adopted at first, $p^2 - 2q^2$; for since $t = 3$, and $u = 2$, we have $r = 9$, and $s = 4$; wherefore $p = 81 - 32 = 49$, and $q = 72$; whence $p^2 - 2q^2 = 2401 - 10368 = - 7967$.

CHAP. XIV.

Solution *of some* Questions *that belong to this part of* Algebra.

212. We have hitherto explained such artifices as occur in this part of Algebra, and such as are necessary for resolving any question belonging to it: it remains to make them still more clear, by adding here some of those questions with their solutions.

213. *Question* 1. To find such a number, that if we add unity to it, or subtract unity from it, we may obtain in both cases a square number.

Let the number sought be x; then both $x + 1$, and $x - 1$ must be squares. Let us suppose for the first case $x+1 = p^2$, we shall have $x = p^2 - 1$, and $x - 1 = p^2 - 2$, which must likewise be a square. Let its root, therefore, be represented by $p - q$; and we shall have $p^2 - 2 = p^2 - 2pq + q^2$; consequently, $p = \frac{q^2 + 2}{2q}$. Hence we obtain $x = \frac{q^4 + 4}{4q^2}$, in which we may give q any value whatever, even a fractional one.

If we therefore make $q = \frac{r}{s}$, so that $x = \frac{r^4 + 4s^4}{4r^2s^2}$, we shall have the following values for some small numbers:

If $r = 1$,	2,	1,	3,	4,
and $s = 1$,	1,	2,	1,	1,
we have $x = \frac{5}{4}$,	$\frac{5}{4}$,	$\frac{65}{16}$,	$\frac{85}{36}$,	$\frac{65}{16}$.

214. *Question* 2. To find such a number x, that if we add to it any two numbers, for example, 4 and 7, we obtain in both cases a square.

According to this enunciation, the two formulæ, $x + 4$ and $x + 7$, must become squares. Let us therefore suppose the first $x + 4 = p^2$, which gives us $x = p^2 - 4$, and the second will become $x + 7 = p^2 + 3$; and, as this last formula must also be a square, let its root be represented by $p + q$, and we shall have $p^2 + 3 = p^2 + 2pq + q^2$; whence we obtain $p = \frac{3-q^2}{2q}$, and, consequently, $x = \frac{9-22q^2+q^4}{4q^2}$; and if we also take a fraction $\frac{r}{s}$ for q, we find

$x = \frac{9s^4-22r^2s^2+r^4}{4r^2s^2}$, in which we may substitute for r and s any integer numbers whatever.

If we make $r = 1$, and $s = 1$, we find $x = -3$; therefore $x + 4 = 1$, and $x + 7 = 4$.

If x were required to be a positive number, we might make $s = 2$, and $r = 1$; we should then have $x = \frac{57}{16}$, whence $x + 4 = \frac{121}{16}$, and $x + 7 = \frac{169}{16}$.

If we make $s = 3$, and $r = 1$, we have $x = \frac{133}{9}$; whence $x + 4 = \frac{169}{9}$, and $x + 7 = \frac{196}{9}$.

In order that the last term of the formula, which expresses x, may exceed the middle term, let us make $r = 5$, and $s = 1$, and we shall have $x = \frac{21}{25}$; consequently, $x + 4 = \frac{121}{25}$, and $x + 7 = \frac{196}{25}$.

215. *Question* 3. Required such a fractional value of x, that if added to 1, or subtracted from 1, it may give in both cases a square.

Since the two formulæ $1 + x$, and $1 - x$, must become squares, let us suppose the first $1 + x = p^2$, and we shall have $x = p^2 - 1$; also, the second formula will then be $1 - x = 2 - p^2$. As this last formula must become a square, and neither the first nor the last term is a square, we must endeavour to find a case, in which the formula does become a □, and we soon perceive one, namely, when $p = 1$. If we therefore make $p = 1 - q$, so that $x = q^2 - 2q$, we have $2 - p^2 = 1 + 2q - q^2$; and supposing its root to be $1 - qr$, we shall have $1 + 2q - q^2 = 1 - 2qr + q^2r^2$; so that $2 - q = -2r + qr^2$, and $q = \frac{2r+2}{r^2+1}$; whence results

$x = \frac{4r-4r^3}{(r^2+1)^2}$; and since r is a fraction, if we make $r = \frac{t}{u}$,

we shall have $x = \frac{4tu^3-4t^3u}{(t^2+u^2)^2} = \frac{4tu(u^2-t^2)}{(t^2+u^2)^2}$, where it is evident that u must be greater than t.

Let therefore $u = 2$, and $t = 1$, and we shall find $x = \frac{24}{25}$.

Let $u = 3$, and $t = 2$; we shall then have $x = \frac{120}{169}$; and the formulæ $1 + x = \frac{289}{169}$, and $1 - x = \frac{49}{169}$, will both be squares.

216. *Question* 4. To find such numbers x, that whether they be added to 10, or subtracted from 10, the sum and the difference may be squares.

It is required therefore, to transform into squares the formulæ $10 + x$, and $10 - x$, which might be done by the method that has just been employed; but let us explain another mode of proceeding. It will be immediately perceived, that the product of these two formulæ, or $100 - x^2$, must likewise become a square. Now, its first term being already a square, we may suppose its root to be $10 - px$, by which means we shall have $100 - x^2 = 100 - 20px + p^2x^2$;

therefore $p^2x + x = 20p$, and $x = \frac{20p}{p^2+1}$; now, from this it is only the product of the two formulæ which becomes a square, and not each of them separately. But provided one becomes a square, the other will necessarily be also a square.

Now $10 + x = \frac{10p^2 + 20p + 10}{p^2+1} = \frac{10(p^2 + 2p + 1)}{p^2+1}$, and since $p^2 + 2p + 1$ is already a square, the whole is reduced to making the fraction $\frac{10}{p^2+1}$, or $\frac{10p^2+10}{(p^2+1)^2}$, a square also.

For this purpose we have only to make $10p^2 + 10$ a square, and here it is necessary to find a case in which that takes place. It will be perceived that $p = 3$ is such a case; for which reason we shall make $p = 3 + q$, and shall have $100 + 60q + 10q^2$. Let the root of this be $10 + qt$, and we shall have the final equation,

$$100 + 60q + 10q^2 = 100 + 20qt + q^2t^2,$$

which gives $q = \frac{60-20t}{t^2-10}$, by which means we shall determine $p = 3 + q$, and $x = \frac{20p}{p^2+1}$.

Let $t = 3$, we shall then find $q = 0$, and $p = 3$; therefore $x = 6$, and our formulæ $10 + x = 16$, and $10 - x = 4$.

But if $t = 1$, we have $q = -\frac{40}{9}$, and $p = -\frac{13}{9}$, so that $x = -\frac{234}{25}$; now it is of no consequence if we also make $x = +\frac{234}{25}$; therefore $10 + x = \frac{484}{25}$, and $10 - x = \frac{16}{25}$, which quantities are both squares.

217. *Remark.* If we wished to generalise this question,

by demanding such numbers, x, for any number, a, that both $a + x$, and $a - x$ may be squares, the solution would frequently become impossible; namely, in all cases in which a was not the sum of two squares. Now, we have already seen, that, between 1 and 50, there are only the following numbers that are the sums of two squares, or that are contained in the formula $x^2 + y^2$:

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50.

So that the other numbers, comprised between 1 and 50, which are,

3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 38, 39, 42, 43, 44, 46, 47, 48, cannot be resolved into two squares; consequently, whenever a is one of these last numbers, the question will be impossible; which may be thus demonstrated: Let $a + x = p^2$, and $a - x = q^2$, then the addition of the two formulæ will give $2a = p^2 + q^2$; therefore $2a$ must be the sum of two squares. Now, if $2a$ be such a sum, a will be so likewise *; consequently, when a is not the sum of two squares, it will always be impossible for $a + x$, and $a - x$, to be each squares at the same time.

218. As 3 is not the sum of two squares, it follows, from what has been said, that, if $a = 3$, the question is impossible. It might, however, be objected, that there are, perhaps, two fractional squares whose sum is 3; but we answer that this also is impossible: for if $\frac{p^2}{q^2} + \frac{r^2}{s^2} = 3$, and we were to multiply by q^2s^2, we should have $3q^2s^2 = p^2s^2 + q^2r^2$; and the second side of this equation, which is the sum of two squares, would be divisible by 3; but we have already seen (Art. 170) that the sum of two squares, that are prime to each other, can have no divisors, except numbers, which are themselves sums of two squares.

The numbers 9 and 45, it is true, are divisible by 3, but they are also divisible by 9, and even each of the two squares that compose both the one and the other, is divisible by 9, since $9 = 3^2 + 0^2$, and $45 = 6^2 + 3^2$; which is therefore a different case, and does not enter into consideration here. We may rest assured, therefore, of this conclusion; that if a number, a, be not the sum of two squares in integer numbers, it will not be so in fractions.

