$b + cx + dx^2 = 2fp + p^2x^2$, an equation of the second degree, which gives

$$x = \frac{p^2 - c + \sqrt{(p^4 - 2cp^2 + 8dfp + c^2 - 4bd)}}{2d}.$$

So that the question is now reduced to finding such values of p, as will make the formula $p^4 - 2cp^2 + 8dfp + c^2 - 4bd$ become a square. But as it is the fourth power of the required number p which occurs here, this case belongs to the following chapter.

CHAP. IX.

Of the Method *of rendering* Rational *the incommensurable* Formula $\sqrt{(a + bx + cx^2 + dx^3 + ex^4)}$.

128. We are now come to formulæ, in which the indeterminate number, x, rises to the fourth power; and this must be the limit of our researches on quantities affected by the sign of the square root; since the subject has not yet been prosecuted far enough to enable us to transform into squares any formulæ, in which higher powers of x are found.

Our new formula furnishes three cases: the first, when the first term, a, is a square; the second, when the last term, ex^4, is a square; and the third, when both the first term and the last are squares. We shall consider each of these cases separately.

129. 1st. Resolution of the formula

$$\sqrt{(f^2 + bx + cx^2 + dx^3 + ex^4)}.$$

As the first term of this is a square, we might, by the first method, suppose the root to be $f + px$, and determine p in such a manner, that the first two terms would disappear, and the others be divisible by x^2; but we should not fail still to find x^2 in the equation, and the determination of x would depend on a new radical sign. We shall therefore have recourse to the second method; and represent the root by $f + px + qx^2$; and then determine p and q, so as to remove the first three terms, and then dividing by x^3, we shall arrive at a simple equation of the first degree, which will give x without any radical signs.

130. If, therefore, the root be $f + px + qx^2$, and for that reason

$$f^2 + bx + cx^2 + dx^3 + ex^4 =$$
$$f^2 + 2fpx + p^2x^2 + 2fqx^2 + 2pqx^3 + q^2x^4,$$

the first terms disappear of themselves; with regard to the second, we shall remove them by making $b = 2fp$, or $p = \frac{b}{2f}$; and, for the third, we must make $c = 2fq + p^2$, or $q = \frac{c - p^2}{2f}$. This being done, the other terms will be divisible by x^3, and will give the equation $d + ex = 2pq + q^2x$, from which we find

$$x = \frac{d - 2pq}{q^2 - e}, \text{ or } x = \frac{2pq - d}{e - q^2}.$$

131. Now, it is easy to see that this method leads to nothing, when the second and third terms are wanting in our formula; that is to say, when $b = 0$, and $c = 0$; for then $p = 0$, and $q = 0$; consequently, $x = -\frac{d}{e}$, from which we can commonly draw no conclusion, because this case evidently gives $dx^3 + ex^4 = 0$; and, therefore, our formula becomes equal to the square f^2. But it is chiefly with respect to such formulæ as $f^2 + ex^4$, that this method is of no advantage, since in this case we have $d = 0$, which gives $x = 0$, and this leads no farther. It is the same, when $b = 0$, and $d = 0$; that is to say, the second and fourth terms are wanting, in which case the formula is $f^2 + cx^2 + ex^4$; for, then $p = 0$, and $q = \frac{c}{2f}$, whence $x = 0$, as we may immediately perceive, from which no further advantage can result.

132. 2d. Resolution of the formula

$$\sqrt{(a + bx + cx^2 + dx^3 + g^2x^4)}.$$

We might reduce this formula to the preceding case, by supposing $x = \frac{1}{y}$; for, as the formula

$$a + \frac{b}{y} + \frac{c}{y^2} + \frac{d}{y^3} + \frac{g^2}{y^4}$$

must then be a square, and remain a square if multiplied by the square y^4, we have only to perform this multiplication, in order to obtain the formula

$$ay^4 + by^3 + cy^2 + dy + g^2,$$

which is quite similar to the former, only inverted.

