or $3r = 6s + 2$: consequently, $r = \frac{6s+2}{3} = 2s + \frac{2}{3}$.

Now, it is evident, that this can never become an integer number, because s is necessarily an integer; which shews the impossibility of such questions *.

CHAP. II.

Of the Rule *which is called* Regula Cæci, *for determining by means of two* Equations, *three or more* Unknown Quantities.

24. In the preceding chapter, we have seen how, by means of a single equation, two unknown quantities may be determined, so far as to express them in integer and positive numbers. If, therefore, we had two equations, in order that the question may be indeterminate, those equations must contain more than two unknown quantities. Questions of this kind occur in the common books of arithmetic; and are resolved by the rule called *Regula Cæci*, *Position*, or *The Rule of False*; the foundation of which we shall now explain, beginning with the following example:

25. *Question* 1. Thirty persons, men, women, and children, spend 50 crowns in a tavern; the share of a man is 3 crowns, that of a woman 2 crowns, and that of a child is 1 crown; how many persons were there of each class?

If the number of men be p, of women q, and of children r, we shall have the two following equations;

$$1.\quad p + q + r = 30, \text{ and}$$
$$2.\quad 3p + 2q + r = 50,$$

from which it is required to find the value of the three letters p, q, and r, in integer and positive numbers. The first equation gives $r = 30 - p - q$; whence we immediately conclude that $p + q$ must be less than 30; and, substituting this value of r in the second equation, we have $2p + q + 30 = 50$; so that $q = 20 - 2p$, and $p + q =$

* See the Appendix to this chapter, at Art. 3. of the Additions by De la Grange. p. 530.

$20 - p$, which evidently is also less than 30. Now, as we may, in this equation, assume all numbers for p which do not exceed 10, we shall have the following eleven answers: the number of men p, of women q, and of children r, being as follow:

$$p = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10;$$
$$q = 20,\ 18,\ 16,\ 14,\ 12,\ 10,\ 8,\ 6,\ 4,\ 2,\ 0;$$
$$r = 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20;$$

and, if we omit the first and the last, there will remain 9.

26. *Question 2.* A certain person buys hogs, goats, and sheep, to the number of 100, for 100 crowns; the hogs cost him $3\frac{1}{2}$ crowns apiece; the goats, $1\frac{1}{3}$ crown; and the sheep, $\frac{1}{2}$ a crown. How many had he of each?

Let the number of hogs be p, that of the goats q, and of the sheep r, then we shall have the two following equations:

$$1.\quad p + q + r = 100,$$
$$2.\quad 3\tfrac{1}{2}p + 1\tfrac{1}{3}q + \tfrac{1}{2}r = 100;$$

the latter of which being multiplied by 6, in order to remove the fractions, becomes, $21p + 8q + 3r = 600$. Now, the first gives $r = 100 - p - q$; and if we substitute this value of r in the second, we have $18p + 5q = 300$, or $5q = 300 - 18p$, and $q = 60 - \frac{18p}{5}$: consequently, $18p$ must be divisible by 5, and therefore, as 18 is not divisible by 5, p must contain 5 as a factor. If we therefore make $p = 5s$, we obtain $q = 60 - 18s$, and $r = 13s + 40$; in which we may assume for the value of s any integer number whatever, provided it be such, that q does not become negative: but this condition limits the value of s to 3; so that if we also exclude 0, there can only be three answers to the question; which are as follow:

$$\text{When}\quad s = 1,\ 2,\ 3,$$

$$\text{We have}\begin{cases} p = 5,\ 10,\ 15, \\ q = 42,\ 24,\ 6, \\ r = 53,\ 66,\ 79. \end{cases}$$

27. In forming such examples for practice, we must take particular care that they may be possible; in order to which, we must observe the following particulars:

Let us represent the two equations, to which we were just now brought, by

$$1.\quad x + y + z = a, \text{ and}$$
$$2.\quad fx + gy + hz = b,$$

in which f, g, and h, as well as a and b, are given numbers.

