CHAP. VIII.

Of the Method *of rendering the* Irrational Formula, $\sqrt{(a + bx + cx^2 + dx^3)}$ Rational.

112. We shall now proceed to a formula, in which x rises to the third power; after which we shall consider also the fourth power of x, although these two cases are treated in the same manner.

Let it be required, therefore, to transform into a square the formula $a + bx + cx^2 + dx^3$, and to find proper values of x for this purpose, expressed in rational numbers. As this investigation is attended with much greater difficulties than any of the preceding cases, more artifice is requisite to find even fractional values of x; and with such we must be satisfied, without pretending to find values in integer numbers.

It must here be previously remarked also, that a general solution cannot be given, as in the preceding cases; and that, instead of the number here employed leading to an infinite number of solutions, each operation will exhibit but one value of x.

113. As in considering the formula $a + bx + cx^2$, we observed an infinite number of cases, in which the solution becomes altogether impossible, we may readily imagine that this will be much oftener the case with respect to the present formula, which, besides, constantly requires that we already know, or have found, a solution. So that here we can only give rules for those cases, in which we set out from one known solution, in order to find a new one; by means of which, we may then find a third, and proceed, successively in the same manner, to others.

It does not, however, always happen, that, by means of a known solution, we can find another; on the contrary, there are many cases, in which only one solution can take place; and this circumstance is the more remarkable, as in the analyses which we have before made, a single solution led to an infinite number of other new ones.

114. We just now observed, that in order to render the transformation of the formula, $a + bx + cx^2 + dx^3$, into a square, a case must be presupposed, in which that solution is possible. Now, such a case is clearly perceived, when the

first term is itself a square already, and the formula may be expressed thus, $f^2 + bx + cx^2 + dx^3$; for it evidently becomes a square, if $x = 0$.

We shall therefore enter upon the subject, by considering this formula; and shall endeavour to see how, by setting out from the known case $x = 0$, we may arrive at some other value of x. For this purpose, we shall employ two different methods, which will be separately explained: in order to which, it will be proper to begin with particular cases.

115. Let, therefore, the formula $1 + 2x - x^2 + x^3$ be proposed, which ought to become a square. Here, as the first term is a square, we shall adopt for the root required such a quantity as will make the first two terms vanish. For which purpose, let $1 + x$ be the root, whose square is to be equal to our formula; and this will give $1 + 2x - x^2 + x^3 = 1 + 2x + x^2$, of which equation the first two terms destroy each other; so that we have $x^2 = - x^2 + x^3$, or $x^3 = 2x^2$, which, being divided by x^2, gives $x = 2$; so that the formula becomes $1 + 4 - 4 + 8 = 9$.

Likewise, in order to make a square of the formula, $4 + 6x - 5x^2 + 3x^3$, we shall first suppose its root to be $2 + nx$, and seek such a value of n as will make the first two terms disappear; hence,

$$4 + 6x - 5x^2 + 3x^3 = 4 + 4nx + n^2x^2;$$

therefore we must have $4n = 6$, and $n = \frac{3}{2}$; whence results the equation $-5x^2 + 3x^3 = n^2x^2 = \frac{9}{4}x^2$, or $3x^3 = \frac{29}{4}x^2$, which gives $x = \frac{29}{12}$; and this is the value which will make a square of the proposed formula, whose root will be

$$2 + \tfrac{3}{2}x = \tfrac{45}{8}.$$

116. The second method consists in giving the root three terms, as $f + gx + hx^2$, such, that the first three terms in the equation may vanish.

Let there be proposed, for example, the formula $1 - 4x + 6x^2 - 5x^3$, the root of which we shall suppose to be $1 - 2x + hx^2$, and we shall thus have

$$1 - 4x + 6x^2 - 5x^3 = 1 - 4x + 4x^2 - 4hx^3 + h^2x^4 + 2hx^2.$$

The first two terms, as we see, are immediately destroyed on both sides; and, in order to remove the third, we must make $2h + 4 = 6$; consequently, $h = 1$; by these means, and transposing $2hx^2 = 2x^2$, we obtain $- 5x^3 = - 4x^3 + x^4$, or $- 5 = - 4 + x$, so that $x = - 1$.

117. These two methods, therefore, may be employed, when the first term a is a square. The first is founded on expressing the root by two terms, as $f + px$, in which f is

the square root of the first term, and p is taken such, that the second term must likewise disappear; so that there remains only to compare p^2x^2 with the third and fourth term of the formula, namely $cx^2 + dx^3$; for then that equation, being divisible by x^2, gives a new value of x, which is $x = \frac{p^2 - c}{d}$.

