem before us we must realize that, when it says the parallelograms are equal, it means the areas, not the sides, are equal, for the statement is about the included spaces, the areas; and also that in the demonstration of this theorem our author for the first time mentions trapezia.

I.36: "Parallelograms which are on equal bases and in the same parallels are equal to one another."

Proof. Let $ABCD, EFGH$ be parallelograms which are on equal bases BC, FG and in the same parallels AH, BG. Let BE, CH be joined (Postulate 1). Since by hypothesis BC is equal to FG, and FG is equal to EH (I.34), thus BC is equal to EH (Common Notion 1). But the straight lines BC, EH are equal and parallel, so the straight lines EB, HC which join the extremities on the same sides are themselves also equal and parallel (I.33); therefore $EBCH$ is a parallelogram. Because the parallelograms $EBCH, ABCD$ are on the same base BC and in the same parallels BC, AH, they are equal (I.35).

It can be proved likewise that $EFGH$ is a parallelogram; and because the parallelograms $EFGH, EBCH$ are on the same base EH and in the same parallels EH, BG, they are equal (I.35).

Therefore the parallelograms $ABCD, EFGH$ are equal (Common Notion 1). □

I.37: "Triangles which are on the same base and in the same parallels are equal to one another."

Proof. Let ABC, DBC be triangles on the same base BC and in the same parallels AD, BC. Let AD be produced in both directions to E, F (Postulate 2). Then through B let BE be drawn parallel to CA, and through C let CF be drawn parallel to BD (I.31). Then each of the figures $EBCA, DBCF$ is a parallelogram; and because they are on the same base BC and in the same parallels BC, EF, the parallelograms $EBCA, DBCF$ are equal (I.35).

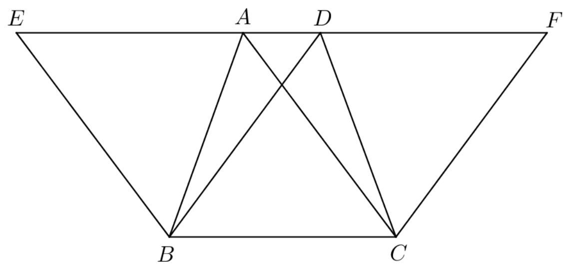

Figure 25: I.37: P 33r, F 11r, B 26r, V 39

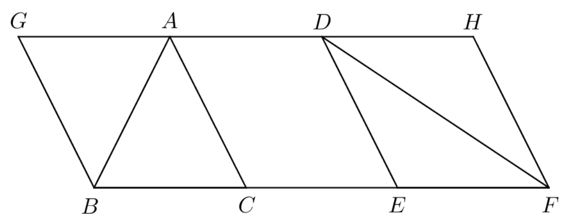

Figure 26: I.38: P 33r, F 11r, B 26v, V 40

In the parallelogram $EBCA$, the diameter AB bisects the area (I.34); thus the triangle ABC is half of the parallelogram $EBCA$. And in the parallelogram $DBCF$, the diameter DC bisects the area (I.34); thus the triangle DBC is half of the parallelogram $DBCF$. But the parallelogram $EBCA$ is equal to the parallelogram $DBCF$; therefore the triangle ABC is equal to the triangle DBC. □

I.38: "Triangles which are on equal bases and in the same parallels are equal to one another."

Proof. Let ABC, DEF be triangles on equal bases BC, EF and in the same parallels BF, AD. Let AD be produced in both directions to G, H (Postulate 2). Through B let BG be drawn parallel to CA and through F let FH be drawn parallel to DE (I.31). Then each of the figures $GBCA, DEFH$ is a parallelogram; and because they are on equal bases BC, EF and in the same parallels BF, GH, the parallelograms $GBCA, DEFH$ are equal (I.36).

In the parallelogram $GBCA$, the diameter AB bisects the area (I.34); thus the triangle ABC is half of the parallelogram $GBCA$. And in the parallelogram $DEFH$, the diameter DF bisects the area (I.34); thus the triangle FED is

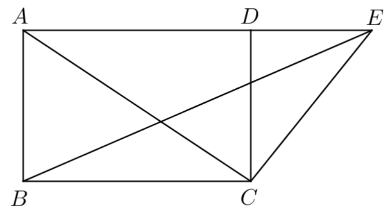

Figure 27: I.41: P 34r, F 11v, B 28r, V 42

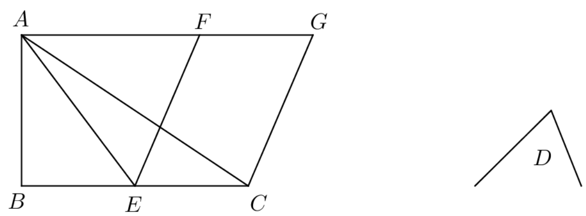

Figure 28: I.42: P 34v, F 12r, B 28r, V 42

half of the parallelogram $DEFH$. But the parallelogram $GBCA$ is equal to the parallelogram $DEFH$; therefore the triangle ABC is equal to the triangle FED. □

I.41: "If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle."

Proof. Let the parallelogram $ABCD$ have the same base BC with the triangle EBC and let it be in the same parallels BC, AE. Let AC be joined (Postulate 1). Then the triangles ABC, EBC are on the same base BC and in the same parallels BC, AE; thus the triangle ABC is equal to the triangle EBC (I.37). But in the parallelogram $ABCD$ the diameter AC bisects the area (I.34); so the parallelogram $ABCD$ is double of the triangle ABC. And the triangle ABC has been proved equal to the triangle EBC; therefore the parallelogram $ABCD$ is double of the triangle EBC. □

I.42: "To construct, in a given rectilineal angle, a parallelogram equal to a given triangle."

Proof. Let ABC be the given triangle and D the given rectilineal angle. Let BC be bisected at E (I.10) and let AE be joined (Postulate 1). On the straight line EC at the point E on it let angle CEF be constructed equal to the angle

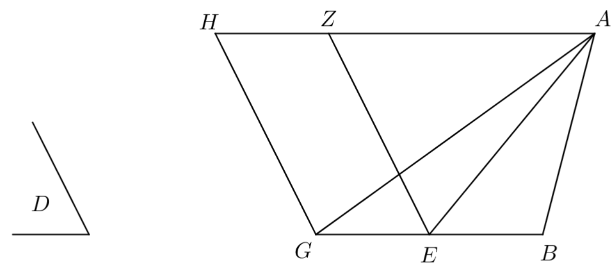

Figure 29: Al-Nayrizi, I.42

D (I.23). Through the point A let the line AG be drawn parallel to EC and through the point C let the line CG be drawn parallel to EF (I.31). Then the figure $FECG$ is a parallelogram.

The triangles ABE, AEC are on equal bases BE, EC and are in the same parallels BC, AG; therefore they are equal (I.38). Thus the triangle ABC is double of the triangle AEC.

The parallelogram $FECG$ is on the same base EC with the triangle AEC and is in the same parallels BC, AG; therefore the parallelogram $FECG$ is double of the triangle AEC (I.42). But the triangle ABC was proved double of the triangle AEC; therefore the parallelogram $FECG$ is equal to the triangle ABC. And $FECG$ has the angle CEF equal to the given angle D. □

Al-Nayrizi [24, p. 185], I.42, *protasis*:

> We want to demonstrate how to construct a surface that is a parallelogram whose angle is equal to a known angle and [which is] equal to a known triangle.