But it is not necessary to go through this process; we have only to suppose the root to be $gx^2 + px + q$, or, inversely, $q + px + gx^2$, and we shall thus have

$$a + bx + cx^2 + dx^3 + g^2x^4 =$$
$$q^2 + 2pqx + 2gqx^2 + p^2x^2 + 2gpx^3 + g^2x^4.$$

Now, the fifth and sixth terms destroying each other, we shall first determine p so, that the fourth terms may also destroy each other; which happens when $d = 2gp$, or $p = \frac{d}{2g}$; we shall then likewise determine q, in order to remove the third terms, making for this purpose

$$c = 2gq + p^2, \text{ or } q = \frac{c - p^2}{2g};$$

which done, the first two terms will furnish the equation $a + bx = q^2 + 2pqx$; whence we obtain

$$x = \frac{a - q^2}{2pq - b}, \text{ or } x = \frac{q^2 - a}{b - 2pq}.$$

133. Here, again, we find the same imperfection that was before remarked, in the case where the second and fourth terms are wanting; that is to say, $b = 0$, and $d = 0$; because we then find $p = 0$, and $q = \frac{c}{2g}$; therefore $x = \frac{a - q^2}{0}$: now, this value being infinite, leads no farther than the value, $x = 0$, in the first case; whence it follows, that this method cannot be at all employed with respect to expressions of the form $a + cx^2 + g^2x^4$.

134. 3d. Resolution of the formula

$$\sqrt{(f^2 + bx + cx^2 + dx^3 + g^2x^4)}.$$

It is evident that we may employ for this formula both the methods that have been made use of; for, in the first place, since the first term is a square, we may assume $f + px + qx^2$ for the root, and make the first three terms vanish; then, as the last term is likewise a square, we may also make the root $q + px + gx^2$, and remove the last three terms; by which means we shall find even two values of x.

But this formula may be resolved also by two other methods, which are peculiarly adapted to it.

In the first, we suppose the root to be $f + px + gx^2$, and

p is determined such, that the second terms destroy each other; that is to say,

$$f^2 + bx + cx^2 + dx^3 + g^2x^4 =$$
$$f^2 + 2fpx + 2fgx^2 + p^2x^2 + 2gpx^3 + g^2x^4.$$

Then, making $b = 2fp$, or $p = \frac{b}{2f}$; and since by these means both the second terms, and the first and last, are destroyed, we may divide the others by x^2, and shall have the equation $c + dx = 2fg + p^2 + 2gpx$, from which we obtain $x = \frac{c-2fg-p^2}{2gp-d}$, or $x = \frac{p^2+2fg-c}{d-2gp}$. Here, it ought to be particularly observed, that as g is found in the formula only in the second power, the root of this square, or g, may be taken negatively as well as positively; and, for this reason, we may obtain also another value of x; namely,

$$x = \frac{c+2fg-p^2}{-2gp-d}, \text{ or } x = \frac{p^2-2fg-c}{2gp+d}.$$

135. There is, as we observed, another method of resolving this formula; which consists in first supposing the root, as before, to be $f + px + gx^2$, and then determining p in such a manner, that the fourth terms may destroy each other; which is done by supposing in the fundamental equation, $d = 2gp$, or $p = \frac{d}{2g}$; for, since the first and the last terms disappear likewise, we may divide the other by x, and there will result the equation $b + cx = 2fp + 2fgx + p^2x$, which gives $x = \frac{b-2fp}{2fg+p^2-c}$. We may farther remark, that as the square f^2 is found alone in the formula, we may suppose its root to be $-f$, from which we shall have $x = \frac{b+2fp}{p^2-2fg-c}$. So that this method also furnishes two new values of x; and, consequently, the methods we have employed give, in all, six new values.

136. But here again the inconvenient circumstance occurs, that, when the second and the fourth terms are wanting, or when $b = 0$, and $d = 0$, we cannot find any value of x which answers our purpose; so that we are unable to resolve the formula $f^2 + cx^2 + gx^4$. For, if $b = 0$, and

$d = 0$, we have, by both methods, $p = 0$; the former giving $x = \frac{c - 2fg}{0}$, and the other giving $x = 0$; neither of which are proper for furnishing any further conclusions.

137. These then are the three formulæ, to which the methods hitherto explained may be applied; and, if in the formula proposed neither term be a square, no success can be expected, until we have found one such value of x as will make the formula a square.

Let us suppose, therefore, that our formula becomes a square in the case of $x = h$, or that

$$a + bh + ch^2 + dh^3 + eh^4 = k^2;$$

if we make $x = h + y$, we shall have a new formula, the first term of which will be k^2; that is to say, a square, which will, consequently, fall under the first case: and we may also use this transformation, after having determined by the preceding methods one value of x, for instance, $x = h$; for we have then only to make $x = h + y$, in order to obtain a new equation, with which we may proceed in the same manner. And the values of x, that may be found in this manner, will furnish new ones; which will also lead to others, and so on.

