we obtain $y = 124 - \frac{14z}{5}$: so that z must be divisible by 5. If therefore we make $z = 5u$, we shall have $y = 124 - 14u$; which values of y and z being substituted in the first equation, we have $3x - 35u + 620 = 560$; or $3x = 35u - 60$, and $x = \frac{35u}{3} - 20$; therefore we shall make $u = 3t$, from which we obtain the following answer, $x = 35t - 20$, $y = 124 - 42t$, and $z = 15t$, in which we must substitute for t an integer number greater than 0 and less than 3: so that we are limited to the two following answers:

$$\text{If} \left\{ \begin{array}{l} t = 1, \\ t = 2, \end{array} \right. \text{we have} \left\{ \begin{array}{l} x = 15, y = 82, z = 15. \\ x = 50, y = 40, z = 30. \end{array} \right.$$

CHAP. III.

Of Compound Indeterminate Equations, *in which one of the* Unknown Quantities *does not exceed the* First Degree.

31. We shall now proceed to indeterminate equations, in which it is required to find two unknown quantities, one of them being multiplied by the other, or raised to a power higher than the first, whilst the other is found only in the first degree. It is evident that equations of this kind may be represented by the following general expression:

$$a + bx + cy + dx^2 + exy + fx^3 + gx^2y + hx^4 + kx^3y + , \&c. = 0.$$

As in this equation y does not exceed the first degree, that letter is easily determined; but here, as before, the values both of x and of y must be assigned in integer numbers.

We shall consider some of those cases, beginning with the easiest.

32. *Question* 1. To find two such numbers, that their product added to their sum may be 79.

Call the numbers sought x and y: then we must have $xy + x + y = 79$; so that $xy + y = 79 - x$, and

$y = \frac{79 - x}{x + 1} = \frac{79}{x+1} + \frac{-x}{x+1} = -1 + \frac{80}{x+1}$, from which

we see that $x + 1$ must be a divisor of 80. Now, 80 having several divisors, we shall also have several values of x, as the following Table will shew:

The divisors of 80 are	1	2	4	5	8	10	16	20	40	80
therefore $x =$	0	1	3	4	7	9	15	19	39	79
and $y =$	79	39	19	15	9	7	4	3	1	0

But as the answers in the bottom line are the same as those in the first, inverted, we have, in reality, only the five following; viz.

$$x = 0,\ 1,\ 3,\ 4,\ 7, \text{ and}$$
$$y = 79,\ 39,\ 19,\ 15,\ 9.$$

33. In the same manner, we may also resolve the general equation $xy + ax + by = c$; for we shall have $xy + by = c - ax$, and $y = \frac{c - ax}{x + b}$, or $y = \frac{ab + c}{x + b} - a$; that is to say, $x + b$ must be a divisor of the known number $ab + c$; so that each divisor of this number gives a value of x. If we therefore make $ab + c = fg$, we have

$y = \frac{fg}{x + b} - a$; and supposing $x + b = f$, or $x = f - b$, it is evident, that $y = g - a$; and, consequently, that we have also two answers for every method of representing the number $ab + c$ by a product, such as fg. Of these two answers, one is $x = f - b$, and $y = g - a$, and the other is obtained by making $x + b = g$, in which case $x = g - b$, and $y = f - a$.

If, therefore, the equation $xy + 2x + 3y = 42$ were proposed, we should have $a = 2$, $b = 3$, and $c = 42$; consequently, $y = \frac{48}{x + 3} - 2$. Now, the number 48 may be represented in several ways by two factors, as fg: and in each of those cases we shall always have either $x = f - 3$, and $y = g - 2$; or else $x = g - 3$, and $y = f - 2$. The analysis of this example is as follows:

Factors	1 × 48		2 × 24		3 × 16		4 × 12		6 × 8	
	x	y	x	y	x	y	x	y	x	y
Numbers	−2	46	−1	22	0	14	1	10	3	6
or	45	−1	21	0	13	1	9	2	5	4

34. The equation may be expressed still more generally, by writing $mxy = ax + by + c$; where a, b, c, and m, are

given numbers, and it is required to find integers for x and y that are not known.

If we first separate y, we shall have $y = \frac{ax + c}{mx - b}$; and removing x from the numerator, by multiplying both sides by m, we have

$$my = \frac{max + mc}{mx - b} = a + \frac{mc + ab}{mx - b}.$$

We have here a fraction whose numerator is a known number, and whose denominator must be a divisor of that number; let us therefore represent the numerator by a product of two factors, as fg (which may often be done in several ways), and see if one of these factors may be compared with $mx - b$, so that $mx - b = f$. Now, for this purpose, since $x = \frac{f + b}{m}$, $f + b$ must be divisible by m; and hence it follows, that out of the factors of $mc + ab$, we can employ only those which are of such a nature, that, by adding b to them, the sums will be divisible by m. We shall illustrate this by an example.