Proof. ekthesis: So let the known angle be the triangle D and the known triangle the triangle ABG.

diorismos: We want to construct a parallelogram whose angle is equal to the angle D and which is equal to the triangle ABG.

kataskeuē: So let us choose one of the sides of the triangle and let us cut it into two halves according to I.10. So let us suppose that we cut the side BG into two halves at the point E, and let us draw the line AE. Then let us construct at the point E of the line GE an angle equal to D, according to I.23, and let this angle be GEZ. And let us draw from the point G a line parallel to EZ and from the point A a line parallel to BG, according to I.23; let the line from G parallel to EZ be GH and let the line from A parallel to BG be AZH.

apodeixis: Then, since the two triangles ABE, AEG are upon equal bases BE, EG and between the same parallel lines BG, AH, the triangle AEG is equal

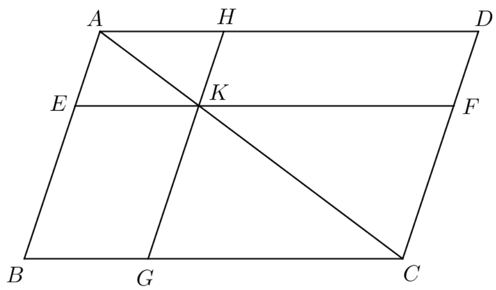

Figure 30: I.43: P 34v, F 12r, B 28v, V 43

to the triangle ABE, according to I.38. So the triangle ABG is double of the triangle AEG. But the surface $GEZH$ is a parallelogram, and its base EG is the base of the triangle AEG, and the two of them are between the two parallel lines BG, AH; so the surface $GEZH$ is double of the triangle AGE, according to I.41. And we have proved that the triangle AGB is double the triangle AGE, and doubles of the same thing are equal, so the parallelogram $GEZH$ is equal to the triangle AGB.

sumperasma: So we have constructed a surface $GEZH$ that is a parallelogram equal to the known triangle ABG and whose angle GEZ is equal to the known angle D, which is what we wanted to demonstrate. □

I.43: "In any parallelogram the complements of the parallelograms about the diameter are equal to one another."

Proof. Let $ABCD$ be a parallelogram and AC its diameter; and about AC let EH, FG be parallelograms and BK, KD the so-called complements. Since $ABCD$ is a parallelogram, the diameter AC bisects its area (I.34); thus the triangle ABC is equal to the triangle ACD. And since EH is a parallelogram, the diameter AK bisects its area (I.34); thus the triangle AEK is equal to the triangle AHK. And since FG is a parallelogram, the diameter KC bisects its area (I.34); thus the triangle KFC is equal to the triangle KGC Then, since the triangle AEK is equal to the triangle AHK and the triangle KFC is equal to the triangle KGC, the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC (Common Notion 2). The triangle ABC is equal to the triangle ADC. Let the triangles AEK, KGC be subtracted from the triangle ABC and let the triangles AHK, KFC be subtracted from the triangle ADC; then the complement BK which remains is equal to the complement KD which remains (Common Notion 3). □

Scholia for I.43 [19, pp. 201–203].

Proclus 418–419 [26, p. 331]:

> The term "complements" was derived by the author of the *Elements* from the thing itself, since complements fill the whole of the area outside the two parallelograms. This is why he does not regard it as deserving of special mention in the Definitions. It would have required a complicated explanation to make us understand what a parallelogram is and what are the parallelograms that are constructed about the same diameter as the whole; for only after these had been explained would the meaning of "complement" have become clear. Those parallelograms are about the same diameter which have a segment of the entire diameter as their diameter; otherwise they are not about the same diameter.

Szabo [30, pp. 344–345]:

> If, for example, the Pythagoreans had not known that the *parapleromata* were equal, they would not have been able to develop their method of *application of areas.* This method was mentioned in *Example 1* above, where it was pointed out that finding a fourth proportional to three given *numbers* or *magnitudes* (i.e. an x such that $a : b = c : x$) could be construed as the problem of finding a rectangle with a given side (a) which has the same area as a given rectangle (bc).
>
> If we think of the given rectangle (bc) as a *parapleroma* and the rectangle which the line a makes with one of its sides (c or b) as a 'parallelogram about the diagonal', then the second 'parallelogram about the diagonal' (bx or cx) can be constructed by extending the side (c or b) of the original rectangle until it meets the continuation of the diagonal of ac or of ab. Now these three rectangles together determine uniquely the second *parapleroma* and, in particular, its side x which we set out to find. (This is sometimes called *parabolic* application of areas, because the area bc is *applied* to the line a; cf. the word παραβάλλειν.)

Dalimier [12] on *parapleromata*.

I.44: "To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle."

Proof. ekthesis: Let AB be the given straight line, C the given rectangle, and D the given rectilineal angle.

diorismos: Thus, it is required to apply a parallelogram equal to the given triangle C, to the given straight line AB, in the given angle D.

kataskeuē: Let the parallelogram $BEFG$ equal to the triangle C be constructed in the angle EBG equal to D (I.42); let it be placed so that BE is in a straight line with AB.[2] Through the point A let the straight line AH be drawn parallel either to BG or EF (I.31).

[2] What is used here is more than what I.42 provides.

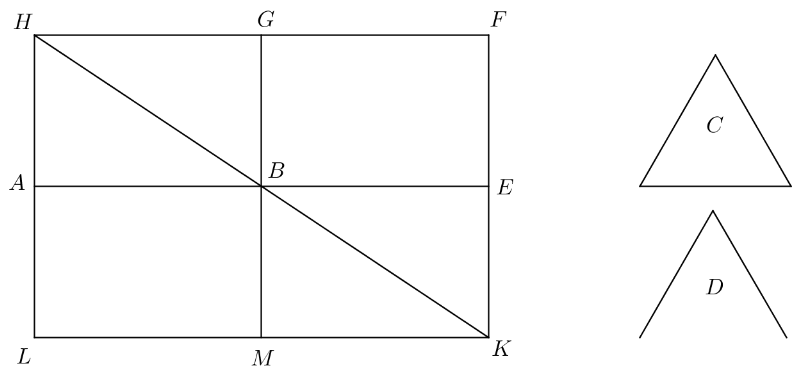

Figure 31: I.44: P 36r, F 12v, B 29r, V 45

apodeixis: Then, since the straight line HF falls upon the parallel straight lines, the interior angles on the same side AHF, HFE are equal to two right angles (I.29). Therefore the angles BHG, GFE are less than two right angles; thus the straight line HF falling on the the two straight lines HB, FE makes BHG, GFE, the interior angles on the same side, less than two right angles, so the straight lines HB, FE if produced will meet on the same side of HF as are the angles BHG, GFE (Postulate 5). Let the lines HB, FE be produced at meet at K. Through the point K let the straight line KL be drawn parallel to either EA or FH (I.31); then let the straight lines HA, GB be produced to the points L, M (Postulate 2). Then $HLKF$ be a parallelogram; HK is its diameter, AG, ME are parallelograms about its diameter, and LB, BF are the so-called complements; therefore LB is equal to BF (I.43). But BF is equal to the triangle C; therefore LB is also equal to C (Common Notion 1). And the lines AE, GM cutting each other make the vertical angles GBE, ABM equal (I.15); but the angle GBE is equal to the angle D, therefore the angle ABM is also equal to the angle D (Common Notion 1).

superasma: Therefore the parallelogram LB equal to the given triangle C has been applied to the given line AB in the angle ABM equal to D. $\square$

Scholia for I.44 [19, pp. 203–207].

