CHAP. IV.

On the Method *of rendering* Surd Quantities *of the form* $\surd(a + bx + cx^2)$ Rational.

38. It is required in the present case to determine the values which are to be adopted for x, in order that the formula $a + bx + cx^2$ may become a real square; and, consequently, that a rational root of it may be assigned. Now, the letters a, b, and c, represent given numbers; and the determination of the unknown quantity depends chiefly on the nature of these numbers; there being many cases in which the solution becomes impossible. But even when it is possible, we must content ourselves at first with being able to assign rational values for the letter x, without requiring those values also to be integer numbers; as this latter condition produces researches altogether peculiar.

39. We suppose here that the formula extends no farther than the second power of x; the higher dimensions require different methods, which will be explained in their proper places.

We shall observe first, that if the second power were not in the formula, and c were $= 0$, the problem would be attended with no difficulty; for if $\surd(a + bx)$ were the given formula, and it were required to determine x, so that $a + bx$ might be a square, we should only have to make $a + bx = y^2$, whence we should immediately obtain $x = \frac{y^2 - a}{b}$. Now, whatever number we substitute here for y, the value of x would always be such, that $a + bx$ would be a square, and consequently, $\surd(a + bx)$ would be a rational quantity.

40. We shall therefore begin with the formula $\surd(1 + x^2)$; that is to say, we are to find such values of x, that, by adding unity to their squares, the sums may likewise be squares; and as it is evident that those values of x cannot be integers, we must be satisfied with finding the fractions which express them.

41. If we supposed $1 + x^2 = y^2$, since $1 + x^2$ must be a square, we should have $x^2 = y^2 - 1$, and $x = \surd(y^2 - 1$; so that in order to find x we should have to seek numbers for y, whose squares, diminished by unity, would also leave squares; and, consequently, we should be led to a question as difficult as the former, without advancing a single step.

It is certain, however, that there are real fractions, which, being substituted for x, will make $1 + x^2$ a square; of which we may be satisfied from the following cases:

1. If $x = \frac{3}{4}$, we have $1 + x^2 = \frac{25}{16}$; and consequently $\sqrt{(1 + x^2)} = \frac{5}{4}$.

2. $1 + x^2$ becomes a square likewise, if $x = \frac{4}{3}$, which gives $\sqrt{(1 + x^2)} = \frac{5}{3}$.

3. If we make $x = \frac{5}{12}$, we obtain $1 + x^2 = \frac{169}{144}$, the square root of which is $\frac{13}{12}$.

But it is required to shew how to find these values of x, and even all possible numbers of this kind.

42. There are two methods of doing this. The first requires us to make $\sqrt{(1 + x^2)} = x + p$; from which supposition we have $1 + x^2 = x^2 + 2px + p^2$, where the square x^2 destroys itself; so that we may express x without a radical sign. For, cancelling x^2 on both sides of the equation, we obtain $2px + p^2 = 1$; whence we find $x = \frac{1 - p^2}{2p}$; a quantity in which we may substitute for p any number whatever less than unity.

Let us therefore suppose $p = \frac{m}{n}$, and we have

$x = \frac{1 - \frac{m^2}{n^2}}{\frac{2m}{n}}$, and if we multiply both terms of this fraction by n^2, we shall find $x = \frac{n^2 - m^2}{2mn}$.

43. In order, therefore, that $1 + x^2$ may become a square, we may take for m and n all possible integer numbers, and in this manner find an infinite number of values for x.

Also, if we make, in general, $x = \frac{n^2 - m^2}{2mn}$, we find, by squaring, $1 + x^2 = 1 + \frac{n^4 - 2m^2n^2 + m^4}{4m^2n^2}$, or, by putting $1 = \frac{4m^2}{4m^2}$ in the numerator, $1 + x^2 = \frac{n^4 + 2m^2n^2 + m^4}{4m^2n^2}$; a fraction which is really a square, and gives

$$\sqrt{(1 + x^2)} = \frac{n^2 + m^2}{2mn}.$$

We shall exhibit, according to this solution, some of the least values of x.