138. But it is to be particularly remarked, that we can in no way hope to resolve those formulæ in which the second and fourth terms are wanting, until we have found one solution; and, with regard to the process that must be followed after that, we shall explain it by applying it to the formula $a + ex^4$, which is one of those that most frequently occur.

Suppose, therefore, we have found such a value of $x = h$, that $a + eh^4 = k^2$; then if we would find, from this, other values of x, we must make $x = h + y$, and the following formula, $a + eh^4 + 4eh^3y + 6eh^2y^2 + 4ehy^3 + ey^4$, must be a square. Now, this formula being reducible to $k^2 + 4eh^3y + 6eh^2y^2 + 4ehy^3 + ey^4$, it therefore belongs to the first of our three cases; so that we shall represent its square root by $k + py + qy^2$; and, consequently, the formula itself will be equal to the square

$$k^2 + 2kpy + p^2y^2 + 2kqy^2 + 2pqy^3 + q^2y^4;$$

from which we must first remove the second term by determining p, and consequently q; that is to say, by making $4eh^3 = 2kp$, or $p = \frac{2eh^3}{k}$; and $6eh^2 = 2kq + p^2$, or

$$q = \frac{6eh^2 - p^2}{2k} = \frac{3eh^2k^2 - 2e^2h^6*}{k^3} = \frac{eh^2(3k^2 - 2eh^4)}{k^3};$$

or, lastly, $q = \frac{eh^2(k^2+2a)\dagger}{k^3}$, because $eh^4 = k^2 - a$; after which, the remaining terms, $4ehy^3 + ey^4$, being divided by y^3, will give $4eh + ey = 2pq + q^2y$, whence we find $y = \frac{4eh - 2pq}{q^2 - e}$; and the numerator of this fraction may be thrown into the form $\frac{4ehk^4 - 4e^2h^5(k^2+2a)}{k^4}$‡,

or, because $eh^4 = k^2 - a$, into this,

$$\frac{4ehk^4 - 4eh(k^2-a)\times(k^2+2a)}{k^4} = \frac{4eh(-ak^2+2a^2)}{k^4} = \frac{4aeh(2a-k^2)}{k^4}.$$

With regard to the denominator $q^2 - e$, since

$q = \frac{eh^2(k^2+2a)}{k^3}$, and $eh^4 = k^2 - a$, it becomes

$$\frac{e(k^2-a)\times(k^2+2a)^2 - ek^6}{k^6} = \frac{e(3ak^4-4a^3)}{k^6} = \frac{ea(3k^4 - 4a^2)}{k^6},$$

so that the value sought will be

$$y = \frac{4aeh(2a-k^2)}{k^4} \times \frac{k^6}{ae(3k^4-4a^2)}, \text{ or,}$$

$$y = \frac{4hk^2(2a-k^2)}{3k^4-4a^2}; \text{ and, consequently,}$$

$$x = y + h = \frac{h(8ak^2-k^4-4a^2)}{3k^4-4a^2}, \text{ or}$$

$$x = \frac{h(k^4-8ak^2+4a^2)}{4a^2-3k^4},$$

* By multiplying $6eh^2-p^2$ by k^2, and substituting for k^2p^2 its equal, $2eh^3$.

† For since $k^2 = a + eh^4$, therefore $3k^2 - 2eh^4 = 3a + eh^4 = k^2 + 2a$.

‡ Here $4eh = \frac{4ehk^4}{k^4}$, also $q = \frac{eh^2(k^2+2a)}{k^3}$, and $p = \frac{2eh^3}{k}$;

therefore $2pq = \frac{4e^2h^5(k^2+2a)}{k^4}$; and, consequently,

$$4eh - 2pq = \frac{4ehk^4 - 4e^2h^5(k^2+2a)}{k^4}. \quad \text{B.}$$

If, therefore, we substitute this value of x in the formula $a + ex^4$, it becomes a square; and its root, which we have supposed to be $k + py + qy^2$, will have this form,

$$k + \frac{8k(k^2-a)\times(2a-k^2)}{3k^4-4a^2} + \frac{16k(k^2-a)\times(k^2+2a)\times(2a-k^2)^2}{(3k^4-4a^2)^2};$$

because, as we have seen, $p = \frac{2eh^3}{k}$, $q = \frac{eh^2(k^2+2a)}{k^3}$,

$y = \frac{4hk^2(2a-k^2)}{3k^4-4a^2}$, and $eh^4 = k^2 - a$ *.