Vitrac [34, p. 276] writes:

> Compte-tenu de l'égalité des compléments établie dans la Prop. 43, le triangle C et l'angle D étant donnés, il suffit de construire un parallélogramme équivalent à C admettant un angle égal à D grâce à la Prop. 42, de la placer de telle façon que l'un de ses côtés soit un alignement avec la droite AB donnée – ce qui suppose déjà que l'on accorde une certaine latitude à ce parallélogramme quant

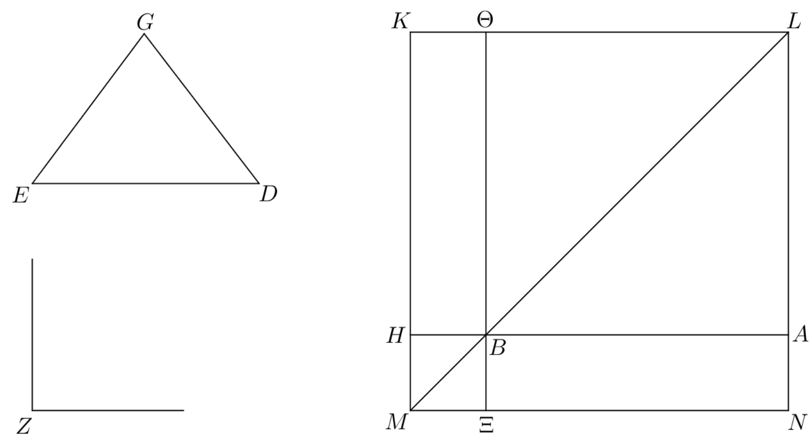

Figure 32: Al-Nayrizi, I.44

> à sa position ou qu l'on autorise le ≪déplacement≫ des figures – puis de compléter la figure grâce à la théorie des parallèles pour obtenir une figure du même type que celle de la Prop. 43. L'autre complément sera donc un parallélogramme équivalent, construit sur la droite donnée et équiangle au premier complément.

Al-Nayrizi [24, p. 188], I.44, *protasis*:

> We want to demonstrate how to construct, upon a known straight line, a surface that is a parallelogram equal to a known triangle and whose angle is equal to a known angle.

Proof. ekthesis: So let the known line be the line AB, the known triangle the triangle GDE, and the known angle the angle Z.

diorismos: We want to demonstrate how to construct on the line AB a surface that is a parallelogram equal to the triangle GDE and whose angle is equal to the angle Z.

kataskeuē: Extend AB (Postulate 2) and cut off BH equal to half DE (I.3). Use I.42 to construct on BH the parallelogram $B\Theta KH$ equal to the triangle GDE and the angle $HB\Theta$ equal to the angle Z.[3] Extend ΘK to L (Postulate 2). Use I.31 to draw through A a line parallel to $B\Theta$. Let L be the intersection of this line and $K\Theta L$. Then draw LB.

[3]What is used here is more than what I.42 provides.

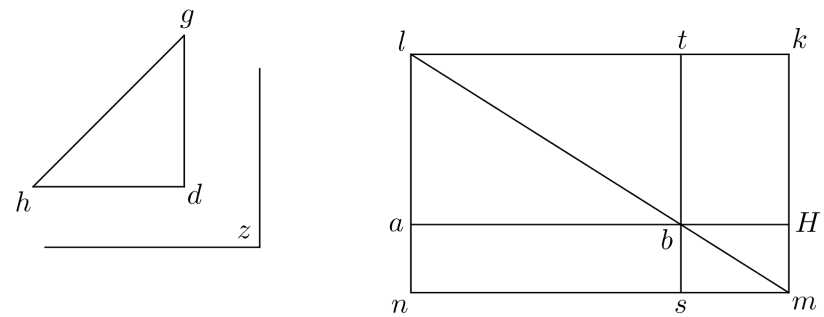

Figure 33: Adelard of Bath, I.44

The lines KH, AL are parallel and the line LK falls on them, so according to I.29, the interior angles LKM, KLN are equal to two right angles. The line LK falling on the lines LB, KH makes the interior angles LKM, KLM, which are less than two right angles. Thus by I.29, the lines KH, LB extended meet at some point; let M be this point. By I.31, draw MN parallel to KL. Extend the line LA and let N be the point at which it meets the line MN. Then extend ΘB and let Ξ be the point at which it meets the line MN.

apodeixis: The surface LM is a parallelogram with diameter LM and the surfaces $A\Theta, \Xi H$ are parallelograms about the diameter. By I.43, the complements NB, BK are equal. The parallelogram $B\Theta KH$ was constructed equal to the triangle GDE, so the parallelogram $AB\Xi N$ is equal to the triangle GDE. Now, the lines $AH, \Theta\Xi$ cut each and make vertical angles $HB\Theta, AB\Xi$, and by I.15 these angles are equal. The angle $HB\Theta$ was constructed equal to the angle Z, so the angle $AB\Xi$ is equal to the angle Z.

superasma: We have constructed on the line AB the parallelogram $A\Xi$ equal to the given triangle GDE and whose angle $AB\Xi$ is equal to the angle Z. □

The proof of I.44 in Adelard of Bath [6, pp. 66–67] is similar to the proof in Al-Nayrizi. The figure for I.44 in Adelard of Bath is given in Figure 33.

Robert of Chester [9, p. 129], I.44:

> Proposita linea recta super eam superficiem equidistancium laterum, cuius angulus sit angulo assignato equalis, ipsa vero superficies triangulo assignato equalis, designare.