In the second method, three terms are given to the root; that is to say, if the first term $a = f^2$, we express the root by $f + px + qx^2$; after which, p and q are determined such, that the first three terms of the formula may vanish, which is done in the following manner: since

$f^2 + bx + cx^2 + dx^3 = f^2 + 2pfx + 2fqx^2 + p^2x^2 + 2pqx^3 + q^2x^4$,

we must have $b = 2fp$; and, consequently, $p = \frac{b}{2f}$; farther, $c = 2fq + p^2$; or $q = \frac{c - p^2}{2f}$; after this, there remains the equation $dx^3 = 2pqx^3 + q^2x^4$; and, as it is divisible by x^3, we obtain from it $x = \frac{d - 2pq}{q^2}$.

118. It may frequently happen, however, even when $a = f^2$, that neither of these methods will give a new value of x; as will appear, by considering the formula $f^2 + dx^3$, in which the second and third terms are wanting.

For if, according to the first method, we suppose the root to be $f + px$, that is,

$$f^2 + dx^3 = f^2 + 2fpx + p^2x^2,$$

we shall have $2fp = 0$, and $p = 0$; so that $dx^3 = 0$; and therefore $x = 0$, which is not a new value of x.

If, according to the second method, we were to make the root $f + px + qx^2$, or

$f^2 + dx^3 = f^2 + 2fpx + p^2x^2 + 2fqx^2 + 2pqx^3 + q^2x^4$,

we should find $2fp = 0$, $p^2 + 2fq = 0$, and $q^2 = 0$; whence $dx^3 = 0$, and also $x = 0$.

119. In this case, we have no other expedient, than to endeavour to find such a value of x, as will make the formula a square; if we succeed, this value will then enable us to find new values, by means of our two methods: and this will apply even to the cases in which the first term is not a square.

If, for example, the formula $3 + x^3$ must become a square; as this takes place when $x = 1$, let $x = 1 + y$, and we shall thus have $4 + 3y + 3y^2 + y^3$, the first term of which is a square. If, therefore, we suppose, according to the first method, the root to be $2+py$, we shall have

$$4+3y+3y^2+y^3=4+4py+p^2y^2.$$

In order that the second term may disappear, we must make $4p=3$; and, consequently, $p=\frac{3}{4}$; whence $3+y=p^2$, and $y=p^2-3=\frac{9}{16}-\frac{48}{16}=\frac{-39}{16}$; therefore $x=\frac{-23}{16}$, which is a new value of x.

If, again, according to the second method, we represent the root by $2+py+qy^2$, we shall have

$$4+3y+3y^2+y^3=4+4py+4qy^2+p^2y^2+2pqy^3+q^2y^4,$$

from which the second term will be removed, by making $4p=3$, or $p=\frac{3}{4}$; and the fourth, by making $4q+p^2=3$, or $q=\frac{3-p^2}{4}=\frac{39}{64}$; so that $1=2pq+q^2y$; whence we obtain $y=\frac{1-2pq}{q^2}$, or $y=\frac{352}{1521}$; and, consequently,

$$x=\frac{1873}{1521}.$$

120. In general, if we have the formula

$$a+bx+cx^2+dx^3,$$

and know also that it becomes a square when $x=f$, or that $a+bf+cf^2+df^3=g^2$, we may make $x=f+y$, and shall hence obtain the following new formula:

$$\begin{array}{l} a \\ +bf+by \\ +cf^2+2cfy+cy^2 \\ +df^3+3df^2y+3dfy^2+dy^3 \\ \hline g^2+(b+2cf+3df^2)y+(c+3df)y^2+dy^3. \end{array}$$

In this formula, the first term is a square; so that the two methods above given may be applied with success, as they will furnish new values of y, and consequently of x also, since $x=f+y$.

121. But often, also, it is of no avail even to have found a value of x. This is the case with the formula $1+x^3$, which becomes a square when $x=2$. For if, in consequence of this, we make $x=2+y$, we shall get the formula $9+12y+6y^2+y^3$, which ought also to become a square.

Now, by the first rule, let the root be $3+py$, and we shall have $9+12y+6y^2+y^3=9+6py+p^2y^2$, in which we must have $6p=12$, and $p=2$; therefore $6+y=p^2=4$, and $y=-2$, which, since we made $x=2+y$, this gives $x=0$; that is to say, a value from which we can derive nothing more.

Let us also try the second method, and represent the root by $3 + py + qy^2$; this gives

$$9+12y+6y^2+y^3=9+6py+6qy^2+p^2y^2+2pqy^3+q^2y^4,$$

in which we must first have $6p = 12$, and $p = 2$; then $6q + p^2 = 6q + 4 = 6$, and $q = \frac{1}{3}$; farther,

$$1 = 2pq + q^2y = \tfrac{4}{3} + \tfrac{1}{9}y;$$

hence $y = -3$, and, consequently, $x = -1$, and $1 + x^3 = 0$; from which we can draw no further conclusion, because, if we wished to make $x = -1 + z$, we should find the formula, $3z - 3z^2 + z^3$, the first term of which vanishes; so that we cannot make use of either method.