139. Let us continue the investigation of the formula $a + ex^4$; and, since the case $a + eh^4 = k^2$ is known, let us consider it as furnishing two different cases; because $x = +h$, and $x = -h$; for which reason we may transform our formula into another of the third class, in which the first term and the last are squares. This transformation is made by an artifice, which is often of great utility, and which consists in making $x = \frac{h(1+y)}{1-y}$: by which means the formula becomes

$$\frac{a(1-y)^4+eh^4(1+y)^4}{(1-y)^4}, \text{ or rather}$$

$$\frac{k^2+4(k^2-2a)y+6k^2y^2+4(k^2-2a)y^3+k^2y^4}{(1-y)^4}.$$

Now, let us suppose the root of this formula, according to the third case, to be $\frac{k+py-ky^2}{(1-y)^2}$; so that the numerator of our formula must be equal to the square

$$k^2 + 2kpy + p^2y^2 - 2k^2y^2 - 2kpy^3 + k^2y^4;$$

and, removing the second terms, by making

$4k^2 - 8a = 2kp$, or $p = \frac{2k^2-4a}{k}$; and dividing the

* Thus,

$$py = \frac{2eh^3}{k}\times\frac{4hk^2(2a-k^2)}{3k^4-4a^2} = \frac{8eh^4k(2a-k^2)}{3k^4-4a^2} = \frac{8k(k^2-a)\times(2a-k^2)}{3k^4-4a^2};$$

also, $qy^2 = \frac{eh^2(k^2+2a)}{k^3}\times\frac{16h^2k^4(2a-k^2)}{(3k^4-4a^2)^2} = \frac{16eh^4k(k^2+2a)\times(2a-k^2)^2}{(3k^4-4a^2)^2}$

$= \frac{16k(k^2-a)\times(k^2+2a)\times(2a-k^2)^2}{(3k^4-4a^2)^2}$, by substituting $eh^4 = k^2 - a$.

B.

other terms by y^2, we shall have

$$6k^2 + 4y(k^2 - 2a) = -2k^2 + p^2 - 2kpy, \text{ or}$$
$$y(4k^2 - 8a + 2kp) = p^2 - 8k^2; \text{ or}$$
$$p = \frac{2k^2 - 4a}{k}, \text{ and } pk = 2k^2 - 4a; \text{ so that}$$
$$y(8k^2 - 16a) = \frac{-4k^4 - 16ak^2 + 16a^2}{k^2}, \text{ and}$$
$$y = \frac{-k^4 - 4ak^2 + 4a^2}{k^2(2k^2 - 4a)}.$$

If we now wish to find x, we have, first,

$$1 + y = \frac{k^4 - 8ak^2 + 4a^2}{k^2(2k^2 - 4a)};$$

and, in the second place,

$$1 - y = \frac{3k^4 - 4a^2}{k^2(2k^2 - 4a)}; \text{ so that}$$
$$\frac{1+y}{1-y} = \frac{k^4 - 8ak^2 + 4a^2}{3k^4 - 4a^2}; \text{ and, consequently,}$$
$$x = \frac{h(k^4 - 8ak^2 + 4a^2)}{3k^4 - 4a^2};$$

but this is just the same value that we found before, with regard to the even powers of x.

140. In order to apply this result to an example, let it be required to make the formula $2x^4 - 1$ a square. Here, we have $a = -1$, and $e = 2$; and the known case when the formula becomes a square, is that in which $x = 1$; so that $h = 1$, and $k^2 = 1$; that is, $k = 1$; therefore, we shall have the new value, $x = \frac{1+8+4}{3-4} = -13$; and since the fourth power of x is found alone, we may also write $x = +13$, whence $2x^4 - 1 = 57121 = (239)^2$.

If we now consider this as the known case, we have $h = 13$ and $k = 239$; and shall obtain a new value of x, namely,

$$\frac{13 \times (239^4 + 8 \times 239^2 + 4)}{3 \times 239^4 - 4} = \frac{42422452969}{9788425919}.$$

141. We shall consider, in the same manner, a formula rather more general, $a + cx^2 + ex^4$, and shall take for the known case, in which it becomes a square, $x = h$; so that $a + ch^2 + eh^4 = k^2$.