* For, let $x^2 + y^2 = 2a$; and put $x = s + d$, and $y = s - d$; then $(s+d)^2 + (s-d)^2 = 2s^2 + 2d^2$: that is, $x^2 + y^2 = 2s^2 + 2d^2 = 2a$, or $s^2 + d^2 = a$. B.

On the contrary, when the number a is the sum of two squares in fractional numbers, it is also the sum of two squares in integer numbers an infinite number of ways: and this we shall illustrate.

219. *Question* 5. To resolve, in as many ways as we please, a number, which is the sum of two squares, into another, that shall also be the sum of two squares.

Let $f^2 + g^2$ be the given number, and let two other squares, x^2 and y^2, be required, whose sum $x^2 + y^2$ may be equal to the number $f^2 + g^2$. Here it is evident, that if x is either greater or less than f, y, on the other hand, must be either less or greater than g: if, therefore, we make $x = f + pz$, and $y = g - qz$, we shall have

$$f^2 + 2fpz + p^2z^2 + g^2 - 2gqz + q^2z^2 = f^2 + g^2,$$

where the two terms f^2 and g^2 are destroyed; after which there remain only terms divisible by z. So that we shall have $2fp + p^2z - 2gq + q^2z = 0$, or $p^2z + q^2z = 2gq - 2fp$; therefore $z = \frac{2gq - 2fp}{p^2 + q^2}$, whence we get the following values for x and y, namely, $x = \frac{2gpq + f(q^2 - p^2)}{p^2 + q^2}$, and . . . . . . $y = \frac{2fpq + g(p^2 - q^2)}{p^2 + q^2}$; in which we may substitute all possible numbers for p and q.

If 2, for example, be the number proposed, so that $f = 1$, and $g = 1$, we shall have $x^2 + y^2 = 2$; and because $x = \frac{2pq + q^2 - p^2}{p^2 + q^2}$, and $y = \frac{2pq + p^2 - q^2}{p^2 + q^2}$, if we make $p = 2$, and $q = 1$, we shall find $x = \frac{1}{5}$, and $y = \frac{7}{5}$.

220. *Question* 6. If a be the sum of two squares, to find such a number, x, that $a + x$, and $a - x$, may become squares.

Let $a = 13 = 9 + 4$, and let us make $13 + x = p^2$, and $13 - x = q^2$. Then we shall first have, by addition, $26 = p^2 + q^2$; and, by subtraction, $2x = p^2 - q^2$; consequently, the values of p and q must be such, that $p^2 + q^2$ may become equal to the number 26, which is also the sum of two squares, namely, of $25 + 1$. Now, since the question in reality is to resolve 26 into two squares, the greater of which may be expressed by p^2, and the less by q^2, we shall immediately have $p = 5$, and $q = 1$; so that $x = 12$. But we may resolve the number 26 into two squares in an infinite number of other ways: for, since $p = 5$, and $q = 1$, if we write t and u, instead of p and q, and p and q, instead of x and y, in the formulæ of the foregoing example, we shall find

$$p = \frac{2tu + 5(u^2 - t^2)}{t^2 + u^2}, \text{ and } q = \frac{10tu + t^2 - u^2}{t^2 + u^2}.$$

Here we may now substitute any numbers for t and u, and by those means determine p and q, and, consequently, also the value of $x = \frac{p^2 - q^2}{2}$.

For example, let $t = 2$, and $u = 1$; we shall then have $p = \frac{11}{5}$, and $q = \frac{23}{5}$; wherefore $p^2 - q^2 = \frac{408}{25}$, and $x = \frac{204}{25}$.

221. But, in order to resolve this question generally, let $a = c^2 + d^2$, and put z for the unknown quantity; that is to say, the formulæ, $a + z$, and $a - z$, must become squares.

Let us therefore make $a + z = x^2$, and $a - z = y^2$, and we shall thus have first $2a = 2(c^2 + d^2) = x^2 + y^2$, then $2z = x^2 - y^2$. Therefore the squares x^2 and y^2 must be such, that $x^2 + y^2 = 2(c^2 + d^2)$; where $2(c^2 + d^2)$ is really the sum of two squares, namely, $(c + d)^2 + (c - d)^2$; and, in order to abbreviate, let us suppose $c + d = f$, and $c - d = g$; then we must have $x^2 + y^2 = f^2 + g^2$; and this will happen, according to what has been already said, when

$$x = \frac{2gpq + f(q^2 - p^2)}{p^2 + q^2}, \text{ and } y = \frac{2fpq + g(p^2 - q^2)}{p^2 + q^2};$$

from which we obtain a very easy solution, by making $p = 1$, and $q = 1$; for we find $x = \frac{2g}{2} = g = c - d$, and $y = f = c + d$; consequently, $z = 2cd$; and it is evident that $a + z = c^2 + 2cd + d^2 = (c + d)^2$, and

$$a - z = c^2 - 2cd + d^2 = (c - d)^2.$$

Let us attempt another solution, by making $p = 2$, and $q = 1$; we shall then have $x = \frac{c - 7d}{5}$, and $y = \frac{7c + d}{5}$, where c and d, as well as x and y, may be taken *minus*, because we have only to consider their squares. Now, since x must be greater than y, let us make d negative, and we shall have $x = \frac{c + 7d}{5}$, and $y = \frac{7c - d}{5}$: hence . . . . . .

$z = \frac{24d^2 + 14cd - 24c^2}{25}$; and this value being added to $a = c^2 + d^2$, gives $\frac{c^2 + 14cd + 49d^2}{25}$, the square root of which is $\frac{c + 7d}{5}$. If we then subtract z from a, there remains $\frac{49c^2 - 14cd + d^2}{25}$, which is the square of $\frac{7c - d}{5}$; the former of these two square roots being x, and the latter y.

222. *Question* 7. Required such a number, x, that whether we add unity to itself, or to its square, the result may be a square.

It is here required to transform the two formulæ $x + 1$, and $x^2 + 1$, into squares. Let us therefore suppose the first $x + 1 = p^2$; and, because $x = p^2 - 1$, the second, $x^2 + 1 = p^4 - 2p^2 + 2$, must be a square: which last formula is of such a nature as not to admit of a solution, unless we already know a satisfactory case; but such a case readily occurs, namely, that of $p = 1$: therefore let $p = 1 + q$, and we shall have $x^2 + 1 = 1 + 4q^2 + 4q^3 + q^4$, which may become a square in several ways.

1. If we suppose its root to be $1 + q^2$, we shall have $1 + 4q^2 + 4q^3 + q^4 = 1 + 2q^2 + q^4$; so that $4q + 4q^2 = 2q$, or $4 + 4q = 2$, and $q = -\frac{1}{2}$; therefore $p = \frac{1}{2}$, and $x = -\frac{3}{4}$.

2. Let the root be $1 - q^2$, and we shall find $1 + 4q^2 + q^3 + q^4 = 1 - 2q^2 + q^4$; consequently $q = -\frac{3}{2}$, and $p = -\frac{1}{2}$, which gives $x = -\frac{3}{4}$, as before.

3. If we represent the root by $1 + 2q + q^2$, in order to destroy the first, and the last two terms, we have

$$1 + 4q^2 + 4q^3 + q^4 = 1 + 4q + 6q^2 + 4q^3 + q^4,$$

whence we get $q = -2$, and $p = -1$; and therefore $x = 0$.

4. We may also adopt $1 - 2q - q^2$ for the root, and in this case shall have

$$1 + 4q^2 + 4q^3 + q^4 = 1 - 4q + 2q^2 + 4q^3 + q^4;$$

but we find, as before, $q = -2$.

5. We may, if we choose, destroy the first two terms, by making the root equal to $1 + 2q^2$; for we shall then have $1 + 4q^2 + 4q^3 + q^4 = 1 + 4q^2 + 4q^4$; also, $q = \frac{4}{3}$, and $p = \frac{7}{3}$; consequently, $x = \frac{40}{9}$; lastly, $x + 1 = \frac{49}{9} = (\frac{7}{3})^2$, and $x^2 + 1 = \frac{1681}{81} = (\frac{41}{9})^2$.