Let the equation be $5xy = 2x + 3y + 18$. Here, we have

$$y = \frac{2x + 18}{5x - 3}, \text{ and } 5y = \frac{10x + 90}{5x - 3} = 2 + \frac{96}{5x - 3};$$

it is therefore required to find those divisors of 96 which, added to 3, will give sums divisible by 5. Now, if we consider all the divisors of 96, which are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96, it is evident that only these three of them, viz. 2, 12, 32, will answer this condition.

Therefore,

1. If $5x - 3 = 2$, we obtain $5y = 50$, and consequently $x = 1$, and $y = 10$.
2. If $5x - 3 = 12$, we obtain $5y = 10$, and consequently $x = 3$, and $y = 2$.
3. If $5x - 3 = 32$, we obtain $5y = 5$, and consequently $x = 7$, and $y = 1$.

35. As in this general solution we have

$$my - a = \frac{mc + ab}{mx - b},$$

it will be proper to observe, that if a number, contained in the formula $mc + ab$, have a divisor of the form $mx - b$, the quotient in that case must necessarily be contained in the formula $my - a$: we may therefore express the number $mc + ab$ by a product, such as $(mx - b) \times (my - a)$. For example, let $m = 12$, $a = 5$, $b = 7$, and $c = 15$, and we have $12y - 5 = \frac{215}{12x - 7}$. Now, the divisors of 215 are 1, 5, 43, 215; and we must select from these such as are contained in the formula $12x - 7$; or such as, by adding 7 to them, the sum may be divisible by 12: but 5 is the only divisor that satisfies this condition; so that $12x - 7 = 5$, and $12y - 5 = 43$. In the same manner, as the first of these equations gives $x = 1$, we also find y, in integer numbers, from the other, namely, $y = 4$. This property is of the greatest importance with regard to the theory of numbers, and therefore deserves particular attention.

36. Let us now consider also an equation of this kind, $xy + x^2 = 2x + 3y + 29$. First, it gives us

$$y = \frac{2x - x^2 + 29}{x - 3}, \text{ or } y = -x - 1 + \frac{26}{x-3}; \text{ and}$$

$$y + x + 1 = \frac{26}{x-3}:$$ so that $x - 3$ must be a divisor of 26; and, in this case, the divisors of 26 being 1, 2, 13, 26, we obtain the three following answers:

1. $x - 3 = 1$, or $x = 4$; so that
   $y + x + 1 = y + 5 = 26$, and $y = 21$;
2. $x - 3 = 2$, or $x = 5$; so that
   $y + x + 1 = y + 6 = 13$, and $y = 7$;
3. $x - 3 = 13$, or $x = 16$; so that
   $y + x + 1 = y + 17 = 2$, and $y = -15$.

This last value, being negative, must be omitted; and, for the same reason, we cannot include the last case, $x - 3 = 26$.

37. It would be unnecessary to analyse any more of these formulæ, in which we find only the first power of y, and higher powers of x; for these cases occur but seldom, and, besides, they may always be resolved by the method which we have explained. But when y also is raised to the second power, or to a degree still higher, and we wish to determine its value by the above rules, we obtain radical signs, which contain the second, or higher powers of x; and it is then necessary to find such values of x, as will destroy the radical signs, or the irrationality. Now, the great art of *Indeterminate Analysis* consists in rendering those surd, or incommensurable formulæ rational: the methods of performing which will be explained in the following chapters*.

* See the Appendix to this chapter, at Art. 4, of the Additions by De la Grange. p. 534.

QUESTIONS FOR PRACTICE.

1. Given $24x = 13y + 16$, to find x and y in whole numbers. *Ans.* $x = 5$, and $y = 8$.

2. Given $87x + 256y = 15410$, to find the least value of x, and the greatest of y, in whole positive numbers.
Ans. $x = 30$, and $y = 12800$.

3. What is the number of all the possible values of x, y, and z, in whole numbers, in the equation $5x + 7y + 11z = 224$? *Ans.* 60.

4. How many old guineas at 21*s*. 6*d*; and pistoles at 17*s*, will pay 100*l*.? and in how many ways can it be done?
Ans. Three different ways; that is,
19, 62, 105 pistoles, and 78, 44, 10 guineas.

5. A man bought 20 birds for 20 pence; consisting of geese at 4 pence, quails at $\frac{1}{2}d$. and larks at $\frac{1}{4}d$. each; how many had he of each?
Ans. Three geese, 15 quails, and 2 larks.

6. A, B, and C, and their wives P, Q, and R, went to market to buy hogs; each man and woman bought as many hogs, as they gave shillings for each; A bought 25 hogs more than Q, and B bought 11 more than P. Also each man laid out three guineas more than his wife. Which two persons were, respectively, man and wife?
Ans. B and Q, C and P, A and R.

7. To determine whether it be possible to pay 100*l*. in guineas and moidores only? *Ans.* It is not possible.

8. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis d'ors, valued at 17*s*. each; how must I acquit myself of the debt?
Ans. I must pay him 13 guineas, and he must give me 16 louis d'ors.

9. In how many ways is it possible to pay 1000*l*. with crowns, guineas, and moidores only? *Ans.* 70734.

10. To find the least whole number, which being divided by the nine whole digits respectively, shall leave no remainders. *Ans.* 2520.