And, in order to find other values from this, let us suppose $x = h + y$, and our formula will assume the following form:

$$\begin{array}{l} a \\ ch^2 + 2chy + cy^2 \\ eh^4 + 4eh^3y + 6eh^2y^2 + 4ehy^3 + ey^4 \\ \hline k^2 + (2ch + 4eh^3)y + (c + 6eh^2)y^2 + 4ehy^3 + ey^4. \end{array}$$

The first term being a square, we shall suppose the root of this formula to be $k + py + qy^2$; and the formula itself will necessarily be equal to the square

$$k^2 + 2kpy + p^2y^2 + 2kqy^2 + 2pqy^3 + q^2y^4;$$

then determining p and q, in order to expunge the second and third terms, we shall have for this purpose

$$2ch + 4eh^3 = 2kp; \text{ or } p = \frac{ch + 2eh^3}{k}; \text{ and}$$

$$c + 6eh^2 = 2kq + p^2; \text{ r } q = \frac{c + 6eh^2 - p^2}{2k}.$$

Now, the last two terms of the general equation being divisible by y^3, they are reduced to

$$4eh + ey = 2pq + q^2y;$$

which gives $y = \dfrac{4eh - 2pq}{q^2 - e}$, and, consequently, the value also of $x = h + y$. If we now consider this new case as the given one, we shall find another new case, and may proceed, in the same manner, as far as we please.

142. Let us illustrate the preceding article, by applying it to the formula $1 - x^2 + x^4$, in which $a = 1$, $c = -1$, and $e = 1$. The known case is evidently $x = 1$; and, therefore, $h = 1$, and $k = 1$. If we make $x = 1 + y$, and the square root of our formula $1 + py + qy^2$, we must first have $p = \dfrac{ch + 2eh^3}{k} = 1$, and then $q = \dfrac{c + 6eh^2 - p^2}{2k} = \frac{4}{2} = 2$. These values give $y = 0$, and $x = 1$. Now, this is the known case, and we have not arrived at a new one; but it is because we may prove, from other considerations, that the proposed formula can never become a square, except in the cases of $x = 0$, and $x = \pm 1$.

143. Let there be given, also, for an example, the formula $2 - 3x^2 + 2x^4$; in which $a = 2$, $c = -3$, and $e = 2$. The known case is readily found; that is, $x = 1$; so that $h = 1$, and $k = 1$: if, therefore, we make $x = 1 + y$, and the root $= 1 + py + qy^2$, we shall have $p = 1$, and

$q=4$; whence $y = 0$, and $x=1$; which, as before, leads to nothing new.

144. Again, let the formula be $1 + 8x^2 + x^4$; in which $a = 1$, $c = 8$, and $e = 1$. Here a slight consideration is sufficient to point out the satisfactory case, namely, $x = 2$; for, by supposing $h = 2$, we find $k = 7$; so that making $x = 2 + y$, and representing the root by $7 + py + qy^2$, we shall have $p = \frac{32}{7}$, and $q = \frac{272}{343}$; whence

$$y = -\tfrac{5880}{2911},\text{ and } x = -\tfrac{58}{2911};$$

and we may omit the sign *minus* in these values. But we may observe, farther, in this example, that, since the last term is already a square, and must therefore remain a square also in the new formula, we may here apply the method which has been already taught for cases of the third class. Therefore, as before, let $x = 2 + y$, and we shall have

$$\begin{array}{l} 1 \\ 32 + 32y + 8y^2 \\ 16 + 32y + 24y^2 + 8y^3 + y^4 \\ \hline 49 + 64y + 32y^2 + 8y^3 + y^4, \end{array}$$

an expression which we may now transform into a square in several ways. For, in the first place, we may suppose the root to be $7 + py + y^2$; and, consequently, the formula equal to the square

$$49 + 14py + p^2y^2 + 14y^2 + 2py^3 + y^4;$$

but then, after destroying $8y^3$, and $2py^3$, by supposing $2p = 8$, or $p = 4$, dividing the other terms by y, and deriving from the equation

$$64 + 32y = 14p + 14y + p^2y = 56 + 30y,$$

the value of $y = - 4$, and of $x = - 2$, or $x = + 2$, we come only to the case that is already known.