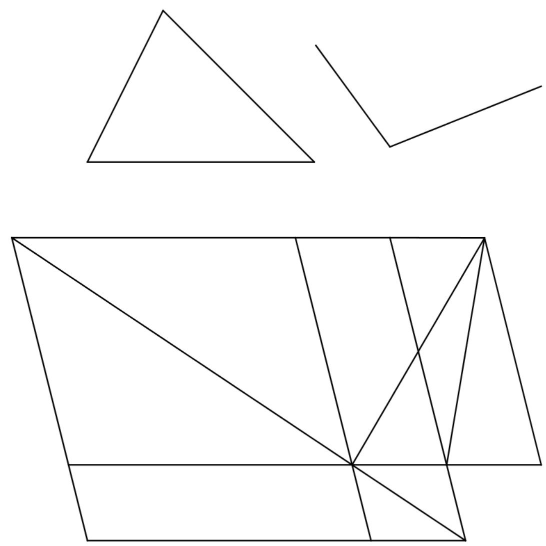

Figure 34: Robert of Chester (?), I.44

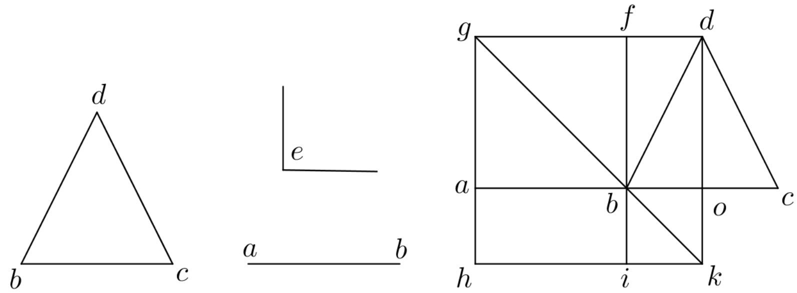

Figure 35: Johannes de Tinemue, I.44

> Date linee tamquam dimidium basis dati trianguli adicito directe atque super adiectam paralellogramum equale dato triangulo, cuius angulus super communem terminum date atque addite linee statutus dato angulo fiat equus, compleaturque figura, in cuius spacio inventum paralellogramum sit unum supplementum et alterum sit super datam lineam. Itaque per XLI$^{\text{am}}$ atque per premissam promissum provenire necesse est.
>
> Sic autem super adiectam statues paralellogramum equum dato triangulo: Protrahe eam, donec equetur toti basi dati trigoni. Deinde super duos terminos tocius adiecte linee fac duos angulos equales illis duobus qui sunt super basim dati trianguli, et conclude triangulum, quem ex XXVl$^{\text{a}}$ convinces esse equalem dato triangulo. Itaque per XLII$^{\text{am}}$ perfice.

The figure for I.44 in Robert of Chester is given in Figure 34.
Johannes de Tinemue [7, pp. 67–68], I.44:

> Proposita linea recta super eam superficiem equidistantium laterum, cuius angulus sit angulo assignato equalis, ipsa vero superficies triangulo assignato equalis, designare.
>
> Esto exemplum ab linea et bcd triangulus et e angulus.
>
> Dispositio. Protrahatur itaque ab in continuum et directum ad equilitatem bc basis trianguli et bc quoque dividatur equaliter in o linea dividente protracta ad d. Erit itaque o angulus dexter superior vel maior vel minor vel equalis e secundum quod tripliciter variatur.

Figure 35.

> Dispositio. Sit itaque primo equalis. Deinde describatur ad parallelogramum secundum exigentiam ao, od. A b vero ducatur bf equidistanter ad od, gb diametro interiecta et protracta in occursum ga, fb

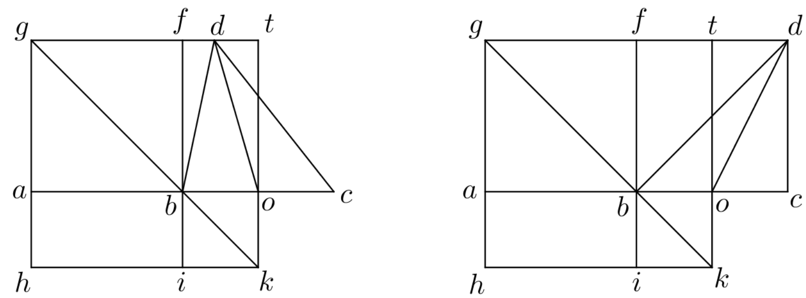

Figure 36: Johannes de Tinemue, I.44

> vero protracta in occursum hk et obveniet hd parallelogramum quasi distinctum per af et io parallelograma circa diametrum et hb, bd supplementa. Ergo secundum premissam hb, bd supplementa sunt equalia. Sed bd parallelogramum et bcd triangulus sunt equalia secundum tenorem antepremisse ratiocinationis, ergo hb et bcd sunt equalia. Preterea o angulus dexter superior est equalis e angulo dato. Sed o sinister inferior est equalis o dextra superiori secundum XV$^{\text{am}}$ et b extrinsecus reest equalis o inferiori secundum 29$^{\text{am}}$. Ergo b angulus est equalis e sicque super ab lineam describitur ai parallelogramum equale bcd cuius b angulus est equalis e angulo proposito. Quod proposuiumus.

Figure 36.

> Sit deinde o dexter superior maior quam e et ad eius equalitatem resecetur ot linea protracta in occursum dt linee equidistantis ad bc. A b angulo vero educatur bf equidistans ad ot. Deinde tota dispositio inclinetur secundum ot dextrorsum et obveniet propositum secundum superiorem ratiocinationem ad consequentiam protractam.
>
> Sit denique o angulus superior dexter minor quam e et augeatur ad equalitatem eiusdem e. Deinde secundum priorem ratiocinationem disponendi tota machina inclinetur sinistrorsum secundum ot. Et exibit propositum.

Proof. Let ab be the line, let bcd be the triangle, and let e be the angle.

Extend ab in a straight line so that bc is equal to the base of the triangle, let bc be divided equally at o, and let a line be drawn from o to d. Then the top right angle at o is either greater than or less than or equal to e.

First take the case where the angle is equal to e. Let the parallelogram ad be described with the sides ao, od. Let bf be drawn from b parallel to od. Extend

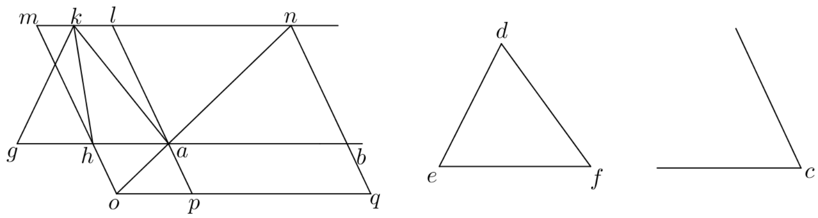

Figure 37: Campanus, I.44

gb to intersect od at some point k. Draw kh parallel to gd and suppose it meets the line ga at some point h. Extend fb to meet hk at some point i. Then the parallelogram gk is divided into the parallelograms af and io about the diameter and the complements hb, bd. So according to I.43, the complements hb, bd are equal. But the parallelogram bd and the triangle bcd are equal. Hence the parallelogram hb and the triangle bcd are equal. Moreover the top right angle at o is equal to the given angle e. But the bottom left angle at o is equal to the top right angle at o by I.15; so bok is equal to the given angle e. And the angle abi is equal to the angle bok by I.29; so abi is equal to the given angle e, i.e., the angle b is equal to the given angle e. Therefore, on the line ab the parallelogram ai has been constructed equal to the given triangle bcd and with the angle at b equal to the given angle e. □

Hermann of Carinthia [5]
Gerard of Cremona [23]
Campanus [8, pp. 91–92]:

> Proposita recta linea super eam superficiem equidistantium laterum, cuius angulus sit angulo assignato equalis, ipsa vero superficies triangulo assignato equalis, designare.

> Designare superficiem equidistantium laterum super lineam aliquam est lineam ipsam facere latus unum ipsius superficiei. Sit ergo data linea ab et datus angulus c et datus triangulus def. Super lineam ab volo designare superficiem unam equidistantium laterum ita quod linea ab sit unum ex lateribus eius cuius uterque duorum angulorum contra se positorum sit equalis angulo c et ipsa totalis superficies sit equalis triangulo def. Differt autem hec a 42 quia hic datum latus unius superficiei describende scilicet linea ab, ibi autem nullum.