A greater number of values will be found for q, by making use of those which we have already determined. Thus, having found $q = -\frac{1}{2}$; let now $q = -\frac{1}{2} + r$, and we shall have $p = \frac{1}{2} + r$; also, $p^2 = \frac{1}{4} + r + r^2$, and $p^4 = \frac{1}{16} + \frac{1}{2}r + \frac{3}{2}r^2 + 2r^3 + r^4$; whence the expression

$$p^4 - 2p^2 + 2 = \frac{25}{16} - \frac{3}{2}r - \frac{1}{2}r^2 + 2r^3 + r^4,$$

to which our formula, $x^2 + 1$, is reduced, must be a square, and it must also be so when multiplied by 16; in which case, we have $25 - 24r - 8r^2 + 32r^3 + 16r^4$ to be a square. For which reason, let us now represent

1. The root by $5 + fr \pm 4r^2$; so that

$$25 - 24r - 8r^2 + 32r^3 + 16r^4 =$$
$$25 + 10fr \pm 40r^2 + f^2r^2 \pm 8fr^3 + 16r^4.$$

The first and the last terms destroy each other; and we may destroy the second also, if we make $10f = -24$, and, consequently, $f = -\frac{12}{5}$; then dividing the remaining terms by r^2, we have $-8 + 32r = \pm 40 + f^2 \pm 8fr$; and, admitting the upper sign, we find $r = \dfrac{48 + f^2}{32 - 8f}$. Now, because $f = -\frac{12}{5}$, we have $r = \frac{21}{20}$; therefore $p = \frac{31}{20}$, and $x = \frac{561}{400}$; so that $x + 1 = (\frac{31}{20})^2$, and $x^2 + 1 = (\frac{689}{400})^2$.

2. If we adopt the lower sign, we have

$$-8 + 32r = -40 + f^2 - 8fr,$$

whence $r = \dfrac{f^2 - 32}{32 + 8f}$; and since $f = -\frac{12}{5}$, we have $r = -\frac{41}{20}$; therefore $p = \frac{31}{20}$, which leads to the preceding equation.

3. Let $4r^2 + 4r \pm 5$ be the root; so that

$$16r^4 + 32r^3 - 8r^2 - 24r + 25 =$$
$$16r^4 + 32r^3 \pm 40r^2 + 16r^2 \pm 40r + 25:$$

and as on both sides the first two terms and the last destroy each other, we shall have

$$-8r - 24 = \pm 40r + 16r \pm 40, \text{ or}$$
$$-24r - 24 = \pm 40r \pm 40.$$

Here, if we admit the upper sign, we shall have

$$-24r - 24 = 40r + 40, \text{ or } 0 = 64r + 64, \text{ or}$$

$0 = r + 1$, that is $r = -1$, and $p = -\frac{1}{2}$; but this is a case already known to us, and we should not have found a different one by making use of the other sign.

4. Let now the root be $5 + fr + gr^2$, and let us determine f and g so, that the first three terms may vanish: then, since

$$25 - 24r - 8r^2 + 32r^3 + 16r^4 =$$
$$25 + 10fr + 10f^2r^2 + 10gr^2 + 2fgr^3 + g^2r^4,$$

we shall first have $10f = -24$, so that $f = -\frac{12}{5}$; then

$$10g + f^2 = -8, \text{ or } g = \frac{-8-f^2}{10} = \frac{-344}{250} = \frac{-172}{125}.$$

When, therefore, we have substituted and divided the remaining terms by r^3, we shall have

$$32 + 16r = 2fg + g^2r, \text{ and } r = \frac{2fg - 32}{16 - g^2}.$$

Now, the numerator $2fg - 32$ becomes here

$$\frac{+24 \times 172 - 32 \times 625}{5 \times 125} = \frac{-32 \times 496}{625} = \frac{-16 \times 32 \times 31}{625}, \text{ and}$$

the denominator

$$16 - g^2 = (4 - g) \times (4 + g) = \tfrac{328}{125} \times \tfrac{672}{125} = \frac{8 \times 32 \times 41 \times 21}{25 \times 625};$$

so that $r = -\frac{1550}{861}$; and hence we conclude that $p = -\frac{22392}{17221}$, by means of which we obtain a new value of $x = p^2 - 1$.

223. *Question* 8. To find a number, x, which, added to each of the given numbers, a, b, c, produces a square.

Since here the three formulæ $x + a$, $x + b$, and $x + c$, must be squares, let us make the first $x + a = z^2$, and we shall have $x = z^2 - a$, and the two other formulæ will be changed into $z^2 + b - a$, and $z^2 + c - a$.

It is now required for each of these to be a square; but this does not admit of a general solution: the problem is frequently impossible, and its possibility entirely depends on the nature of the numbers $b - a$, and $c - a$. For example, if $b - a = 1$, and $c - a = -1$, that is to say, if $b = a + 1$, and $c = a - 1$, it would be required to make $z^2 + 1$, and $z^2 - 1$ squares, and, consequently, that z should be a fraction; so that we should make $z = \frac{p}{q}$, and it would be necessary that the two formulæ $p^2 + q^2$, and $p^2 - q^2$, should be squares, and, consequently, that their product also, $p^4 - q^4$, should be a square. Now, we have already shewn (Art. 202) that this is impossible.

Were we to make $b - a = 2$, and $c - a = -2$, that is, $b = a + 2$, and $c = a - 2$; and also, if $z = \frac{p}{q}$, we should have the two formulæ, $p^2 + 2q^2$, and $p^2 - 2q^2$, to transform into squares; consequently, it would also be necessary for their product, $p^4 - 4q^4$, to become a square; but this we have likewise shewn to be impossible. (Art. 209.)

In general, let $b - a = m$, $c - a = n$, and $z = \frac{p}{q}$: then the formulæ $p^2 + mq^2$, and $p^2 + nq^2$, must become squares; but we have seen that this is impossible, both when $m = +1$, and $n = -1$, and when $m = +2$, and $n = -2$.

It is also impossible, when $m = f^2$, and $n = -f^2$; for, in that case, we should have two formulæ, whose product would be $= p^4 - f^4q^4$, that is to say, the difference of two biquadrates; and we know that such a difference can never become a square.

Likewise, when $m = 2f^2$, and $n = -2f^2$, we have the two formulæ $p^2 + 2f^2q^2$, and $p^2 - 2f^2q^2$, which cannot both become squares, because their product $p^4 - 4f^4q^4$ must become a square. Now, if we make $fq = r$, this product is changed into $p^4 - 4r^4$, a formula, the impossibility of which has been already demonstrated.

If we suppose $m = 1$, and $n = 2$, so that it is required to reduce to squares the formulæ $p^2 + q^2$, and $p^2 + 2q^2$, we shall make $p^2 + q^2 = r^2$, and $p^2 + 2q^2 = s^2$; the first equation will give $p^2 = r^2 - q^2$, and the second will give $r^2 + q^2 = s^2$; and therefore both $r^2 - q^2$, and $r^2 + q^2$, must be squares: but the impossibility of this is proved, since the product of these formulæ, or $r^4 - q^4$, cannot become a square.

These examples are sufficient to shew, that it is not easy to choose such numbers for m and n as will render the solution possible. The only means of finding such values of m and n, is to imagine them, or to determine them by the following method.

Let us make $f^2 + mg^2 = h^2$, and $f^2 + ng^2 = k^2$; then we have, by the former equation, $m = \frac{h^2 - f^2}{g^2}$, and, by the latter, $n = \frac{k^2 - f^2}{g^2}$; this being done, we have only to take for f, g, h, and k, any numbers at pleasure, and we shall have values of m and n that will render the solution possible.

For example, let $h = 3$, $k = 5$, $f = 1$, and $g = 2$, we shall have $m = 2$, and $n = 6$; and we may now be certain that it is possible to reduce the formulæ $p^2 + 2q^2$ and $p^2 + 6q^2$ to squares, since it takes place when $p = 1$, and $q = 2$. But the former formula generally becomes a square, if $p = r^2 - 2s^2$, and $q = 2rs$; for then $p^2 + 2q^2 =$

$(r^2 + 2s^2)^2$. The latter formula also becomes $p^2 + 6q^2 = r^4 + 20r^2s^2 + 4s^4$; and we know a case in which it becomes a square, namely, when $p = 1$, and $q = 2$, which gives $r = 1$, and $s = 1$; or, generally, $r = s$; so that the formula is $25s^4$. Knowing this case, therefore, let us make $r = s + t$; and we shall then have $r^2 = s^2 + 2st + t^2$, or $r^4 = s^4 + 4s^3t + 6s^2t^2 + 4st^3 + t^4$; so that our formula will become $25s^4 + 44s^3t + 26s^2t^2 + 4st^3 + t^4$: and, supposing its root to be $5s^2 + fst + t^2$, we shall make it equal to the square $25s^4 + 10fs^3t + f^2s^2t^2 + 10s^2t^2 + 2fst^3 + t^4$, by which means the first and last terms will be destroyed. Let us likewise make $2f = 4$, or $f = 2$, in order to remove the last terms but one, and we shall obtain the equation
$44s + 26t = 10fs + 10t + f^2t = 20s + 14t$, or $2s = -t$,

and $\frac{s}{t} = -\frac{1}{2}$; therefore $s = -1$, and $t = 2$, or $t = -2s$;

and, consequently, $r = -s$, also $r^2 = s^2$, which is nothing more than the case already known.