Now, if we suppose that among the numbers f, g, and h, the first, f, is the greatest, and h the least, since we have $fx + fy + fz$, or $(x + y + z)f = fa$, (because $x + y + z = a$) it is evident, that $fx + fy + fz$ is greater than $fx + gy + hz$; consequently, fa must be greater than b, or b must be less than fa. Farther, since $hx + hy + hz$, or $(x + y + z)h = ha$, and $hx + hy + hz$ is undoubtedly less than $fx + gy + hz$, ha must be less than b, or b must be greater than ha. Hence it follows, that if b be not less than fa, and also greater than ha, the question will be impossible: which condition is also expressed, by saying that b must be contained between the limits fa and ha; and care must also be taken that it may not approach either limit too nearly, as that would render it impossible to determine the other letters.

In the preceding example, in which $a = 100$, $f = 3\frac{1}{2}$, and $h = \frac{1}{2}$, the limits were 350 and 50. Now, if we suppose $b = 51$, instead of 100, the equations will become

$$x + y + z = 100, \text{ and } 3\tfrac{1}{2}x + 1\tfrac{1}{3}y + \tfrac{1}{2}z = 51;$$

or, removing the fractions, $21x + 8y + 3z = 306$; and if the first be multiplied by 3, we have $3x + 3y + 3z = 300$. Now, subtracting this equation from the other, there remains $18x + 5y = 6$; which is evidently impossible, because x and y must be integer and positive numbers *.

28. Goldsmiths and coiners make great use of this rule, when they propose to make, from three or more kinds of metal, a mixture of a given value, as the following example will shew.

Question 3. A coiner has three kinds of silver, the first of 7 ounces, the second of $5\frac{1}{2}$ ounces, the third of $4\frac{1}{2}$ ounces, fine per marc †; and he wishes to form a mixture of the weight of 30 marcs, at 6 ounces: how many marcs of each sort must he take?

If he take x marcs of the first kind, y marcs of the second, and z marcs of the third, he will have $x + y + z = 30$, which is the first equation.

Then, since a marc of the first sort contains 7 ounces of fine silver, the x marcs of this sort will contain $7x$ ounces of such silver. Also, the y marcs of the second sort will contain $5\frac{1}{2}y$ ounces, and the z marcs of the third sort will contain $4\frac{1}{2}z$ ounces, of fine silver; so that the whole mass will contain $7x + 5\frac{1}{2}y + 4\frac{1}{2}z$ ounces of fine silver. As this mixture is to weigh 30 marcs, and each of these marcs must contain 6 ounces of fine silver, it follows that the whole mass

* Vide Article 22.

† A *marc* is eight ounces.

will contain 180 ounces of fine silver; and thence results the second equation, $7x + 5\frac{1}{2}y + 4\frac{1}{2}z = 180$, or $14x + 11y + 9z = 360$. If we now subtract from this equation nine times the first, or $9x + 9y + 9z = 270$, there remains $5x + 2y = 90$, an equation which must give the values of x and y in integer numbers; and with regard to the value of z, we may derive it from the first equation $z = 30 - x - y$. Now, the former equation gives $2y = 90 - 5x$, and

$y = 45 - \frac{5x}{2}$; therefore, if $x = 2u$, we shall have $y = 45 - 5u$, and $z = 3u - 15$; which shews that u must be greater than 4, and yet less than 10. Consequently, the question admits of the following solutions:

$$\text{If } u = 5,\ 6,\ 7,\ 8,\ 9,$$

$$\text{Then } \begin{cases} x = 10,\ 12,\ 14,\ 16,\ 18, \\ y = 20,\ 15,\ 10,\ 5,\ 0, \\ z = 0,\ 3,\ 6,\ 9,\ 12. \end{cases}$$

29. Questions sometimes occur, containing more than three unknown quantities; but they are also resolved in the same manner, as the following example will shew.

Question 4. A person buys 100 head of cattle for 100 pounds; viz. oxen at 10 pounds each, cows at 5 pounds, calves at 2 pounds, and sheep at 10 shillings each. How many oxen, cows, calves, and sheep, did he buy?