> Cum ergo hoc volo facere ad lineam ab, adiungo secundum rectitudinem lineam ag quam pono equalem linee ef basi trianguli dati super quam constituo triangulum unum ei equalem et equilaterum. Quod hoc modo facio. Constituo angulum agk equalem angulo e et angulum gak equalem angulo f per 23. Et quia ga posita fuerat equalis ef, erit per 26 triangulus gak equalis et equilaterus triangulo efd. Dividam ergo ga per equalia in puncto h et protraham kh et producam a puncto k lineam mkn equidistantem linee gh eritque per 38 triangulus ahk equalis triangulo ghk. Tunc super punctum a linee ga faciam angulum gal equalem angulo c dato et complebo super basim ah et inter lineas gh et mn equidistantes superficiem equidistantium laterum $mlha$ que per 41 dupla erit ad triangulum kha, quare equalis totali triangulo kga quare et triangulo def proposito. Protraham ergo bn equidistantem al et producam diametrum na quam protraham quousque concurrat cum mh in puncto o et complebo superficiem equidistantium laterum $monq$ et protraham la usque ad p. Eritque per precedentem supplementum $abpq$ equale supplemento $mlha$, quare triangulo def et quia per 15 angulus lah est equalis angulo bap et ideo angulus bap equalis angulo c, patet super datam lineam ab descriptam superficiem esse equidistantium laterum $abpq$ equalem dato triangulo def cuius uterque duorum angulorum contra se positorum qui sunt a et q est equalis dato angulo c. Quod fuit propositum.

Proof. Let ab be the given line, c the given angle, and def the given triangle. Extend the line ab by the line ag equal to ef. Apply I.23 to make angle agk equal to angle e; apply I.23 to make angle gak equal to angle f; let k be the point of intersection of the lines gk and ak. Because ga was put equal to ef, by I.27, the triangle gak is equal to the triangle efd. Divide ga into equal halves at the point h and join kh; draw through k a line mkn parallel to the line gh; then by I.38, the triangle ahk is equal to the triangle ghk. Therefore the triangle kga is double the triangle kha. At the point a on the line ga make an angle gal equal to the given angle c. Then on the base ah and between the parallel lines gh and mn complete the parallelogram $mlha$; by I.41 this is double the triangle kha and therefore the parallelogram $mlha$ is equal to the whole triangle kga which itself is equal to the given triangle def. Draw the line bn parallel to the line al and produce na until it intersects mh, and let o be the point of intersection. Then complete the parallelogram $monq$ and produce la to p. Then by I.43, the complement $abpq$ is equal to the complement $mlha$, hence the parallelogram $abpq$ is equal to the given triangle def. By I.15, the lines bh, lp cutting each other make equal vertical angles lah, bap. But gal is equal to the given angle c, so bap is equal to the given angle c. Therefore, on the given line ab the parallelogram $abpq$ has been described that is equal to the given triangle def and both of the opposite angles a and q are equal to the given angle c. □

Clavius, *Euclidis Elementorum libri XV*, *Opera mathematica*, pp. 72–73, scholium to I.43, gives two different constructions for I.44.

Proof. Ad datam rectam lineam, dato triangulo aequale parallelogrammum applicare in dato angulo rectilineo.

Let the given line be A, let the given triangle be B, and let the given angle be C. *Quod si quis optet, lineam ipsam A, datam, esse unum latus parallelogrammi, non difficile erit transferre parallelogrammum $FMLH$, ad rectam A, ex iis, quae in scholio propos. 31. huius lib. docuimus.* □

Proclus 419–420 [26, pp. 332–333]:

> Eudemus and his school tell us that these things – the application (*parabole*) of areas, their exceeding (*huperbole*), and their falling short (*elleipsis*) – are ancient discoveries of the Pythagorean muse. It is from these procedures that later geometers took these terms and applied them to the so-called conic lines, calling one of them "parabola," another "hyperbola," and the third "ellipse," although these godlike men of old saw the significance of these terms in the describing of plane areas along a finite straight line. For when, given a straight line, you make the given area extend along the whole of the line, they say you "apply" the area; when you make the length of the area greater than the straight line itself, then it "exceeds"; and when less, so that there is a part of the line extending beyond the area described, then it "falls short." Euclid too in his sixth book speaks in this sense of "exceeding" and "falling short"; but here he needed "application," since he wished to apply to a given straight line an area equal to a given triangle, in order that we might be able not only to construct a parallelogram equal to a given triangle, but also to apply it to a given finite straight line. For example, when a triangle is given having an area of twelve feet and we posit a straight line whose length is four feet, we apply to the straight line an area equal to the triangle when we take its length as the whole four feet and find how many feet in breadth it must be in order that the parallelogram may be equal to the triangle. Then when we have found, let us say, a breadth of three feet and multiplied the length by the breadth, we shall have the area, that is, if the angle assumed is a right angle. Something like this is the method of "application" which has come down to us from the Pythagoreans.

Proclus 421 [26, pp. 333–334]:

> Application and construction are not the same thing, as we have said. Construction brings the whole figure into being, both its area and all its sides, whereas application starts with one side given and constructs the area along it, neither falling short of the length of the line nor exceeding it, but using it as one of the sides enclosing the area.

I.45: "To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure."

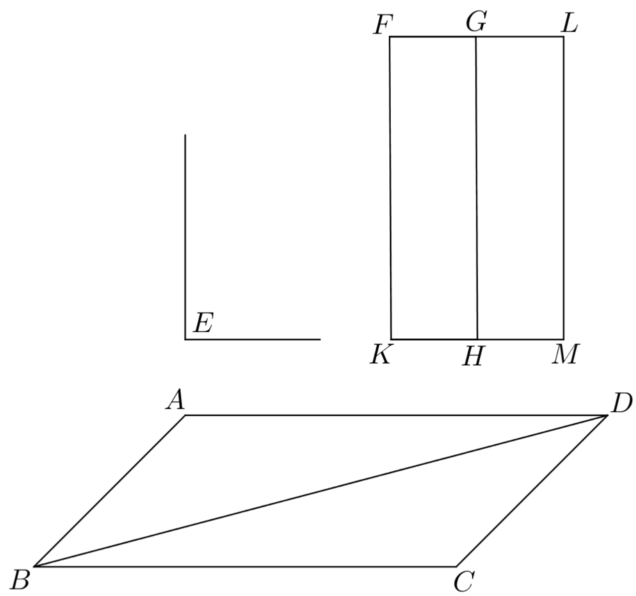

Figure 38: I.45: P 38r, F 13r, B 30r, V 46

Scholia for I.45 [19, pp. 207–209].
Proclus 422–423 [26, pp. 334–335], on *Elements* I.45:

> For any rectilineal figure, as we said earlier, is as such divisible into triangles, and we have given the method by which the number of its triangles can be found. Therefore by dividing the given rectilineal figure into triangles and constructing a parallelogram equal to one of them, then applying parallelograms equal to the others along the given straight line – that line to which we made the first application – we shall have the parallelogram composed of them equal to the rectilineal figure composed of the triangles, and the assigned task will have been accomplished. That is, if the rectilineal figure has ten sides, we shall divide it into eight triangles, construct a parallelogram equal to one of them, and then by applying in seven steps parallelograms equal to each of the others, we shall have what we wanted.
>
> It is my opinion that this problem is what led the ancients to attempt the squaring of the circle. For if a parallelogram can be found equal to any rectilineal figure, it is worth inquiring whether it is not possible to prove that a rectilineal figure is equal to a circular area. Indeed Archimedes proved that a circle is equal to a right-angled triangle when its radius is equal to one of the sides about the right angle and its perimeter is equal to the base.