We have therefore sufficient grounds to suppose, after what has been attempted, that the formula $1 + x^3$ can never become a square, except in these three cases; namely, when

1. $x = 0$, 2. $x = -1$, and 3. $x = 2$.

But of this we may satisfy ourselves from other reasons.

122. Let us consider, for the sake of practice, the formula $1 + 3x^3$, which becomes a square in the following cases; when

1. $x = 0$, 2. $x = -1$, 3. $x = 2$,

and let us see whether we shall arrive at other similar values.

Since $x = 1$ is one of the satisfactory values, let us suppose $x = 1 + y$, and we shall thus have

$$1 + 3x^3 = 4 + 9y + 9y^2 + 3y^3.$$

Now, let the root of this new formula be $2 + py$, so that $4 + 9y + 9y^2 + 3y^3 = 4 + 4py + p^2y^2$. We must have $9 = 4p$, and $p = \frac{9}{4}$, and the other terms will give $9 + 3y = p^2 = \frac{81}{16}$, and $y = -\frac{21}{16}$; consequently, $x = -\frac{5}{16}$, and $1+3x^3$ becomes a square, namely, $-\frac{3721}{4096}$, the root of which is $-\frac{61}{64}$, or $+\frac{61}{64}$: and, if we chose to proceed, by making $x = -\frac{5}{16} + z$, we should not fail to find new values.

Let us also apply the second method to the same formula, and suppose the root to be $2 + py + qy^2$; which supposition gives

$$4 + 9y + 9y^2 + 3y^3 = \left\{\begin{matrix} 4+4py+4qy^2+2pqy^3+q^2y^4; \\ +p^2y^2 \end{matrix}\right\}$$

therefore, we must have $4p = 9$, or $p = \frac{9}{4}$, and $4q + p^2 = 9 = 4q + \frac{81}{16}$, or $q = \frac{63}{64}$: and the other terms will give $3 = 2pq + q^2y = \frac{567}{128} + q^2y$, or $567 + 128q^2y = 384$, or $128q^2y = -183$; that is to say,

$$128 \times (\tfrac{63}{64})^2y = -183, \text{ or } \frac{63^2}{32}y = -183.$$

So that $y = -\frac{1952}{1323}$, and $x = -\frac{629}{1323}$; and these values will furnish new ones, by following the methods which have been pointed out.

123. It must be remarked, however, that if we gave ourselves the trouble of deducing new values from the two, which the known case of $x = 1$ has furnished, we should arrive at fractions extremely prolix; and we have reason to be surprised that the case, $x = 1$, has not rather led us to the other, $x = 2$, which is no less evident. This, indeed, is an imperfection of the present method, which is the only mode of proceeding hitherto known.

We may, in the same manner, set out from the case $x = 2$, in order to find other values. Let us, for this purpose, make $x = 2 + y$, and it will be required to make a square of the formula, $25 + 36y + 18y^2 + 3y^3$. Here, if we suppose its root, according to the first method, to be $5 + py$, we shall have

$$25 + 36y + 18y^2 + 3y^3 = 25 + 10py + p^2y^2;$$

and, consequently, $10p = 36$, or $p = \frac{18}{5}$: then expunging the terms which destroy each other, and dividing the others by y^2, there results $18 + 3y = p^2 = \frac{324}{25}$; consequently, $y = -\frac{42}{25}$, and $x = \frac{8}{25}$; whence it follows, that $1 + 3x^3$ is a square, whose root is $5 + py = -\frac{131}{125}$, or $+\frac{131}{125}$.

In the second method, it would be necessary to suppose the root $= 5 + py + qy^2$, and we should then have

$$25 + 36y + 18y^2 + 3y^3 = \left\{ \begin{array}{r} 25 + 10py + 10qy^2 + 2pqy^3 \\ + p^2y^2 + q^3yy^3; \end{array} \right\}$$

the second and third terms would disappear by making $10p = 36$, or $p = \frac{18}{5}$, and $10q + p^2 = 18$, or $10q = 18 - \frac{324}{25} = \frac{126}{25}$, or $q = \frac{63}{125}$; and then the other terms, divided by y^3, would give $2pq + q^2y = 3$, or $q^2y = 3 - 2pq = -\frac{393}{625}$; that is, $y = -\frac{3275}{1323}$, and $x = -\frac{629}{1323}$.