If $n = 2, \ 3, \ 3, \ 4, \ 4, \ 5, \ 5, \ 5, \ 5,$
and $m = 1, \ 1, \ 2, \ 1, \ 3, \ 1, \ 2, \ 3, \ 4,$
We have $x = \frac{3}{4}, \ \frac{4}{3}, \ \frac{5}{12}, \ \frac{15}{8}, \ \frac{7}{24}, \ \frac{12}{5}, \ \frac{21}{20}, \ \frac{8}{15}, \ \frac{9}{40}.$

44. We have, therefore, in general,

$$1 + \frac{(n^2-m^2)^2}{(2mn)^2} = \frac{(n^2+m^2)^2}{(2mn)^2};$$

and, if we multiply this equation by $(2mn)^2$, we find

$$(2mn)^2 + (n^2 - m^2)^2 = (n^2 + m^2)^2;$$

so that we know, in a general manner, two squares, whose sum gives a new square. This remark will lead to the solution of the following question:

To find two square numbers, whose sum is likewise a square number.

We must have $p^2 + q^2 = r^2$; we have therefore only to make $p = 2mn$, and $q = n^2 - m^2$, then we shall have $r = n^2 + m^2$.

Farther, as $(n^2 + m^2)^2 - (2mn)^2 = (n^2 - m^2)^2$, we may also resolve the following question:

To find two squares, whose difference may also be a square number.

Here, since $p^2 - q^2 = r^2$, we have only to suppose $p = n^2 + m^2$, and $q = 2mn$, and we obtain $r = n^2 - m^2$. We might also make $p = n^2 + m^2$, and $q = n^2 - m^2$, from which we should find $r = 2mn$.

45. We spoke of two methods of giving the form of a square to the formula $1 + x^2$. The other is as follows:

If we suppose $\sqrt{(1+x^2)} = 1 + \frac{mx}{n}$, we shall have

$1 + x^2 = 1 + \frac{2mx}{n} + \frac{m^2x^2}{n^2}$; subtracting 1 from both sides, we have $x^2 = \frac{2mx}{n} + \frac{m^2x^2}{n^2}$. This equation may be divided by x, so that we have $x = \frac{2m}{n} + \frac{m^2x}{n^2}$, or $n^2x = 2mn + m^2x$, whence we find $x = \frac{2mn}{n^2 - m^2}$. Having found this value of x, we have

$1 + x^2 = 1 + \frac{4m^2n^2}{n^4 - 2m^2n^2 + m^4} = \frac{n^4 + 2m^2n^2 + m^4}{n^4 - 2m^2n^2 + m^4}$, which is the square of $\frac{n^2 + m^2}{n^2 - m^2}$. Now, as we obtain from that, the equation $1 + \frac{(2mn)^2}{(n^2-m^2)^2} = \frac{(n^2+m^2)^2}{(n^2-m^2)^2}$, we shall have, as before, $(n^2-m^2)^2 + (2mn)^2 = (n^2+m^2)^2$;

that is, the same two squares, whose sum is also a square.

46. The case which we have just analysed furnishes two methods of transforming the general formula $a + bx + cx^2$ into a square. The first of these methods applies to all the cases in which c is a square; and the second to those in which a is a square. We shall consider both these suppositions.