Vitrac [34, p. 278]:

> La Prop. 45 montre donc comment construire, dans un angle rectiligne donné, un parallélogramme équivalent à une figure rectiligne donnée : celle-ci est divisée en triangles et l'on construit pour chacun d'eux un parallélogramme équivalent; il faut les agencer de manière à ce qu'ils forment, ensemble, un parallélogramme, ce qu'Euclide vérifie soigneusement grâce aux angles (Prop. 14) et à la theorie des parallèles (Prop. 29, 30, 33, 34). En fait la question de l'agencement est résolue seulement pour deux parallélogrammes, mais le lecteur doit comprendre que la méthode peut donner lieu à itération un nombre n de fois, bien qu'à chaque étape on ne traite que deux parallélogrammes à la fois. Dans les Livres arithmétiques, nous retrouverons cette façon de procéder qui n'est certainement pas un raisonnement par induction, mais qui, en indiquant un schéma général par le biais de la première étape, justifie néanmoins une énonciation qui, quant à elle, est bien «universelle».

I.46: "On a given straight line to describe a square."

Proof. ekthesis: Let AB be the given straight line.
diorismos: It is required to describe a square on the straight line AB.

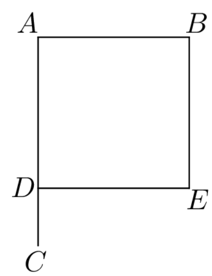

Figure 39: I.46: P 38r, F 13r, B 30v, V 46

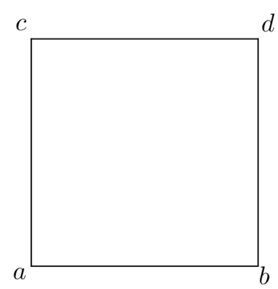

Figure 40: Campanus, I.46 (I.45 in Campanus)

kataskeuē: By I.11, draw a stright line AC at right angles to the straight line AB. By I.3, cut off AD from AC equal to AB. By I.31, draw DE parallel to AB and by I.3 make this equal to AB.
apodeixis:
sumperasma: □

Scholia for I.46 [19, pp. 209–212].
Campanus [8, p. 92], I.46 (I.45 in Campanus; Campanus does not include our I.45):

> Ex data linea quadratam describere.

Proof. Let the given line be ab. It is required to construct a square on the line ab. By I.11, at the point a let the line ac be drawn at right angles to the line ab; likewise by I.11, at the point b let bd be drawn at right angles to the line ab.

The line ab falling on the lines ac, bd makes interior angles cab and dba on one side; each is a right angle, so the interior angles are equal to two right angles and then by I.28, the lines ca, bd are parallel. By I.3, cut down each of these to equal ab. Join cd. By I.33, because the lines ca, bd are equal and parallel, the lines cd, ab are equal and parallel. By I.29, because both of the two angles a and b are right, both of the two angles c and d are right. Therefore by definition $abcd$ is a square.

Another proof. By I.11, let ac be perpendicular to ab and let it be equal to ab. By I.31, at the point c, draw cd parallel with ab and let it be equal to ab. Then draw db, which by I.33 it equal and parallel to ac. By I.29 all the angles are right angles, and so by definition $abcd$ is a square. □

Talking about Postulates 1–3 of the *Elements*, Mueller [27, p. 16] says:

> Much more insight is obtained by examining the central proposition or propositions of book I and showing how the book builds to its or their proof. In fact, almost the entire content of book I can be explained by reference to the construction of a parallelogram in a given angle and equal (in area) to a given rectilineal figure in proposition 45. This proposition makes it possible to represent any rectilineal area as a rectangle. Euclid could have proved a stronger result, namely that any rectilineal area can be represented as a rectangle with a given base. (Compare I,44.) From our point of view this result would be more interesting, since the areas of rectangles on equal bases are proportional to the lengths of their sides. For the Greeks, however, the important representation of an area seems to be as a square, and I,45 is sufficient for Euclid to be able to show in II,14 how to construct a square equal to any given rectilineal figure. This proposition represents the true culmination of the geometry of the area of rectilineal figures. Euclid postpones it to book II because its proof involves methods which he introduces there and which he wishes, presumably for purposes of exposition, to separate from the methods of book I.

I.2 Construct at a point a line equal to given line.

I.11–12 Construct at a point a line perpendicular to given line.

I.23 Construct at a point and on a line an angle equal to a given angle.

I.31 Construct at a point a line parallel to a given line.

I.35 Parallelograms on the same base in the same parallels are equal.

I.41 A parallelogram on the same base and in the same parallels as a given triangle is double the given triangle.

I.42 Construct in an angle a parallelogram equal to a given triangle.

I.43 In a parallelogram, given two parallelograms about the diameter, the complements are equal.

I.44 Construct in an angle and applied to a line a parallelogram equal to a given triangle.

I.45 Triangulation and induction, invokes I.44.

Mugler [28, p. 324]: *paraballo*, *paraballein*
Plato, *Meno* 86e–87a [16, p. 140]:

> Just grant me one small relaxation of your sway, and allow me, in considering whether or not it can be taught, to make use of a hypothesis – the sort of thing, I mean, that geometers often use in their inquiries. When they are asked, for example, about a given area, whether it is possible for this area to be inscribed as a triangle in a given circle, they will probably reply: 'I don't know yet whether it fulfils the conditions, but I think I have a hypothesis which will help us in the matter. It is this. If the area is such that, when one has applied it [sc. as a rectangle] to the given line [i.e. the diameter] of the circle, it is deficient by another rectangle similar to the one which is applied, then, I should say, one result follows; if not, the result is different. If you ask me, then, about the inscription of the figure in the circle – whether it is possible or not – I am ready to answer you in this hypothetical way.'

Bluck [3, p. 442] writes in his commentary on the *Meno*: "If a rectangle $ABCD$ is applied to a line BH which is greater than the base of the rectangle, it was said to 'fall short' by the area enclosed when DCH is completed as a rectangle. The same is true, *mutatis mutandis*, in the case of any parallelogram. This appears not only from Euclid, but from a passage of Proclus (*Comm. in Eucl.* I, 44), in which this use of *ἐλλείπειν* is attributed, on the authority of *οἱ περὶ τὸν Εὔδημον*, to early Pythagoreans."