Let us rather, therefore, determine f in such a manner, that the second terms may vanish. We must make $10f = 44$, or $f = \frac{22}{5}$; and then dividing the other terms by st^2, we shall have $26s + 4t = 10s + f^2s + 2ft$, that is, $-\frac{84}{25}s = \frac{24}{5}t$;

which gives $t = -\frac{7}{10}s$, and $r = s + t = \frac{3}{10}s$, or $\frac{r}{s} = \frac{3}{10}$; so that

$r = 3$, and $s = 10$; by which means we find $p = 2s^2 - r^2 = 191$, and $q = 2rs = 60$, and our formulæ will be

$$p^2 + 2q^2 = 43681 = (209)^2 \text{ and}$$
$$p^2 + 6q^2 = 58081 = (241)^2.$$

224. *Remark.* In the same manner, other numbers may be found for m and n, that will make our formulæ squares; and it is proper to observe, that the ratio of m to n is arbitrary.

Let this ratio be as a to b, and let $m = az$, and $n = bz$; it will be required to know how z is to be determined, in order that the two formulæ $p^2 + azq^2$, and $p^2 + bzq^2$, may be transformed into squares: the method of doing which we shall explain in the solution of the following problem.

225. *Question* 9. Two numbers, a and b, being given, to find the number z such, that the two formulæ, $p^2 + azq^2$, and $p^2 + bzq^2$, may become squares; and, at the same time, to determine the least possible values of p and q.

Here, if we make $p^2 + azq^2 = r^2$, and $p^2 + bzq^2 = s^2$, and multiply the first equation by a, and the second by b, the difference of the two products will furnish the equation

$(b-a)p^2=br^2-as^2$, and, consequently, $p^2=\frac{br^2-as^2}{b-a}$; which formula must be a square: now, this happens when $r=s$. Let us, therefore, in order to remove the fractions, suppose $r=s+(b-a)t$, and we shall have

$$p^2=\frac{br^2-as^2}{b-a}=\frac{bs^2+2b(b-a)st+b(b-a)^2t^2-as^2}{b-a}=$$
$$\frac{(b-a)s^2+2b(b-a)st+b(b-a)^2t^2}{b-a}=$$
$$s^2+2bst+b(b-a)t^2.$$

Let us now make $p=s+\frac{x}{y}t$, and we shall have

$$p^2=s^2+\frac{2x}{y}st+\frac{x^2}{y^2}t^2=s^2+2bst+b(b-a)t^2,$$

in which the terms s^2 destroy each other; so that the other terms being divided by t, and multiplied by y^2, give $2sxy+tx^2=2bsy^2+b(b-a)ty^2$; whence

$$t=\frac{2sxy-2sby^2}{b(b-a)y^2-x^2}, \text{ and } \frac{t}{s}=\frac{2xy-2by^2}{b(b-a)y^2-x^2}.$$

So that $t=2xy-2by^2$, and $s=b(b-a)y^2-x^2$; farther, $r=2(b-a)xy-b(b-a)y^2-x^2$; and, consequently,

$$p=s+\frac{x}{y}t=b(b-a)y^2+x^2-2bxy=(x-by)^2-aby^2.$$

Having therefore found p, r, and s, it remains to determine z; and, for this purpose, let us subtract the first equation, $p^2+azq^2=r^2$, from the second, $p^2+bzq^2=s^2$; the remainder will be $zq^2(b-a)=s^2-r^2=(s+r)\times(s-r)$.

Now, $s+r=2(b-a)xy-2x^2$, and
$s-r=2b(b-a)y^2-2(b-a)xy$, or
$s+r=2x((b-a)y-x)$, and
$s-r=2(b-a)\times(by-x)y$; so that
$(b-a)zq^2=2x((b-a)y-x)\times 2(b-a)\times(by-x)y$, or
$zq^2=2x((b-a)y-x)\times(by-x)2y$, or
$zq^2=4xy((b-a)y-x)\times(by-x)$;

consequently, $z=\frac{4xy((b-a)y-x)\times(by-x)}{q^2}$.

We must therefore take the greatest square for q^2, that will divide the numerator; but let us observe, that we have already found $p=b(b-a)y^2+x^2-2bxy=(x-by)^2-aby^2$; and therefore we may simplify, by making $x=v+by$, or $x-by=v$; for then $p=v^2-aby^2$, and

$$z = \frac{4(v+by) \times vy \times (v+ay)}{q^2}, \text{ or } z = \frac{4vy(v+ay) \times (v+by)}{q^2}.$$

By these means we may take any numbers for v and y, and assuming for q^2 the greatest square contained in the numerator, we shall easily determine the value of z; after which, we may return to the equations $m = az$, $n = bz$, and $p = v^2 - aby^2$, and shall obtain the formulæ required.

1. $p^2 + azq^2 = (v^2 - aby^2)^2 + 4avy(v + ay) \times (v + by)$, which is a square, whose root is $r = - v^2 - 2avy - aby^2$.

2. The second formula becomes

$p^2 + bzq^2 = (v^2 - aby^2)^2 + 4bvy(v + ay) \times (v + by)$, which is also a square, whose root is $s = -v^2 - 2bvy - aby^2$, and the values both of r and s may be taken positive.

It may be proper to analyse these results in some examples.

226. *Example* 1. Let $a = -1$, and $b = +1$, and let us endeavour to seek such a number for z, that the two formulæ $p^2 - zq^2$, and $p^2 + zq^2$, may become squares; namely, the first r^2, and the second s^2.

We have therefore $p = v^2 + y^2$; and, in order to find z, we have only to consider the formula

$z = \frac{4vy(v-y) \times (v+y)}{q^2}$; and, by giving different values to v and y, we shall see those that result for z.

	1	2	3	4	5	6
v	2	3	4	5	16	8
y	1	2	1	4	9	1
$v - y$	1	1	3	1	7	7
$v + y$	3	5	5	9	25	9
zq^2	4×6	4×30	16×15	$9 \times 16 \times 5$	$36 \times 25 \times 16 \times 7$	$16 \times 9 \times 14$
q^2	4	4	16	9×16	$36 \times 25 \times 16$	16×9
z	6	30	15	5	7	14
p	5	13	17	41	337	65

And by means of these values, we may resolve the following formulæ, and make squares of them:

1. We may transform into squares the formulæ $p^2 - 6q^2$, and $p^2 + bq^2$; which is done by supposing $p = 5$, and $q = 2$; for the first becomes $25 - 24 = 1$, and the second

$$25 + 24 = 49.$$

2. Likewise, the two formulæ $p^2 - 30q^2$, and $p^2 + 30q^2$;

namely, by making $p = 13$, and $q = 2$; for the first becomes $169 - 120 = 49$, and the second $169 + 120 = 289$.

3. Likewise the two formulæ $p^2 - 15q^2$, and $p^2 + 15q^2$; for if we make $p = 17$, and $q = 4$, we have, for the first; $289 - 240 = 49$, and for the second $289 + 240 = 529$.

4. The two formulæ $p^2 - 5q^2$, and $p^2 + 5q^2$, become likewise squares: namely, when $p = 41$, and $q = 12$; for then $p^2 - 5q^2 = 1681 - 720 = 961 = 31^2$, and

$$p^2 + 5q^2 = 1681 + 720 = 2401 = 49^2.$$

5. The two formulæ $p^2 - 7q^2$, and $p^2 + 7q^2$, are squares, if $p = 337$, and $q = 120$; for the first is then $113569 - 100800 = 12769 = 113^2$, and the second is $113569 + 100800 = 214369 = 463^2$.

6. The formulæ $p^2 - 14q^2$, and $p^2 + 14q^2$, become squares in the case of $p = 65$, and $q = 12$; for then

$$p^2 - 14q^2 = 4225 - 2016 = 2209 = 47^2, \text{ and}$$
$$p^2 + 14q^2 = 4225 + 2016 = 6241 = 79^2.$$

227. *Example* 2. When the two numbers m and n are in the ratio of 1 to 2; that is to say, when $a = 1$, and $b = 2$, and therefore $m = z$, and $n = 2z$, to find such values for z, that the formulæ $p^2 + zq^2$ and $p^2 + 2zq^2$ may be transformed into squares.

Here it would be superfluous to make use of the general formulæ already given, since this example may be immediately reduced to the preceding. In fact, if $p^2 + zq^2 = r^2$, and $p^2 + 2zq^2 = s^2$, we have, from the first equation, $p^2 = r^2 - zq^2$; which being substituted in the second, gives $r^2 + zq^2 = s^2$; so that the question only requires, that the two formulæ, $r^2 - zq^2$, and $r^2 + zq^2$, may become squares; and this is evidently the case of the preceding example. We shall consequently have for z the following values: 6, 30, 15, 5, 7, 14, &c.