First, let us suppose that c is a square, or that the given formula is $a + bx + f^2x^2$. Since this must be a square, we shall make $\sqrt{(a + bx + f^2x^2)} = fx + \frac{m}{n}$, and shall thus have $a + bx + f^2x^2 = f^2x^2 + \frac{2mfx}{n} + \frac{m^2}{n^2}$, in which the terms containing x^2 destroy each other, so that $a + bx = \frac{2mfx}{n} + \frac{m^2}{n^2}$. If we multiply by n^2, we obtain $n^2a + n^2bx = 2mnfx + m^2$; hence we find $x = \frac{m^2 - n^2a}{n^2b - 2mnf}$; and, substituting this value for x, we shall have

$$\sqrt{(a + bx + f^2x^2)} = \frac{m^2f - n^2af}{n^2b - 2mnf} + \frac{m}{n} = \frac{mnb - m^2f - n^2af}{n^2b - 2mnf}.$$

47. As we have got a fraction for x, namely, $\frac{m^2 - n^2a}{n^2b - 2mnf}$, let us make $x = \frac{p}{q}$, then $p = m^2 - n^2a$, and $q = n^2b - 2mnf$; so that the formula $a + \frac{bp}{q} + \frac{f^2p^2}{q^2}$ is a square; and as it continues a square, though it be multiplied by the square q^2, it follows, that the formula $aq^2 + bpq + f^2p^2$ is also a square, if we suppose $p = m^2 - n^2a$, and $q = n^2b - 2mnf$. Hence it is evident, that an infinite number of answers, in integer numbers, may result from this expression, because the values of the letters m and n are arbitrary.

48. The second case which we have to consider, is that in which a, or the first term, is a square. Let there be proposed, for example, the formula $f^2 + bx + cx^2$, which it is required to make a square. Here, let us suppose

$\sqrt{(f^2+bx+cx^2)}=f+\frac{mx}{n}$, and we shall have

$f^2+bx+cx^2=f^2+\frac{2fmx}{n}+\frac{m^2x^2}{n^2}$, in which equation the terms f^2 destroying each other, we may divide the remaining terms by x, so that we obtain

$$b+cx=\frac{2mf}{n}+\frac{m^2x}{n^2}, \text{ or}$$

$$n^2b+n^2cx=2mnf+m^2x, \text{ or}$$

$$x(n^2c-m^2)=2mnf-n^2b; \text{ or, lastly,}$$

$$x=\frac{2mnf-n^2b}{n^2c-m^2}.$$

If we now substitute this value instead of x, we have

$\sqrt{(f^2+bx+cx^2)}=f+\frac{2m^2f-mnb}{n^2c-m^2}=\frac{n^2cf+m^2f-mnb}{n^2c-m^2}$;

and making $x=\frac{p}{q}$, we may, in the same manner as before, transform the expression $f^2q^2+bpq+cp^2$, into a square, by making $p=2mnf-n^2b$, and $q=n^2c-m^2$.

49. Here we have chiefly to distinguish the case in which $a=0$, that is to say, in which it is required to make a square of the formula $bx+cx^2$; for we have only to suppose $\sqrt{(bx+cx^2)}=\frac{mx}{n}$, from which we have the equation $bx+cx^2=\frac{m^2x^2}{n^2}$; which, divided by x, and multiplied by n^2, gives $bn^2+cn^2x=m^2x$; and, consequently,

$$x=\frac{bn^2}{m^2-cn^2}.$$

If we seek, for example, all the triangular numbers that are at the same time squares, it will be necessary that $\frac{x^2+x}{2}$, which is the form of triangular numbers, must be a square; and, consequently, $2x^2+2x$ must also be a square. Let us, therefore, suppose $\frac{m^2x^2}{n^2}$ to be that square, and we shall have $2n^2x+2n^2=m^2x$, and $x=\frac{2n^2}{m^2-2n^2}$; in which value we may substitute, instead of m and n, all possible numbers; but we shall generally find a fraction for x, though sometimes we may obtain an integer number. For example, if $m = 3$, and $n = 2$, we find $x = 8$, the triangular number of which, or 36, is also a square.

We may also make $m = 7$, and $n = 5$; in this case, $x = -50$, the triangle of which, 1225, is at the same time the triangle of $+49$, and the square of 35. We should have obtained the same result by making $n = 7$ and $m = 10$; for, in that case, we should also have found $x = 49$.