Klein [21, p. 206] gives the following translation of this passage:

> Geometricians do often adopt the following kind of procedure. If, for example, one of them has to answer the question whether a certain amount of space (whatever its – rectilinear – boundaries) is capable of being fitted as a triangle into a given circular area (so that the three vertices will touch the circumference of that circular area), he may say: while I do not know whether this particular amount of space has that capability I believe I have something of a supposition (*hosper* ... *tina hypothesin*) at hand which might be useful for the purpose. It is this: if that amount of space (which can always be transformed into a triangular or rectangular area) were to be such that he who "stretches it along" (*parateinanta*) its (*autou*) given line "runs short" (*elleipein*) of a space like the very one which had been

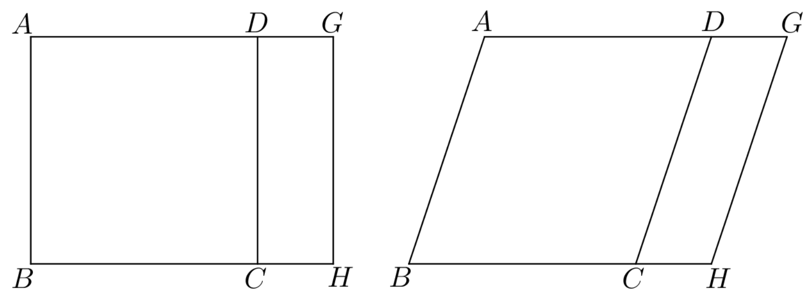

Figure 41: Bluck, on *Meno* 86e–87a

"stretched along" (the given line), then, it seems to me, one thing would be the result, and another again, if it were impossible for him to go through this experience. And so I am disposed to tell you what will happen with regard to the inscription of your amount of space (*autou*) into the circle, whether it is impossible or not impossible, by way of "hypothesizing" (*hypothemenos*).

Plato, *Republic* VII, 526e–527b [11, p. 244]:

> Socrates: So geometry will be suitable or not, according as it makes us contemplate reality or the world of change.
>
> Glaucon: That is our view.
>
> Socrates: In this respect, then, no one who has even a slight acquaintance with geometry will deny that the nature of this science is in flat contradiction with the absurd language used by mathematicians, for want of better terms. They constantly talk of 'operations' like 'squaring,' 'applying,' 'adding,' and so on, as if the whole subject were to *do* something, whereas the true purpose of the whole subject is knowledge – knowledge, moreover, of what eternally exists, not of anything that comes to be this or that at some time and ceases to be.

Plutarch, *Quaestiones Convivales* VIII.2.4, 720A [33, p. 177]:

> Among the most geometrical theorems, or rather problems, is this – given two figures, to apply a third equal to the one and similar to the other; it was in virtue of this discovery they say Pythagoras sacrificed. This is unquestionably more subtle and elegant than the theorem which he proved that the square on the hypotenuse is equal to the squares on the sides about the right angle.

Plutarch, *Non posse suaviter vivi secundum Epicurum* 1094B [13, pp. 65–67]:

> Our love of pleasure, to be sure, takes many forms and is enterprising enough; but no one has so far upon having his way with the woman he loves been so overjoyed that he sacrificed an ox, nor has anyone prayed to die on the spot if he could only eat his fill of royal meat or cakes; whereas Eudoxus prayed to be consumed in flames like Phaëthon if he could but stand next to the sun and ascertain the shape, size, and composition of the planets, and when Pythagoras discovered his theorem he sacrificed an ox in honour of the occasion, as Apollodorus says:
>
> > When for the famous proof Pythagoras
> > Offered an ox in splendid sacrifice–
>
> whether it was the theorem that the square on the hypotenuse is equal to the sum of the squares on the sides of the right angle or a problem about the application of a given area.

Aristotle, *De anima* II.2, 413a13–20 [17, p. 191]:

> It is not enough that the defining statement should make clear the bare fact as most definitions do; it should also include and exhibit the cause. As things are, what is stated in definitions is usually of the nature of a conclusion. For instance, what is 'squaring'? The construction of an equilateral rectangle equal (in area) to a given oblong (rectangle). Such a definition is a statement of the conclusion, whereas, if you say that squaring is the finding of a mean proportional, you state the cause of the thing defined.

Philoponus [10, p. 34] writes the following in his commentary on this passage:

> It is clear that he is speaking of squaring an oblong. A square is both equilateral and a rectangle, that is, an area that has both the four sides equal and the angles right angles; an oblong is rectangular, indeed, but not equilateral. Those, then, who wish to square the oblong seek a mean proportional. What sort of thing do I mean? Let there be an oblong area having one side of eight cubits and the other of two. Clearly the whole is of 16 [square] cubits. For every quadrilateral is measured by multiplying side by side. If, therefore, we wish to make a square equal to this oblong area, so as to be 16 cubits, the size the oblong was, we must find the mean proportional of the two sides of the oblong, so that it may have that ratio to the greater side, which was of 8 cubits, which the side of the oblong which was of 2 cubits has to it, the mean. Such [a mean] would be of 4 cubits. For that same ratio which 4 has to 8, 2 has to 4: each is half the greater. This is the mean proportional. On this, therefore, will be inscribed a square area of 16 cubits equal to the oblong. And

thus should we do with every oblong when we want to inscribe a square equal to it. For again, if there should be an oblong having one side of 16 cubits and the other of 4, it inscribes an area, clearly, of 64. If you should want to make a square equal to this, seek a mean proportional. That is of 8 cubits. For 8 times 8 is 64. For just as the 16 cubit side of the oblong is the double of the 8 cubit [side] that has been found, so too this is the double of the remaining side of the oblong, which was of 4 cubits.

> ‘Finding the mean?, Alexander says, ‘is shown in the second book of Euclid?. But it is not. Nothing of this sort is shown there, but in the sixth. There it is shown: ‘Given two straight lines, to find the mean proportional’ [*Elements* 6.13], and ?If three straight lines are proportional, the [rectangle] contained by the extremes is equal to the [square] on the middle? [*Elements* 6.17].

Aristotle, *Metaphysics* III.2, 996b18–21 [17, pp. 191–192]:

> Again, in the case of other things, namely those which are the subject of demonstration, we consider that we possess knowledge of a particular thing when we know what it is [i.e. its definition], e.g. we know what squaring is because (we know that) it is the finding of the mean proportional.

Iamblichus, *De communi mathematica scientia*, l. 21, Chapter XXIV [14, p. 75]
Becker [2, p. 59f.]
Knorr [22]
Burkert [4, p. 452]:

> The application of areas was known to Plato, but Hippocrates of Chios, for a problem soluble by this method, used the method of “inclination” or “verging” (νεῦσις); it looks as though the application of areas was at least not fully developed in Hippocrates.

Archibald’s reconstruction of Euclid’s *Divisions of Figures* [1]
Pappus, *Collection* 7.30–31 [20, pp. 114–116]:

> [30] Apollonius, filled out Euclid’s four books of *Conics* and added on another four, handing down eight volumes of *Conics*. Aristaeus, who wrote the five volumes of *Solid Loci*, which have been transmitted until the present immediately following the *Conics*, and Apollonius’s (other) predecessors, named the first of the three conic curves ‘section of an acute-angled cone’, the second 1of a right-angled’, the third of an obtuse-angled’. But since the three curves occur in each of these three cones, when cut variously, Apollonius was apparently at a loss to know why on earth his predecessors selectively named the one ‘section of an acute-angled cone’ when it can also be (a section)

of a right-angled and obtuse-angled one, the second (cone), and the third 'of an obtuse-angled' when it can be of an acute-angled and a right-angled (cone), so, replacing the names, he called the (section) of an acute-angled (cone) 'ellipse', that of a right-angled 'parabola', and that of an obtuse-angled 'hyperbola', each from a certain property of its own. For a certain area applied to a certain line, in the section of an acute-angled cone, falls short by a square, in that of an obtuse-angled (cone) exceeds by a square, but in that of a right-angled (cone) neither falls short nor exceeds.