We may also make a similar transformation in a general manner. For, supposing that the two formulæ $p^2 + mq^2$, and $p^2 + nq^2$, may become squares, let us make $p^2 + mq^2 = r^2$, and $p^2 + nq^2 = s^2$; the first equation gives $p^2 = r^2 - mq^2$; the second will become

$s^2 = r^2 - mq^2 + nq^2$, or $r^2 + (n - m)q^2 = s^2$: if, therefore, the first formulæ are possible, these last $r^2 - mq^2$, and $r^2 + (n - m)q^2$, will be so likewise; and as m and n may be substituted for each other, the formulæ $r^2 - nq^2$, and $r^2 + (m - n)q^2$, will also be possible: on the contrary, if the first are impossible, the others will be so likewise.

228. *Example* 3. Let m be to n as 1 to 3, or let $a = 1$, and $b = 3$, so that $m = z$, and $n = 3z$, and let it be required to transform into squares the formulæ $p^2 + zq^2$, and $p^2 + 3zq^2$.

Since $a = 1$, and $b = 3$, the question will be possible in all the cases in which $zq^2 = 4vy(v + y) \times (v + 3y)$, and $p = v^2 - 3y^2$. Let us therefore adopt the following values for v and y:

v	1	3	4	1	16
y	1	2	1	8	9
$v+y$	2	5	5	9	25
$v+3y$	4	9	7	25	43
zq^2	16×2	$4 \times 9 \times 30$	$4 \times 4 \times 35$	$4 \times 9 \times 25 \times 4 \times 2$	$4 \times 9 \times 16 \times 25 \times 43$
q^2	16	4×9	4×4	$4 \times 4 \times 9 \times 25$	$4 \times 9 \times 16 \times 25$
z	2	30	35	2	43
p	2	3	13	191	13

Now, we have here two cases for $z = 2$, which enables us to transform, in two ways, the formulæ $p^2 + 2q^2$, and $p^2 + 6q^2$.

The first is, to make $p = 2$, and $q = 4$, and consequently also $p = 1$, and $q = 2$; for we have then from the last $p^2 + 2q^2 = 9$, and $p^2 + 6q^2 = 25$.

The second is, to suppose $p = 191$, and $q = 60$, by which means we shall have $p^2 + 2q^2 = (209)^2$, and $p^2 + 6q^2 = (241)^2$. It is difficult to determine whether we cannot also make $z = 1$; which would be the case, if zq^2 were a square: but, in order to determine the question, whether the two formulæ $p^2 + q^2$, and $p^2 + 3q^2$, can become squares, the following process is necessary.

229. It is required to investigate, whether we can transform into squares the formulæ $p^2 + q^2$, and $p^2 + 3q^2$, with the same values of p and q. Let us here suppose $p^2 + q^2 = r^2$, and $p^2 + 3q^2 = s^2$, which leads to the investigation of the following circumstances.

1. The numbers p and q may be considered as prime to each other; for if they had a common divisor, the two formulæ would still continue squares, after dividing p and q by that divisor.

2. It is impossible for p to be an even number; for in that case q would be odd; and, consequently, the second formula would be a number of the class $4n + 3$, which cannot become a square; wherefore p is necessarily odd, and p^2 is a number of the class $8n + 1$.

3. Since p therefore is odd, q must in the first formula not only be even, but divisible by 4, in order that q^2 may become a number of the class $16n$, and that $p^2 + q^2$ may be of the class $8n + 1$.

4. Farther, p cannot be divisible by 3; for in that case, p^2 would be divisible by 9, and q^2 not; so that $3q^2$ would

only be divisible by 3, and not by 9; consequently, also, $p^2 + 3q^2$ could only be divisible by 3, and not by 9, and therefore could not be a square; so that p cannot be divisible by 3, and p^2 will be a number of the class $3n + 1$.

5. Since p is not divisible by 3, q must be so; for otherwise q^2 would be a number of the class $3n + 1$, and consequently $p^2 + q^2$ a number of the class $3n + 2$, which cannot be a square: therefore q must be divisible by 3.

6. Nor is p divisible by 5; for if that were the case, q would not be so, and q^2 would be a number of the class $5n + 1$, or $5n + 4$; consequently, $3q^2$ would be of the class $5n + 3$, or $5n + 2$; and as $p^2 + 3q^2$ would belong to the same classes, this formula therefore could not in that case become a square; consequently p must not be divisible by 5, and p^2 must be a number of the class $5n + 1$, or of the class $5n + 4$.

7. But since p is not divisible by 5, let us see whether q is divisible by 5, or not; since if q were not divisible by 5, q^2 must be of the class $5n + 2$, or $5n + 3$, as we have already seen; and since p^2 is of the class $5n + 1$, or $5n + 4$, $p^2 + 3q^2$ must be the same; namely, $5n + 1$, or $5n + 4$; and therefore, of one of the forms $5n + 3$, or $5n + 2$. Let us consider these cases separately.

If we suppose p^2 (F) $5n + 1$ *, then we must have q^2 (F) $5n + 4$, because otherwise $p^2 + q^2$ could not be a square; but we should then have $3q^2$ (F) $5n + 2$ and $p^2 + 3q^2$ (F) $5n + 3$, which cannot be a square.

In the second place, let p^2 (F) $5n + 4$; in this case we must have q^2 (F) $5n + 1$, in order that $p^2 + q^2$ may be a square, and $3q^2$ (F) $5n + 3$; therefore $p^2 + 3q^2$ (F) $5n + 2$, which cannot be a square. It follows, therefore, that q^2 must be divisible by 5.

8. Now, q being divisible first by 4, then by 3, and in the third place by 5, it must be such a number as $4 \times 3 \times 5m$, or $q = 60m$; so that our formulæ would become $p^2 + 3600m^2 = r^2$, and $p^2 + 10800m^2 = s^2$: this being established, the first, subtracted from the second, will give $7200m^2 = s^2 - r^2 = (s + r) \times (s - r)$; so that $s + r$ and $s - r$ must be factors of $7200m^2$, and at the same time

* In the former editions of this work, the sign = is used to express the words, "*of the form.*" This was adopted in order to save the repetition of these words; but as it may occasionally produce ambiguity, or confusion, it was thought proper to substitute (F) instead of =, which is to be read thus: p^2 (F) $5n + 1$, *of the form* $5n + 1$.

it should be observed, that s and r must be odd numbers, and also prime to each other*.

9. Farther, let $7200m^2 = 4fg$, or let its factors be $2f$ and $2g$, supposing $s + r = 2f$, and $s - r = 2g$, we shall have $s = f + g$, and $r = f - g$; f and g, also, must be prime to each other, and the one must be odd and the other even. Now, as $fg = 1800m^2$, we may resolve $1800m^2$ into two factors, the one being even and the other odd, and having at the same time no common divisor.

10. It is to be farther remarked, that since $r^2 = p^2 + q^2$, and since r is a divisor of $p^2 + q^2$, $r = f - g$ must likewise be the sum of two squares (Art. 170); and as this number is odd, it must be contained in the formula $4n + 1$.

11. If we now begin with supposing $m = 1$, we shall have $fg = 1800 = 8 \times 9 \times 25$, and hence the following results: $f = 1800$, and $g = 1$, or $f = 200$, and $g = 9$, or $f = 72$, and $g = 25$, or $f = 225$, and $g = 8$.

$$\left.\begin{matrix}\text{The 1st}\\ \text{2d}\\ \text{3d}\\ \text{4th}\end{matrix}\right\}\text{gives}\left\{\begin{matrix} r = f - g = 1799(\text{F})4n + 3;\\ r = f - g = 191(\text{F})4n + 3;\\ r = f - g = 47(\text{F})4n + 3;\\ r = f - g = 217(\text{F})4n + 1;\end{matrix}\right.$$

So that the first three must be excluded, and there remains only the fourth: from which we may conclude, generally, that the greater factor must be odd, and the less even; but even the value, $r = 217$, cannot be admitted here, because that number is divisible by 7, which is not the sum of two squares†.

12. If $m = 2$, we shall have $fg = 7200 = 32 \times 225$; for which reason we shall make $f = 225$, and $g = 32$, so that $r = f - g = 193$; and this number being the sum of two squares, it will be worth while to try it. Now, as $q = 120$, and $r = 193$, and $p^2 = r^2 - q^2 = (r + q) \times (r - q$, we shall have $r + q = 313$, and $r - q = 73$; but since these factors are not squares, it is evident that p^2 does not become a square. In the same manner, it would be in vain to substitute any other numbers for m, as we shall now shew.

230. *Theorem.* It is impossible for the two formulæ $p^2 + q^2$, and $p^2 + 3q^2$, to be both squares at the same time; so that in the cases where one of them is a square, it is certain that the other is not.

* Because p is odd and q is even; therefore $p^2 + q^2 = r^2$, and $p^2 + 3q^2 = s^2$, must be both odd. B.

† Because the sum of two squares, prime to each other, can only be divided by numbers of the same form. B.