In the same manner, if $m = 17$ and $n = 12$, we obtain $x = 288$, its triangular number is

$$\frac{x(x+1)}{2} = \frac{288 \times 289}{2} = 144 \times 289,$$

which is a square, whose root is $12 \times 17 = 204$.

50. We may remark, with regard to this last case, that we have been able to transform the formula $bx + cx^2$ into a square from its having a known factor, x; this observation leads to other cases, in which the formula $a + bx + cx^2$ may likewise become a square, even when neither a nor c are squares.

These cases occur when $a + bx + cx^2$ may be resolved into two factors; and this happens when $b^2 - 4ac$ is a square: to prove which, we may remark, that the factors depend always on the roots of an equation; and that, therefore, we must suppose $a + bx + cx^2 = 0$. This being laid down, we have $cx^2 = -bx - a$, or

$$x^2 = -\frac{bx}{c} - \frac{a}{c}, \text{ whence we find}$$

$$x = -\frac{b}{2c} \pm \sqrt{\left(\frac{b^2}{4c^2} - \frac{a}{c}\right)}, \text{ or } x = -\frac{b}{2c} \pm \frac{\sqrt{(b^2 - 4ac)}}{2c},$$

and, it is evident, that if $b^2 - 4ac$ be a square, this quantity becomes rational.

Therefore let $b^2 - 4ac = d^2$; then the roots will be $\frac{-b \pm d}{2c}$, that is to say, $x = \frac{-b \pm d}{2c}$; and, consequently, the divisors of the formula $a + bx + cx^2$ are $x + \frac{b - d}{2c}$, and $x + \frac{b + d}{2c}$. If we multiply these factors together, we are brought to the same formula again, except that it is divided by c; for the product is $x^2 + \frac{bx}{c} + \frac{b^2}{4c^2} - \frac{d^2}{4c^2}$; and since $d^2 = b^2 - 4ac$, we have

$x^2 + \frac{bx}{c} + \frac{b^2}{4c^2} - \frac{b^2}{4c^2} + \frac{4ac}{4c^2} = x^2 + \frac{bx}{c} + \frac{a}{c}$; which being multiplied by c, gives $cx^2 + bx + a$. We have, therefore, only to multiply one of the factors by c, and we obtain the formula in question expressed by the product,

$$(cx + \frac{b}{2} - \frac{d}{2}) \times (x + \frac{b}{2c} + \frac{d}{2c});$$

and it is evident that this solution must be applicable whenever $b^2 - 4ac$ is a square.

51. From this results the third case, in which the formula $a + bx + cx^2$ may be transformed into a square; which we shall add to the other two.

52. This case, as we have already observed, takes place, when the formula may be represented by a product, such as $(f + gx) \times (h + kx)$. Now, in order to make a square of this quantity, let us suppose its root, or

$\sqrt{(f + gx) \times (h + kx)} = \frac{m(f+gx)}{n}$; and we shall then have $(f + gx) \times (h + kx) = \frac{m^2(f+gx)^2}{n^2}$; and, dividing this equation by $f + gx$, we have $h + kx = \frac{m^2(f+gx)}{n^2}$; or

$$hn^2 + kn^2x = fm^2 + gm^2x;$$

and, consequently, $x = \frac{fm^2 - hn^2}{kn^2 - gm^2}$.

To illustrate this, let the following questions be proposed.

Question 1. To find all the numbers, x, such, that if 2 be subtracted from twice their square, the remainder may be a square.

Since $2x^2 - 2$ is the quantity which is to be a square, we must observe, that this quantity is expressed by the factors, $2 \times (x + 1) \times (x - 1)$. If, therefore, we suppose its root $= \frac{m(x+1)}{n}$, we have $2(x + 1) \times (x - 1) = \frac{m^2(x + 1)^2}{n^2}$; dividing by $x + 1$, and multiplying by n^2, we obtain

$$2n^2x - 2n^2 = m^2x + m^2, \text{ and } x = \frac{m^2 + 2n^2}{2n^2 - m^2}.$$

If, therefore, we make $m = 1$, and $n = 1$, we find $x = 3$, and $2x^2 - 2 = 16 = 4^2$.