124. This calculation does not become less tedious and difficult, even in the cases where, setting out differently, we can give a general solution; as, for example, when the formula proposed is $1 - x - x^2 + x^3$, in which we may make, generally, $x = n^2 - 1$, by giving any value whatever to n: for, let $n = 2$; we have then $x = 3$, and the formula becomes $1 - 3 - 9 + 27 = 16$. Let $n = 3$, we have then $x = 8$, and the formula becomes $1 - 8 - 64 + 512 = 441$, and so on.

But it should be observed, that it is to a very peculiar circumstance we owe a solution so easy, and this circumstance is readily perceived by analysing our formula into factors; for we immediately see, that it is divisible by

$1 - x$, that the quotient will be $1 - x^2$, that this quotient is composed of the factors $(1 + x) \times (1 - x)$; and, lastly, that our formula,

$$1 - x - x^2 + x^3 = (1 - x) \times (1 + x) \times (1 - x) = (1 - x)^2 \times (1 + x).$$

Now, as it must be a □ [*square*], and as a □, when divisible by a □, gives a □ for the quotient*, we must also have $1 + x =$ □; and, conversely, if $1 + x$ be a □, it is certain that $(1 - x)^2 \times (1 + x)$ will be a square; we have therefore only to make $1 + x = n^2$, and we immediately obtain $x = n^2 - 1$.

If this circumstance had escaped us, it would have been difficult even to have determined only five or six values of x by the preceding methods.

125. Hence we conclude, that it is proper to resolve every formula proposed into factors, when it can be done; and we have already shewn how this is to be done, by making the given formula equal to 0, and then seeking the root of this equation; for each root, as $x = f$, will give a factor $f - x$; and this inquiry is so much the easier, as here we seek only rational roots, which are always divisors of the known term, or the term which does not contain x.

126. This circumstance takes place also in our general formula, $a + bx + cx^2 + dx^3$, when the first two terms disappear, and it is consequently the quantity $cx^2 + dx^3$ that must be a square; for it is evident, in this case, that by dividing by the square x^2, we must also have $c + dx$ a square; and we have therefore only to make $c + dx = n^2$, in order to have $x = \frac{n^2 - c}{d}$, a value which contains an infinite number of answers, and even all the possible answers.

127. In the application of the first of the two preceding methods, if we do not choose to determine the letter p, for the sake of removing the second term, we shall arrive at another irrational formula, which it will be required to make rational.

For example, let $f^2 + bx + cx^2 + dx^3$ be the formula proposed, and let its root $= f + px$. Here we shall have $f^2 + bx + cx^2 + dx^3 = f^2 + 2fpx + p^2x^2$, from which the first terms vanish; dividing, therefore by x, we obtain

* The mathematical student, who may wish to acquire an extensive knowledge of the many curious properties of numbers, is referred, once for all, to the second edition of Legendre's celebrated Essai sur la Theorie des Nombres; or to Mr. Barlow's Elementary Investigation of the same subject.

$b + cx + dx^2 = 2fp + p^2x^2$, an equation of the second degree, which gives

$$x = \frac{p^2 - c + \sqrt{(p^4 - 2cp^2 + 8dfp + c^2 - 4bd)}}{2d}.$$

So that the question is now reduced to finding such values of p, as will make the formula $p^4 - 2cp^2 + 8dfp + c^2 - 4bd$ become a square. But as it is the fourth power of the required number p which occurs here, this case belongs to the following chapter.

CHAP. IX.

Of the Method *of rendering* Rational *the incommensurable* Formula $\sqrt{(a + bx + cx^2 + dx^3 + ex^4)}$.

128. We are now come to formulæ, in which the indeterminate number, x, rises to the fourth power; and this must be the limit of our researches on quantities affected by the sign of the square root; since the subject has not yet been prosecuted far enough to enable us to transform into squares any formulæ, in which higher powers of x are found.

Our new formula furnishes three cases: the first, when the first term, a, is a square; the second, when the last term, ex^4, is a square; and the third, when both the first term and the last are squares. We shall consider each of these cases separately.

129. 1st. Resolution of the formula

$$\sqrt{(f^2 + bx + cx^2 + dx^3 + ex^4)}.$$

As the first term of this is a square, we might, by the first method, suppose the root to be $f + px$, and determine p in such a manner, that the first two terms would disappear, and the others be divisible by x^2; but we should not fail still to find x^2 in the equation, and the determination of x would depend on a new radical sign. We shall therefore have recourse to the second method; and represent the root by $f + px + qx^2$; and then determine p and q, so as to remove the first three terms, and then dividing by x^3, we shall arrive at a simple equation of the first degree, which will give x without any radical signs.