Demonstration. We have seen that p is odd, and q even, because $p^2 + q^2$ cannot be a square, except when $q = 2rs$, and $p = r^2 - s^2$; and $p^2 + 3q^2$ cannot be a square, except when $q = 2tu$, and $p = t^2 - 3u^2$, or $p = 3u^2 - t^2$. Now, as in both cases q must be a double product, let us suppose for both, $q = 2abcd$; and, for the first formula, let us make $r = ab$, and $s = cd$; for the second, let $t = ac$, and $u = bd$. We shall have for the former $p = a^2b^2 - c^2d^2$, and for the latter $p = a^2c^2 - 3b^2d^2$, or $p = 3b^2d^2 - a^2c^2$, and these two values must be equal; so that we have either $a^2b^2 - c^2d^2 = a^2c^2 - 3b^2d^2$, or $a^2b^2 - c^2d^2 = 3b^2d^2 - a^2c^2$; and it will be perceived that the numbers a, b, c, and d, are each less than p and q. We must however consider each case separately: the first gives $a^2b^2 + 3b^2d^2 = c^2d^2 + a^2c^2$, or $b^2(a^2 + 3d^2) = c^2(a^2 + d^2)$, whence $\frac{b^2}{c^2} = \frac{a^2 + d^2}{a^2 + 3d^2}$ a fraction that must be a square.

Now, the numerator and denominator can here have no other common divisor than 2, because their difference is $2d^2$. If, therefore, 2 were a common divisor, both $\frac{a^2 + d^2}{2}$, and $\frac{a^2 + 3d^2}{2}$, must be a square; but the numbers a and d are in this case both odd, so that their squares have the form $8n + 1$, and the formula $\frac{a^2 + 3d^2}{2}$ is contained in the expression $4n + 2$, and cannot be a square; wherefore 2 cannot be a common divisor; the numerator $a^2 + d^2$, and the denominator $a^2 + 3d^2$ are therefore prime to each other, and each of them must of itself be a square.

But these formulæ are similar to the former, and if the last were squares, similar formulæ, though composed of the smallest numbers, would have also been squares; so that we conclude, reciprocally, from our not having found squares in small numbers, that there are none in great.

This conclusion however is not admissible, unless the second case, $a^2b^2 - c^2d^2 = 3b^2d^2 - a^2c^2$, furnishes a similar one. Now, this equation gives $a^2b^2 + a^2c^2 = 3b^2d^2 + c^2d^2$, or $a^2(b^2 + c^2) = d^2(3b^2 + c^2)$; and, consequently, $\frac{a^2}{d^2} = \frac{b^2 + c^2}{3b^2 + c^2} = \frac{c^2 + b^2}{c^2 + 3b^2}$; so that as this fraction ought to be a square, the foregoing conclusion is fully confirmed; for, if in great numbers there were cases in which $p^2 + q^2$, and $p^2 + 3q^2$, were squares, such cases must have also existed with regard to smaller numbers; but this is not the fact.

231. *Question* 10. To determine three numbers, x, y, and z, such, that multiplying them together two and two, and adding 1 to the product, we may obtain a square each time; that is, to transform into squares the three following formulæ:

$$xy + 1,\ xz + 1, \text{ and } yz + 1.$$

Let us suppose one of the last two, as $xz + 1, = p^2$, and the other $yz + 1 = q^2$, and we shall have $x = \frac{p^2-1}{z}$, and $y = \frac{q^2-1}{z}$. The first formula is now transformed to $\frac{(p^2-1)\times(q^2-1)}{z^2} + 1$; which must consequently be a square, and will be no less so, if multiplied by z^2; so that $(p^2 - 1) \times (q^2 - 1) + z^2$, must be a square, which it is easy to form. For, let its root be $z + r$, and we shall have

$$(p^2 - 1) \times (q^2 - 1) = 2rz + r^2, \text{ and}$$

$z = \frac{(p^2-1)\times(q^2-1) - r^2}{2r}$, in which any numbers may be substituted for p, q, and r.

For example, if $r = (pq + 1)$, we shall have

$r^2 = p^2q^2 + 2qp + 1$, and $z = \frac{p^2 + 2pq + q^2}{2pq + 2}$; wherefore

$x = \frac{(p^2-1)\times(2pq+2)}{p^2+2pq+q^2} = \frac{2(pq+1)\times(p^2-1)}{(p+q)^2}$, and

$y = \frac{2(pq+1)\times(q^2-1)}{(p+q)^2}$.

But if whole numbers be required, we must make the first formula $xy + 1 = p^2$, and suppose $z = x + y + q$; then the second formula becomes

$x^2 + xy + xq + 1 = x^2 + qx + p^2$, and the third will be $xy + y^2 + qy + 1 = y^2 + qy + p^2$. Now, these evidently become squares, if we make $q = \pm 2p$; since in that case the second is $x^2 \pm 2px + p^2$, the root of which is $x \pm p$, and the third is $y^2 \pm 2py + p^2$, the root of which is $y \pm p$. We have consequently this very elegant solution: $xy + 1 = p^2$, or $xy = p^2 - 1$, which applies easily to any value of p; and from this the third number also is found, in two ways, since we have either $z = x + y + 2p$, or $z = x + y - 2p$. Let us illustrate these results by some examples.

1. Let $p = 3$, and we shall have $p^2 - 1 = 8$; if we make $x = 2$, and $y = 4$, we shall have either $z = 12$, or $z = 0$; so that the three numbers sought are 2, 4, and 12.

2. If $p = 4$, we shall have $p^2 - 1 = 15$. Now, if $x = 5$, and $y = 3$, we find $z = 16$, or $z = 0$; wherefore the three numbers sought are 3, 5, and 16.

3. If $p = 5$, we shall have $p^2 - 1 = 24$; and if we farther make $x = 3$, and $y = 8$, we find $z = 21$, or $z = 1$; whence the following numbers result; 1, 3, and 8; or 3, 8, and 21.

232. *Question* 11. Required three whole numbers x, y, and z, such, that if we add a given number, a, to each product of these numbers, multiplied two and two, we may obtain a square each time.

Here we must make squares of the three following formulæ,

$$xy + a,\ xz + a, \text{ and } yz + a.$$

Let us therefore suppose the first $xy + a = p^2$, and make $z = x + y + q$; then we shall have, for the second formula, $x^2 + xy + xq + a = x^2 + xq + p^2$; and, for the third, $xy + y^2 + yq + a = y^2 + qy + p^2$; and these both become squares by making $q = \pm 2p$: so that $z = x + y \pm 2p$; that is to say, we may find two different values for z.

233. *Question* 12. Required four whole numbers, x, y, z, and v, such, that if we add a given number, a, to the products of these numbers, multiplied two by two, each of the sums may be a square.

Here, the six following formulæ must become squares:

$$1.\ xy + a,\quad 2.\ xz + a,\quad 3.\ yz + a,$$
$$4.\ xv + a,\quad 5.\ yv + a,\quad 6.\ zv + a.$$

If we begin by supposing the first $xy + a = p^2$, and take $z = x + y + 2p$, the second and third formulæ will become squares. If we farther suppose $v = x + y - 2p$, the fourth and fifth formulæ will likewise become squares; there remains therefore only the sixth formula, which will be $x^2 + 2xy + y^2 - 4p^2 + a$, and which must also become a square. Now, as $p^2 = xy + a$, this last formula becomes $x - 2xy + y^2 - 3a$; and, consequently, it is required to transform into squares the two following formulæ:

$$xy + a = p^2, \text{ and } (x - y)^2 - 3a.$$

If the root of the last be $(x - y) - q$, we shall have $(x - y)^2 - 3a = (x - y)^2 - 2q(x - y) + q^2$; so that

$$-3a = -2q(x - y) + q^2, \text{ and } x - y = \frac{q^2 + 3a}{2q}, \text{ or}$$

$$x = y + \frac{q^2 + 3a}{2q};\ \text{consequently, } p^2 = y^2 + \frac{q^2 + 3a}{2q}y + a.$$

If $p = y + r$, we shall have

$$2ry + r^2 = \frac{q^2+3a}{2q}y + a, \text{ or}$$

$$4qry + 2qr^2 = (q^2 + 3a)y + 2aq, \text{ or}$$
$$2qr^2 - 2aq = (q^2 + 3a)y - 4qry, \text{ and}$$

$$y = \frac{2qr^2-2aq}{q^2+3a-4qr},$$

where q and r may have any values, provided r and y become whole numbers; for since $p = y + r$, the numbers, z and v, will likewise be integers. The whole depends therefore chiefly on the nature of the number a, and it is true that the condition which requires integer numbers might cause some difficulties; but it must be remarked, that the solution is already much restricted on the other side, because we have given the letters, z and v, the values $x + y \pm 2p$, notwithstanding they might evidently have a great number of other values. The following observations, however, on this question, may be useful also in other cases.