If $m = 3$ and $n = 2$, we have $x = -17$. Now, as x is only found in the second power, it is indifferent whether we take $x = -17$, or $x = +17$; either supposition equally gives $2x^2 - 2 = 576 = 24^2$.

53. *Question* 2. Let the formula $6 + 13x + 6x^2$ be proposed to be transformed into a square. Here, we have $a = 6$, $b = 13$, and $c = 6$, in which neither a nor c is a square. If, therefore, we try whether $b^2 - 4ac$ becomes a square, we obtain 25; so that we are sure the formula may be represented by two factors; and those factors are

$$(2 + 3x) \times (3 + 2x).$$

If $\frac{m(2+3x)}{n}$ is their root, we have

$$(2 + 3x) \times (3 + 2x) = \frac{m^2(2+3x)^2}{n^2},$$

which becomes $3n^2 + 2n^2x = 2m^2 + 3m^2x$, whence we find $x = \frac{2m^2-3n^2}{2n^2-3m^2} = \frac{3n^2-2m^2}{3m^2-2n^2}$. Now, in order that the numerator of this fraction may become positive, $3n^2$ must be greater than $2m^2$; and, consequently, $2m^2$ less than $3n^2$: that is to say, $\frac{m^2}{n^2}$ must be less than $\frac{3}{2}$. With regard to the denominator, if it must be positive, it is evident that $3m^2$ must exceed $2n^2$; and, consequently, $\frac{m^2}{n^2}$ must be greater than $\frac{2}{3}$. If, therefore, we would have the positive values of x, we must assume such numbers for m and n, that $\frac{m^2}{n^2}$ may be less than $\frac{3}{2}$, and yet greater than $\frac{2}{3}$.

For example, let $m = 6$, and $n = 5$; we shall then have $\frac{m^2}{n^2} = \frac{36}{25}$, which is less than $\frac{3}{2}$, and evidently greater than $\frac{2}{3}$, whence $x = \frac{3}{58}$.

54. This third case leads us to consider also a fourth, which occurs whenever the formula $a + bx + cx^2$ may be resolved into two such parts, that the first is a square, and the second the product of two factors: that is to say, in this case, the formula must be represented by a quantity of the form $p^2 + qr$, in which the letters p, q, and r express quantities of the form $f + gx$. It is evident that the rule for this

case will be to make $\sqrt{(p^2+qr)} = p + \frac{mq}{n}$; for we shall thus obtain $p^2 + qr = p^2 + \frac{2mpq}{n} + \frac{m^2q^2}{n^2}$, in which the terms p^2 vanish; after which we may divide by q, so that we find $r = \frac{2mp}{n} + \frac{m^2q}{n^2}$, or $n^2r = 2nmp + m^2q$, an equation from which x is easily determined. This, therefore, is the fourth case in which our formula may be transformed into a square; the application of which is easy, and we shall illustrate it by a few examples.

55. *Question* 3. Required a number, x, such, that double its square, shall exceed some other square by unity; that is, if we subtract unity from this double square, the remainder may be a square.

For instance, the case applies to the number 5, whose square 25, taken twice, gives the number 50, which is greater by 1 than the square 49.

According to this enunciation, $2x^2 - 1$ must be a square; and as we have, by the formula, $a = -1$, $b = 0$, and $c = 2$, it is evident that neither a nor c is a square; and farther, that the given quantity cannot be resolved into two factors, since $b^2 - 4ac = 8$ which is not a square; so that none of the first three cases will apply. But, according to the fourth, this formula may be represented by

$$x^2 + (x^2 - 1) = x^2 + (x - 1) \times (x + 1).$$

If, therefore, we suppose its root $= x + \frac{m(x+1)}{n}$, we shall have

$$x^2 + (x + 1) \times (x - 1) = x^2 + \frac{2mx(x+1)}{n} + \frac{m^2(x+1)^2}{n^2}.$$