Let the number of oxen be p, that of the cows q, of calves r, and of sheep s. Then we have the following equations:

$$1.\quad p + q + r + s = 100;$$
$$2.\quad 10p + 5q + 2r + \tfrac{1}{2}s = 100;$$

or, removing the fractions, $20p + 10q + 4r + s = 200$: then subtracting the first equation from this, there remains $19p + 9q + 3r = 100$; whence

$$3r = 100 - 19p - 9q, \text{ and}$$
$$r = 33 + \tfrac{1}{3} - 6p - \tfrac{1}{3}p - 3q; \text{ or}$$
$$r = 33 - 6p - 3q + \frac{1-p}{3};$$

whence $1 - p$, or $p - 1$, must be divisible by 3; therefore if we make

$$p - 1 = 3t, \text{ we have}$$

$$p = 3t + 1$$
$$q = q$$
$$r = 27 - 19t - 3q$$
$$s = 72 + 2q + 16t;$$

whence it follows, that $19t + 3q$ must be less than 27, and that, provided this condition be observed, we may give any value to q and t. We have therefore to consider the following cases:

1. If $t = 0$	2. If $t = 1$
we have $p = 1$	$p = 4$
$q = q$	$q = q$
$r = 27 - 3q$	$r = 8 - 3q$
$s = 72 + 2q.$	$s = 88 + 2q.$

We cannot make $t = 2$, because r would then become negative.

Now, in the first case, q cannot exceed 9; and, in the second, it cannot exceed 2; so that these two cases give the following solutions, the first giving the following ten answers:

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
$p =$	1	1	1	1	1	1	1	1	1	1
$q =$	0	1	2	3	4	5	6	7	8	9
$r =$	27	24	21	18	15	12	9	6	3	0
$s =$	72	74	76	78	80	82	84	86	88	90.

And the second furnishes the three following answers:

	1.	2.	3.
$p =$	4	4	4
$q =$	0	1	2
$r =$	8	5	2
$s =$	88	90	92.

There are, therefore, in all, thirteen answers, which are reduced to ten if we exclude those that contain *zero*, or 0.

30. The method would still be the same, even if the letters in the first equation were multiplied by given numbers, as will be seen from the following example.

Question 5. To find three such integer numbers, that if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products may be 560; and if we multiply the first by 9, the second by 25, and the third by 49, the sum of the products may be 2920.

If the first number be x, the second y, and the third z, we shall have the two equations,

$$1.\ 3x + 5y + 7z = 560$$
$$2.\ 9x + 25y + 49z = 2920.$$

And here, if we subtract three times the first, or $9x + 15y + 21z = 1680$, from the second, there remains $10y + 28z = 1240$; dividing by 2, we have $5y + 14z = 620$; whence we obtain $y = 124 - \frac{14z}{5}$: so that z must be divisible by 5. If therefore we make $z = 5u$, we shall have $y = 124 - 14u$; which values of y and z being substituted in the first equation, we have $3x - 35u + 620 = 560$; or $3x = 35u - 60$, and $x = \frac{35u}{3} - 20$; therefore we shall make $u = 3t$, from which we obtain the following answer, $x = 35t - 20$, $y = 124 - 42t$, and $z = 15t$, in which we must substitute for t an integer number greater than 0 and less than 3: so that we are limited to the two following answers:

If $\left.\begin{matrix} t = 1, \\ t = 2, \end{matrix}\right\}$ we have $\left.\begin{matrix} x = 15, y = 82, z = 15. \\ x = 50, y = 40, z = 30. \end{matrix}\right\}$

CHAP. III.

Of Compound Indeterminate Equations, *in which one of the* Unknown Quantities *does not exceed the* First Degree.

31. We shall now proceed to indeterminate equations, in which it is required to find two unknown quantities, one of them being multiplied by the other, or raised to a power higher than the first, whilst the other is found only in the first degree. It is evident that equations of this kind may be represented by the following general expression:

$$a + bx + cy + dx^2 + exy + fx^3 + gx^2y + hx^4 + kx^3y + \text{, \&c.} = 0.$$

As in this equation y does not exceed the first degree, that letter is easily determined; but here, as before, the values both of x and of y must be assigned in integer numbers.

We shall consider some of those cases, beginning with the easiest.

32. *Question* 1. To find two such numbers, that their product added to their sum may be 79.

Call the numbers sought x and y: then we must have $xy + x + y = 79$; so that $xy + y = 79 - x$, and $y = \frac{79 - x}{x + 1} = \frac{79}{x+1} + \frac{-x}{x+1} = -1 + \frac{80}{x+1}$, from which we see that $x + 1$ must be a divisor of 80. Now, 80 having