1. When $xy + a$ must be a square, or $xy = p^2 - a$, the numbers x and y must always have the form $r^2 - as^2$ (Art. 176); if, therefore, we suppose

$$x = b^2 - ac^2, \text{ and } y = d^2 - ae^2,$$

we find $xy = (bd - ace)^2 - a(be - cd)^2$.

If $be - cd = \pm 1$, we shall have $xy = (bd - ace)^2 - a$, and, consequently, $xy + a = (bd - ace)^2$.

2. If we farther suppose $z = f^2 - ag^2$, and give such values to f and g, that $bg - cf = \pm 1$, and also $dg - ef = \pm 1$, the formulæ $xz + a$, and $yz + a$, will likewise become squares. So that the whole consists in giving such values to b, c, d, and e, and also to f and g, that the property which we have supposed may take place.

3. Let us represent these three couples of letters by the fractions $\frac{b}{c}$, $\frac{d}{e}$, and $\frac{f}{g}$; now, they ought to be such, that the difference of any two of them may be expressed by a fraction, whose numerator is 1. For since

$\frac{b}{c} - \frac{d}{e} = \frac{be-dc}{ce}$, this numerator, as has been seen, must be equal to ± 1. Besides, one of these fractions is arbitrary; and it is easy to find another, in order that the given condition may take place. For example, let the first

$\frac{b}{c} = \frac{3}{2}$, the second $\frac{d}{e}$ must be nearly equal to it; if, therefore, we make $\frac{d}{e} = \frac{4}{3}$, we shall have the difference $z = \frac{1}{6}$. We may also determine this second fraction by means of the first, generally; for since $\frac{3}{2} - \frac{d}{e} = \frac{3e-2d}{2e}$, we must have $3e - 2d = 1$, and, consequently, $2d = 3e - 1$, and $d = e + \frac{e-1}{2}$. So that making $\frac{e-1}{2} = m$, or $e = 2m + 1$, we shall have $d = 3m + 1$, and our second fraction will be $\frac{d}{e} = \frac{3m+1}{2m+1}$. In the same manner, we may determine the second fraction for any first whatever, as in the following Table of examples:

$\frac{b}{c} = \frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{3}$	$\frac{8}{5}$	$\frac{11}{4}$	$\frac{13}{8}$	$\frac{17}{7}$
$\frac{d}{e} = \frac{3m+1}{2m+1}$	$\frac{5m+1}{3m+1}$	$\frac{7m+2}{3m+1}$	$\frac{8m+3}{5m+2}$	$\frac{11m+3}{4m+1}$	$\frac{13m+5}{8m+3}$	$\frac{17m+5}{7m+2}$

4. When we have determined, in the manner required, the two fractions, $\frac{b}{c}$, and $\frac{d}{e}$, it will be easy to find a third also analogous to these. We have only to suppose $f = b + d$, and $g = c + e$, so that $\frac{f}{g} = \frac{b+d}{c+e}$; for the two first giving $be - cd = \pm 1$, we have $\frac{f}{g} - \frac{b}{c} = \frac{\pm 1}{c^2 + ce}$; and subtracting likewise the second from the third, we shall have

$$\frac{f}{g} - \frac{d}{e} = \frac{be - cd}{e^2 + ce} = \frac{\pm 1}{ce + e^2}.$$

5. After having determined in this manner the three fractions, $\frac{b}{c}$, $\frac{d}{e}$, and $\frac{f}{g}$, it will be easy to resolve our question for three numbers, x, y, and z, by making the three formulæ $xy + a$, $xz + a$, and $yz + a$, become squares: since we have only to make $x = b^2 - ac^2$, $y = d^2 - ae^2$, and $z = f^2 - ag^2$. For example, in the foregoing Table,

let us take $\frac{b}{c} = \frac{5}{3}$, and $\frac{d}{e} = \frac{7}{4}$, we shall then have $\frac{f}{g} = \frac{12}{7}$; whence $x = 25 - 9a$, $y = 49 - 16a$, and $z = 144 - 49a$; by which means we have

1. $xy + a = 1225 - 840a + 144a^2 = (35 - 12a)^2$;
2. $xz + a = 3600 - 2520a + 441a^2 = (60 - 21a)^2$;
3. $yz + a = 7056 - 4704a + 784a^2 = (84 - 28a)^2$.

234. In order now to determine, according to our question, four letters, x, y, z, and v, we must add a fourth fraction to the three preceding: therefore let the first three be $\frac{b}{c}$, $\frac{d}{e}$, $\frac{f}{g} = \frac{b+d}{c+c}$, and let us suppose the fourth fraction $\frac{h}{k} = \frac{b+d}{e+g} = \frac{2d+b}{2e+c}$, so that it may have the given relation with the third and second; if after this we make $x = b^2 - ac^2$, $y = d^2 - ae^2$, $z = f^2 - ag^2$, and $v = h^2 - ak^2$, we shall have already fulfilled the following conditions:

$$xy + a = \square, \quad xz + a = \square, \quad yz + a = \square,$$
$$yv + a = \square, \quad zy + a = \square.$$

It therefore only remains to make $xv + a$ become a square, which does not result from the preceding conditions, because the first fraction has not the necessary relation with the fourth. This obliges us to preserve the indeterminate number m in the three first fractions; by means of which, and by determining m, we shall be able also to transform the formula $xv + a$ into a square.

6. If we therefore take the first case from our small Table, and make $\frac{b}{c} = \frac{3}{2}$, and $\frac{d}{e} = \frac{3m+1}{2m+1}$; we shall have $\frac{f}{g} = \frac{3m+4}{2m+3}$, and $\frac{h}{k} = \frac{6m+5}{4m+4}$, whence $x = 9 - 4a$, and $v = (6m + 5)^2 - a(4m + 4)^2$;

so that $xv + a = \begin{cases} 9(6m + 5)^2 - 4a(6m + 5)^2 \\ -9a(4m + 4)^2 + 4a^2(4m + 4)^2 \end{cases}$

or $xv + a = \begin{cases} 9(6m + 5)^2 + 4a^2(4m + 4)^2 \\ -a(288m^2 + 528m + 244), \end{cases}$

which we can easily transform into a square, since m^2 will be found to be multiplied by a square; but on this we shall not dwell.

7. The fractions, which have been found to be neces-

sary, may also be represented in a more general manner; for if $\frac{b}{c} = \frac{\beta}{1}$, $\frac{d}{e} = \frac{n\beta - 1}{n}$, we shall have

$\frac{f}{g} = \frac{n\beta + \beta - 1}{n + 1}$, and $\frac{g}{h} = \frac{2n\beta + \beta - 2}{2n + 1}$; if in this last fraction we suppose $2n + 1 = m$, it will become $\frac{\beta m - 2}{m}$; consequently, the first gives $x = \beta^2 - a$, and the last furnishes $v = (\beta m - 2)^2 - am^2$. The only question therefore is, to make $xv + a$ a square. Now, because

$$v = (\beta^2 - a)m^2 - 4\beta m + 4, \text{ we have}$$

$xv + a = (\beta^2 - a)^2m^2 - 4(\beta^2 - a)\beta m + 4\beta^2 - 3a$; and since this must be a square, let us suppose its root to be $(\beta^2 - a)m - p$; the square of which quantity being $(\beta^2 - a)^2m^2 - 2(\beta^2 - a)mp + p^2$, we shall have
$-4(\beta^2 - a)\beta m + 4\beta^2 - 3a = -2(\beta^2 - a)mp + p^2$; wherefore
$m = \frac{p^2 - 4\beta^2 + 3a}{(\beta^2 - a) \times (2p - 4\beta)}$. If $p = 2\beta + q$, we shall find
$m = \frac{4\beta q + q^2 + 3a}{2q(\beta^2 - a)}$, in which we may substitute any numbers whatever for β and q.

For example, if $a = 1$, let us make $\beta = 2$: we shall then have $m = \frac{4q + q^2 + 3}{6q}$; and making $q = 1$, we shall find $m \frac{4}{3}$; farther, $m = 2n + 1$; but without dwelling any longer on this question, let us proceed to another.

235. *Question* 12. Required three such numbers, x, y, and z, that the sums and differences of these numbers, taken two by two, may be squares.