This equation, after having expunged the terms x^2, and divided the other terms by $x + 1$, gives

$$n^2x - n^2 = 2mnx + m^2x + m^2;\ \text{whence we find}$$

$x = \frac{m^2+n^2}{n^2-2mn-m^2}$; and, since in our formula, $2x^2 - 1$, the square x^2 alone is found, it is indifferent whether we take positive or negative values for x. We may at first even write $-m$, instead of $+m$, in order to have

$$x = \frac{m^2+n^2}{n^2+2mn-m^2}.$$

If we make $m = 1$, and $n = 1$, we find $x = 1$, and $2x^2 - 1 = 1$; or if we make $m = 1$, and $n = 2$, we find $x = \frac{5}{7}$, and $2x^2 - 1 = \frac{1}{49}$; lastly, if we suppose $m = 1$, and $n = -2$, we find $x = -5$, or $x = +5$, and $2x^2 - 1 = 49$.

56. *Question* 4. To find numbers whose squares doubled and increased by 2, may likewise be squares.

Such a number, for instance, is 7, since the double of its square is 98, and if we add 2 to it, we have the square 100.

We must, therefore, have $2x^2 + 2$ a square; and as $a = 2$, $b = 0$, and $c = 2$, so that neither a nor c, nor $b^2 - 4ac$, the last being $= -16$, are squares, we must, therefore, have recourse to the fourth rule.

Let us suppose the first part to be 4, then the second will be $2x^2 - 2 = 2(x + 1) \times (x - 1)$, which presents the quantity proposed in the form

$$4 + (x + 1) \times (x - 1).$$

Now, let $2 + \frac{m(x+1)}{n}$ be its root, and we shall have the equation

$$4 + 2(x + 1) \times (x - 1) = 4 + \frac{4m(x+1)}{n} + \frac{m^2(x+1)^2}{n^2},$$

in which the squares 4, are destroyed; so that after having divided the other terms by $x + 1$, we have

$2n^2x - 2n^2 = 4mn + m^2x + m^2$; and consequently,

$$x = \frac{4mn+m^2+2n^2}{2n^2-m^2}.$$

If, in this value, we make $m = 1$, and $n = 1$, we find $x = 7$, and $2x^2 + 2 = 100$. But if $m = 0$, and $n = 1$, we have $x = 1$, and $2x^2 + 2 = 4$.

57. It frequently happens, also, when none of the first three rules applies, that we are still able to resolve the formula into such parts as the fourth rule requires, though not so readily as in the foregoing examples.

Thus, if the question comprises the formula $7 + 15x + 13x^2$, the resolution we speak of is possible, but the method of performing it does not readily occur to the mind. It requires us to suppose the first part to be $(1 - x)^2$ or $1 - 2x + x^2$, so that the other may be $6 + 17x + 12x^2$: and we perceive that this part has two factors, because $17^2 - (4 \times 6 \times 12)$, $= 1$, is a square. The two factors therefore are $(2 + 3x) \times (3 + 4x)$; so that the formula becomes $(1 - x)^2 + (2 + 3x) \times (3 + 4x)$, which we may now resolve by the fourth rule.

But, as we have observed, it cannot be said that this analysis is easily found; and, on this account, we shall explain a general method for discovering, beforehand, whether the resolution of any such formula be possible or not; for there is an infinite number of them which cannot be resolved at all: such, for instance, as the formula $3x^2 + 2$, which can in no case whatever become a square. On the other hand, it is sufficient to know a single case, in which a formula is possible, to enable us to find all its answers; and this we shall explain at some length.

58. From what has been said, it may be observed, that all the advantage that can be expected on these occasions, is to determine, or suppose, any case in which such a formula as $a + bx + cx^2$, may be transformed into a square; and the method which naturally occurs for this, is to substitute small numbers successively for x, until we meet with a case which gives a square.