[31] This was his notion because he did not perceive that by a certain single way of having the plane cut the cone in generating the curves, a different one of the curves is produced in each of the cones, and they named it from the property of the cone. For if the cutting plane is drawn parallel to one side of the cone, one only of the three curves is formed, always the same one, which Aristaeus named a section of the (kind of) cone that was cut.

Taisbak on application of areas [31]
Euclid's *Data* [32], Data 57
Sulba Sutras
Friberg [15]: pp. 114,

References

[1] Raymond Clare Archibald. *Euclid's Book on Division of Figures, with a Restoration Based on Woepcke's Text and on the* Practica Geometriae *of Leonardo Pisano.* Cambridge University Press, 1915.

[2] Oskar Becker. *Das mathematische Denken der Antike*, volume 3 of *Studienhefte zur Altertumswissenschaft.* Vandenhoeck & Ruprecht, Göttingen, second edition, 1966.

[3] R. S. Bluck. *Plato's Meno.* Cambridge University Press, 1961.

[4] Walter Burkert. *Lore and Science in Ancient Pythagoreanism.* Harvard University Press, 1972. Translated from the German by Edwin L. Minar, Jr.

[5] H. L. L. Busard, editor. *The translation of the Elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?).* Brill, Leiden, 1968.

[6] H. L. L. Busard, editor. *The first Latin translation of Euclid's* Elements *commonly ascribed to Adelard of Bath*, volume 64 of *Studies and Texts.* Pontifical Institute of Mediaeval Studies, Toronto, 1983.

[7] H. L. L. Busard, editor. *Johannes de Tinemue's redaction of Euclid's Elements, the so-called Adelard III version*, volume 45,1 of *Boethius: Texte*

und Abhandlungen zur Geschichte der Mathematik und der Naturwissenschaften. Franz Steiner Verlag, Stuttgart, 2001.

[8] H. L. L. Busard, editor. *Campanus of Novara and Euclid's* Elements*, volume I*. Franz Steiner Verlag, Stuttgart, 2005.

[9] Hubert L. L. Busard and Menso Folkerts, editors. *Robert of Chester's (?) redaction of Euclid's* Elements*, the so-called Adelard II version. Volume I*, volume 8 of *Science Networks. Historical Studies*. Springer Basel AG, 1992.

[10] William Charlton. *Philoponus: On Aristotle On the Soul 2.1–6*. Ancient Commentators on Aristotle. Bloomsbury, 2005.

[11] Francis Macdonald Cornford. *The* Republic *of Plato, translated with introduction and notes*. Oxford University Press, 1941.

[12] Catherine Dalimier. Apollonios Dyscole sur la fonction des conjonctions explétives. *Revue des Études Grecques*, 112:719–730, 1999.

[13] Benedict Einarson and Phillip H. De Lacy. *Plutarch, Moralia, volume XIV*. Number 428 in Loeb Classical Library. Harvard University Press, 1967.

[14] Nicola Festa, editor. *Iamblichi de communi mathematica scientia liber*. B. G. Teubner, Lipsiae, 1891.

[15] Jöran Friberg. *Amazing Traces of a Babylonian Origin in Greek Mathematics*. World Scientific, 2007.

[16] W. K. C. Guthrie. *Plato, Protagoras and Meno*. Penguin Classics. Penguin Books, 1956.

[17] Thomas L. Heath. *Mathematics in Aristotle*. Claredon Press, Oxford, 1949.

[18] Thomas L. Heath. *The thirteen books of Euclid's Elements, volume I: Books I-II*. Dover Publications, second edition, 1956.

[19] J. L. Heiberg, editor. *Euclidis Opera omnia, vol. V*. B. G. Teubner, Lipsiae, 1888.

[20] Alexander Jones, editor. *Pappus of Alexandria. Book 7 of the* Collection*. Part 1. Introduction, Text, and Translation*, volume 8 of *Sources in the History of Mathematics and Physical Sciences*. Springer, 1986.

[21] Jacob Klein. *A Commentary on Plato's* Meno. University of Chicago Press, 1989.

[22] Wilbur Richard Knorr. *The Ancient Tradition of Geometric Problems*. Dover Publications, 2012.

[23] Anthony Lo Bello. *Gerard of Cremona's translation of the commentary of al-Nayrizi on Book I of Euclid's* Elements of Geometry, volume 2 of *Ancient Mediterranean and Medieval Texts and Contexts*. Brill, 2003.

[24] Anthony Lo Bello. *The Commentary of al-Nayrizi on Book I of Euclid's* Elements of Geometry*, with an Introduction on the Transmission of Euclid's* Elements *in the Middle Ages*, volume 1 of *Ancient Mediterranean and Medieval Texts and Contexts*. Brill, 2003.

[25] Anthony Lo Bello. *The Commentary of Albertus Magnus on Book I of Euclid's* Elements of Geometry, volume 3 of *Ancient Mediterranean and Medieval Texts and Contexts*. Brill, 2003.

[26] Glenn R. Morrow. *Proclus: A Commentary on the First Book of Euclid's* Elements. Princeton University Press, 1992.

[27] Ian Mueller. *Philosophy of Mathematics and Deductive Structure in Euclid's* Elements. Dover Publications, 2006.

[28] Charles Mugler. *Dictionnaire historique de la terminologie géométrique des Grecs*, volume XXVIII of *Études et commentaires*. Librairie C. Klincksieck, Paris, 1958.

[29] Reviel Netz. *The shaping of deduction in Greek mathematics: a study in cognitive history*, volume 51 of *Ideas in Context*. Cambridge University Press, 1999.

[30] Árpád Szabó. *The beginnings of Greek mathematics*, volume 17 of *Sythese Historical Library*. D. Reidel Publishing Company, Dordrecht, 1978. Translated from the German by A. M. Ungar.

[31] Christian Marinus Taisbak. Exceeding and falling short: Elliptical and hyperbolical application of areas. *Science in Context*, 16(3):299–318, 2003.

[32] Christian Marinus Taisbak. *Euclid's* Data*, or The Importance of Being Given*, volume 45 of *Acta Historica Scientiarum Naturalium et Medicinalium*. Museum Tusculanum Press, Copenhagen, 2010.

[33] Ivor Thomas, editor. *Greek Mathematics, volume I.* Number 362 in Loeb Classical Library. Harvard University Press, 1991.

[34] Bernard Vitrac and Maurice Caveing. *Euclide d'Alexandrie. Les Éléments. Volume 1: Introduction générale et Livres I–IV*. Bibliothèque d'histoire des sciences. Presses Universitaires de France, Paris, 1990.

[35] Robin Waterfield. Hippias Major and Hippias Minor. In Trevor J. Saunders, editor, *Plato, Early Socratic Dialogues*, Penguin Classics, pages 211–293. Penguin Books, 2005.