Farther, if we seek to determine such a value for p, that the second terms may vanish, we shall have $14p = 64$, and $p = \frac{32}{7}$; and the other terms, when divided by y^2, form the equation $14 + p^2 + 2py = 32 + 8y$, or $\frac{1710}{49} + \frac{64}{7}y = 32 + 8y$, whence we find $y = -\frac{71}{28}$; and, consequently, $x = -\frac{15}{28}$, or $x = +\frac{15}{28}$; and this value transforms our formula into a square, whose root is $\frac{1441}{784}$. Farther, as $-y^2$ is no less the root of the last term than $+y^2$, we may suppose the root of the formula to be $7 + py - y^2$, or the formula itself equal to $49 + 14py + p^2y^2 - 14y^2 - 2py^3 + y^4$. And here we shall destroy the last terms but one, by making $-2p = 8$, or $p= - 4$; then, dividing the other terms by y, we shall have

$$64 + 32y = 14p - 14p + p^2y = -56 + 2y,$$

which gives $y = -4$; that is, the known case again. If we chose to destroy the second terms, we should have $64 = 14p$, and $p = \frac{32}{7}$; and, consequently, dividing the other terms by y^2, we should obtain

$$32 + 8y = -14 + p^2 - 2py, \text{ or}$$
$$32 + 8y = \tfrac{338}{49} - \tfrac{64}{7}y; \text{ whence}$$
$$y = -\tfrac{71}{28}, \text{ and } x = -\tfrac{15}{28};$$

that is to say, the same values that we found before.

145. We may proceed, in the same manner, with respect to the general formula

$$a + bx + cx^2 + dx^3 + ex^4,$$

when we know one case, as $x = h$, in which it becomes a square, k^2. The constant method is to suppose $x = h + y$: from this, we obtain a formula of as many terms as the other, the first of them being k^2. If, after that, we express the root by $k + py + qy^2$; and determine p and q so, that the second and third terms may disappear; the last two, being divisible by y^3, will be reduced to a simple equation of the first degree, from which we may easily obtain the value of y, and, consequently, that of x also.

Still, however, we shall be obliged, as before, to exclude a great number of cases in the application of this method; those, for instance, in which the value found for x is no other than $x = h$, which was given, and in which, consequently, we could not advance one step. Such cases shew either that the formula is impossible in itself, or that we have yet to find some other case in which it becomes a square.

146. And this is the utmost length to which mathematicians have yet advanced, in the resolution of formulæ, that are affected by the sign of the square root. No discovery has hitherto been made for those, in which the quantities under the sign exceed the fourth degree; and when formulæ occur which contain the fifth, or a higher power of x, the artifices which we have explained are not sufficient to resolve them, even although a case be given.

That the truth of what is now said may be more evident, we shall consider the formula

$$k^2 + bx + cx^2 + dx^3 + ex^4 + fx^5,$$

the first term of which is already a square. If, as before, we suppose the root of this formula to be $k + px + qx^2$, and determine p and q, so as to make the second and third terms disappear, there will still remain three terms, which, when divided by x^3, form an equation of the second degree; and x evidently cannot be expressed, except by a new irrational quantity. But if we were to suppose the root to be $k + px + qx^2 + rx^3$, its square would rise to the sixth power; and, consequently, though we should even determine p, q, and r, so as to remove the second, third, and fourth terms, there would still remain the fourth, the fifth, and the sixth powers; and, dividing by x^4, we should again have an equation of the second degree, which we could not resolve without a radical sign. This seems to indicate that we have really exhausted the subject of transforming formulæ into squares: we may now, therefore, proceed to quantities affected by the sign of the cube root.

CHAP. X.

Of the Method *of rendering rational the irrational* Formula $\sqrt[3]{(a + bx + cx^2 + dx^3)}$.

147. It is here required to find such values of x, that the formula $a + bx + cx^2 + dx^3$ may become a cube, and that we may be able to extract its cube root. We see immediately that no such solution could be expected, if the formula exceeded the third degree; and we shall add, that if it were only of the second degree, that is to say, if the term dx^3 disappeared, the solution would not be easier. With regard to the case in which the last two terms disappear, and in which it would be required to reduce the formula $a + bx$ to a cube, it is evidently attended with no difficulty; for we have only to make $a + bx = p^3$, to find at once $x = \frac{p^3 - a}{b}$.

148. Before we proceed farther on this subject, we must again remark, that when neither the first nor the last term is a cube, we must not think of resolving the formula, unless we already know a case in which it becomes a cube, whether that case readily occurs, or whether we are obliged to find it out by trial.

So that we have three kinds of formulæ to consider. One is, when the first term is a cube; and as then the formula is expressed by $f^3 + bx + cx^2 + dx^3$, we imme-