Now, as x may be a fraction, let us begin with substituting for x the general fraction $\frac{t}{u}$; and, if the formula $a + \frac{bt}{u} + \frac{ct^2}{u^2}$ which results from it, be a square, it will be so also after having been multiplied by u^2; so that it only remains to try to find such integer values for t and u, as will make the formula $au^2 + btu + ct^2$ a square; and it is evident, that after this, the supposition of $x = \frac{t}{u}$ cannot fail to give the formula $a + bx + cx^2$ equal to a square.

But if, whatever we do, we cannot arrive at any satisfactory case, we have every reason to suppose that it is altogether impossible to transform the formula into a square; which, as we have already said, very frequently happens.

59. We shall now shew, on the other hand, that when one satisfactory case is determined, it will be easy to find all the other cases which likewise give a square; and it will be perceived, at the same time, that the number of those solutions is always infinitely great.

Let us first consider the formula $2 + 7x^2$, in which $a = 2$, $b = 0$, and $c = 7$. This evidently becomes a square, if we suppose $x = 1$; let us therefore make $x = 1 + y$, and, by substitution, we shall have $x^2 = 1 + 2y + y^2$, and our formula becomes $9 + 14y + 7y^2$, in which the first term is a square; so that we shall suppose, conformably to the second rule, the square root of the new formula to be

$3 + \frac{my}{n}$, and we shall thus obtain the equation

$9 + 14y + 7y^2 = 9 + \frac{6my}{n} + \frac{m^2y^2}{n^2}$, in which we may expunge 9 from both sides, and divide by y: which being done, we shall have $14n^2 + 7n^2y = 6mn + m^2y$; whence

$$y = \frac{6mn - 14n^2}{7n^2 - m^2}; \text{ and, consequently,}$$

$$x = \frac{6mn - 7n^2 - m^2}{7n^2 - m^2},$$ in which we may substitute any values we please for m and n.

If we make $m = 1$, and $n = 1$, we have $x = -\frac{1}{3}$; or, since the second power of x stands alone, $x = +\frac{1}{3}$, wherefore $2 + 7x^2 = \frac{25}{9}$.

If $m = 3$, and $n = 1$, we have $x = -1$, or $x = +1$.

But if $m = 3$, and $n = -1$, we have $x = 17$; which gives $2 + 7x^2 = 2025$, the square of 45.

If $m = 8$, and $n = 3$, we shall then have, in the same manner, $x = -17$, or $x = +17$.

But, by making $m = 8$, and $n = -3$, we find $x = 271$; so that $2 + 7x^2 = 514089 = 717^2$.

60. Let us now examine the formula $5x^2 + 3x + 7$, which becomes a square by the supposition of $x = -1$. Here, if we make $x = y - 1$, our formula will be changed into this:

$$\begin{array}{r} 5y^2 - 10y + 5 \\ + \;\; 3y - 3 \\ + 7 \\ \hline 5y^2 - \;\; 7y + 9, \end{array}$$

the square root of which we shall suppose to be $3 - \frac{my}{n}$; by which means we shall have

$$5y^2 - 7y + 9 = 9 - \frac{6my}{n} + \frac{m^2y^2}{n^2}, \text{ or}$$

$5n^2y - 7n^2 = -6mn + m^2y$; whence we deduce

$$y = \frac{7n^2 - 6mn}{5n^2 - m^2}; \text{ and, lastly, } x = \frac{2n^2 - 6mn + m^2}{5n^2 - m^2}.$$

If $m = 2$, and $n = 1$, we have $x = -6$, and consequently $5x^2 + 3x + 7 = 169 = 13^2$.

But if $m = -2$ and $n = 1$, we find $x = 18$, and $5x^2 + 3x + 7 = 1681 = 41^2$.

61. Let us now consider the formula, $7x^2 + 15x + 13$, in which we must begin with the supposition of $x = \frac{t}{u}$. Having substituted and multiplied u^2, we obtain $7t^2 + 15tu + 13u^2$, which must be a square. Let us therefore try to adopt some small numbers as the values of t and u.

$$\left.\begin{array}{l} \text{If } t = 1, \text{ and } u = 1, \\ t = 2, \text{ and } u = 1, \\ t = 2, \text{ and } u = -1, \\ t = 3, \text{ and } u = 1, \end{array}\right\} \text{the formula will become} \left\{\begin{array}{l} = 35 \\ = 71 \\ = 11 \\ = 121. \end{array}\right.$$

Now, 121 being a square, it is proof that the value of $x = 3$ answers the required condition; let us therefore suppose $x = y + 3$, and we shall have, by substituting this value in the formula,

$$7y^2 + 42y + 63 + 15y + 45 + 13, \text{ or}$$
$$7y^2 + 57y + 121.$$

Therefore let the root be represented by $11 + \frac{my}{n}$, and we shall have $7y^2 + 57y + 121 = 121 + \frac{22my}{n} + \frac{m^2y^2}{n^2}$, or $7n^2y + 57n^2 = 22mn + m^2y$; whence

$$y = \frac{57n^2 - 22mn}{m^2 - 7n^2}, \text{ and } x = \frac{36n^2 - 22mn + 3m^2}{m^2 - 7n^2}.$$

Suppose, for example, $m = 3$, and $n = 1$; we shall then find $x = -\frac{3}{2}$, and the formula becomes

$$7x^2 + 15x + 13 = \tfrac{25}{4} = (\tfrac{5}{2})^2.$$

If $m = 1$, and $n = 1$, we find $x = -\frac{17}{6}$; if $m = 3$, and $n = -1$, we have $x = \frac{129}{2}$, and the formula

$$7x^2 + 15x + 13 = \tfrac{120409}{4} = (\tfrac{347}{2})^2.$$

62. But frequently it is only lost labor to endeavour to find a case, in which the proposed formula may become a square. We have already said that $3x^2 + 2$ is one of those unmanageable formulæ; and, by giving it, according to this rule, the form $3t^2 + 2u^2$, we shall perceive that, whatever values we give to t and u, this quantity never becomes a square number. As the formulæ of this kind are very

numerous, it will be worth while to fix on some characters, by which their impossibility may be perceived, in order that we may be often saved the trouble of useless trials; which shall form the subject of the following chapter*.

CHAP. V.

Of the Cases *in which the* Formula $a + bx + cx^2$ *can never become a* Square.

63. As our general formula is composed of three terms, we shall observe, in the first place, that it may always be transformed into another, in which the middle term is wanting. This is done by supposing $x = \frac{y-b}{2c}$; which substitution changes the formula into
$a + \frac{by - b^2}{2c} + \frac{y^2 - 2by + b^2}{4c}$; or $\frac{4ac - b^2 + y^2}{4c}$; and since this must be a square, let us make it equal to $\frac{z^2}{4}$, we shall then have $4ac - b^2 + y^2 = \frac{4cz^2}{4}, = cz^2$, and, consequently, $y^2 = cz^2 + b^2 - 4ac$. Whenever, therefore, our formula is a square, this last $cz^2 + b^2 - 4ac$ will be so likewise; and reciprocally, if this be a square, the proposed formula will be a square also. If therefore we write t, instead of $b^2 - 4ac$, the whole will be reduced to determining whether a quantity of the form $cz^2 + t$ can become a square or not. And as this formula consists only of two terms, it is certainly much easier to judge from that whether it be possible or not; but in any further inquiry we must be guided by the nature of the given numbers c and t.

64. It is evident that if $t = 0$, the formula cz^2 can become a square only when c is a square; for the quotient arising from the division of a square by another square being likewise a square, the quantity cz^2 cannot be a square, unless

* See the Appendix to this chapter, at Article 5. of the Additions by De la Grange. p. 537.