The question requiring us to transform the six following formulæ into squares, *viz.*

$$x + y, \quad x + z, \quad y + z,$$
$$x - y, \quad x - z, \quad y - z,$$

let us begin with the last three, and suppose $x - y = p^2$, $x - z = q^2$, and $y - z = r^2$; the last two will furnish $x = q^2 + z$, and $y = r^2 + z$; so that we shall have $q^2 = p^2 + r^2$, because $x - y = q^2 - r^2 = p^2$; hence, $p^2 + r^2$, or the sum of two squares, must be equal to a square q^2; now, this happens, when $p = 2ab$, and $r = a^2 - b^2$, since then $q = a^2 + b^2$. But let us still preserve the letters p, q, and r, and consider also the first three formulæ. We shall have,

$$1.\ x + y = q^2 + r^2 + 2z;$$
$$2.\ x + z = q^2 + 2z;$$
$$3.\ y + z = r^2 + 2z.$$

Let the first $q^2+r^2+2z, =t^2$, by which means $2z=t^2-q^2-r^2$; we must also have $t^2 - r^2 = \square$, and $t^2 - q^2 = \square$; that is to say, $t^2 - (a^2 - b^2)^2 = \square$, and $t^2 - (a^2 + b^2)^2 = \square$; we shall have to consider the two formulæ $t^2 - a^4 - b^4+2a^2b^2$, and $t^2 - a^4 - b^4 - 2a^2b^2$. Now, as both $c^2 + d^2 + 2cd$, and $c^2 + d^2 - 2cd$, are squares, it is evident that we shall obtain what we want by comparing $t^2 - a^4 - b^4$, with $c^2 + d^2$, and $2a^2b^2$ with $2cd$. With this view, let us suppose $cd = a^2b^2 = f^2g^2h^2k^2$, and take $c = f^2g^2$, and $d = h^2k^2$; $a^2 = f^2h^2$, and $b^2 = g^2k^2$, or $a = fh$, and $b = gk$; the first equation $t^2 - a^4 - b^4 = c^2 + d^2$, will assume the form $t^2 - f^4h^4 - g^4k^4 = f^4g^4 + h^4k^4$; whence
$t^2 = f^4g^4 + f^4h^4 - g^4k^4 + h^4k^4$, or $t^2 = (f^4+k^4) \times (g^4+h^4)$; consequently, this product must be a square; but as the resolution of it would be difficult, let us consider the subject under a different point of view.

If from the first three equations $x - y = p^2$, $x - z = q^2$, $y - z = r^2$, we determine the letters y and z, we shall find $y = x - p^2$, and $z = x - q^2$; whence it follows that $q^2 = p^2 + r^2$. Our first formulæ now become $x+y=2x-p^2$, $x + z = 2x - q^2$, and $y + z = 2x - p^2 - q^2$. Let us make this last $2x - p^2 - q^2 = t^2$, so that $2x = t^2 + p^2 + q^2$, and there will only remain the formulæ $t^2 + q^2$, and $t^2 + p^2$, to transform into squares. But since we must have $q^2 = p^2 + r^2$, let $q = a^2 + b^2$, and $p = a^2 - b^2$; and we shall then have $r = 2ab$, and, consequently, our formulæ will be:

$$1.\ t^2 + (a^2 + b^2)^2 = t^2 + a^4 + b^4 + 2a^2b^2 = \square;$$
$$2.\ t^2 + (a^2 - b^2)^2 = t^2 + a^4 + b^4 - 2a^2b^2 = \square.$$

In order to accomplish our purpose, we have only to compare again $t^2 + a^4 + b^4$ with $c^2 + d^2$, and $2a^2b^2$, with $2cd$. Therefore, as before, let $c = f^2g^2$, $d = h^2k^2$, $a=fh$, and $b = gk$; we shall then have $cd = a^2b^2$, and we must again have

$t^2 + f^4h^4 + g^4k^4 = c^2 + d^2 = f^4g^4 + h^4k^4$; whence
$t^2 = f^4g^4 - f^4h^4 + h^4k^4 - g^4k^4 = (f^4 - k^4) \times (g^4 - h^4)$.

So that the whole is reduced to finding the differences of two pair of biquadrates, namely, $f^4 - k^4$, and $g^4 - h^4$, which, multiplied together, may produce a square.

For this purpose, let us consider the formula $m^4 - n^4$; let us see what numbers it furnishes, if we substitute given numbers for m and n, and attend to the squares that will be found among those numbers; the property of $m^4 - n^4 = (m^2 + n^2) \times (m^2 - n^2)$, will enable us to construct for our purpose the following Table:

A Table of Numbers contained in the Formula $m^4 - n^4$.

m^2	n^2	$m^2 - n^2$	$m^2 + n^2$	$m^4 - n^4$
4	1	3	5	3×5
9	1	8	10	16×5
9	4	5	13	5×13
16	1	15	17	$3 \times 5 \times 17$
16	9	7	25	25×7
25	1	24	26	$16 \times 3 \times 13$
25	9	16	34	$16 \times 2 \times 17$
49	1	48	50	$25 \times 16 \times 2 \times 3$
49	16	33	65	$3 \times 5 \times 11 \times 13$
64	1	63	65	$9 \times 5 \times 7 \times 13$
81	49	32	130	$64 \times 5 \times 13$
121	4	117	125	$25 \times 9 \times 5 \times 13$
121	9	112	130	$16 \times 2 \times 5 \times 7 \times 13$
121	49	72	170	$144 \times 5 \times 17$
144	25	119	169	$169 \times 7 \times 17$
169	1	168	170	$16 \times 3 \times 5 \times 7 \times 17$
169	81	88	250	$25 \times 16 \times 5 \times 11$
225	64	161	289	$289 \times 7 \times 23$

We may already deduce some answers from this. For, if $f^2 = 9$, and $k^2 = 4$, we shall have $f^4 - k^4 = 13 \times 5$; farther, let $g^2 = 81$, and $h^2 = 49$, we shall then have $g^4 - h^4 = 64 \times 5 \times 13$; therefore $t^2 = 64 \times 25 \times 169$, and $t = 520$. Now, since $t^2 = 270400$, $f = 3$, $g = 9$, $k = 2$, $h = 7$, we shall have $a = 21$, and $b = 18$; so that $p = 117$, $q = 765$, and $r = 756$; from which results $2x = t^2 + p^2 + q^2 = 869314$; consequently, $x = 434657$; then $y = x - p^2 = 420968$, and lastly, $z = x - q^2 = -150568$. This last number may also be taken positively; the difference then becomes the sum, and, reciprocally, the sum becomes the difference. Since therefore the three numbers sought are:

$$x = 434657$$
$$y = 420968$$
$$z = 150568$$

we have $x + y = 855625 = (925)^2$
$x + z = 585225 = (765)^2$
and $y + z = 571536 = (756)^2$

also, $x - y = 13689 = (117)^2$
$x - z = 284089 = (533)^2$
and $y - z = 270400 = (520)^2$.

The Table which has been given, would enable us to find other numbers also, by supposing $f^2 = 9$, and $k^2 = 4$, $g^2 = 121$, and $h^2 = 4$; for then $t^2 = 13 \times 5 \times 5 \times 13 \times 9 \times 25 = 9 \times 25 \times 25 \times 169$, and

$$t = 3 \times 5 \times 5 \times 13 = 975.$$

Now, as $f = 3$, $g = 11$, $k = 2$, and $h = 2$, we have $a = fh = 6$, and $b = gk = 22$; consequently, $p = a^2 - b^2 = - 448$, $q = a^2 + b^2 = 520$, and $r = 2ab = 264$; whence $2x = t^2 + p^2 + q^2 = 950625 + 200704 + 270400 = 1421729$, and $x = \frac{1421729}{2}$; wherefore $y = x - p^2 = \frac{1020321}{2}$, and $z = x - q^2 = \frac{880929}{2}$.

Now, it is to be observed, that if these numbers have the property required, they will preserve it by whatever square they are multiplied. If, therefore, we take them four times greater, the following numbers must be equally satisfactory: $x = 2843458$, $y = 2040642$, and $z = 1761858$; and as these numbers are greater than the former, we may consider the former as the least which the question admits of.

236. *Question* 14. Required three such squares, that the difference of every two of them may be a square.

The preceding solution will serve to resolve the present question. In fact, if x, y, and z, are such numbers that the following formulæ, namely,

$$x + y = \square, \quad x - y = \square, \quad x + z = \square,$$
$$x - z = \square, \quad y + z = \square, \quad y - z = \square,$$

may become squares; it is evident, likewise, that the product $x^2 - y^2$ of the first and second, the product $x^2 - z^2$ of the third and fourth, and the product $y^2 - z^2$ of the fifth and sixth, will be squares; and, consequently, x^2, y^2, and z^2, will be three such squares as are sought. But these numbers would be very great, and there are, doubtless, less numbers that will satisfy the question; since, in order that $x^2 - y^2$ may become a square, it is not necessary that $x + y$, and $x - y$, should be squares: for example, $25 - 9$ is a square, although neither 5 + 3, nor 5 − 3, are squares. Let us, therefore, resolve the question independently of this consideration, and remark, in the first place, that we may take 1 for one of the squares sought: the reason for which is, that if the formulæ $x^2 - y^2$, $x^2 - z^2$, and $y^2 - z^2$, are squares, they will continue so, though divided by z^2; consequently, we may suppose that the question is to transform $\left(\frac{x^2}{z^2} - \frac{y^2}{z^2}\right)$, $\left(\frac{x^2}{z^2} - 1\right)$, and $\left(\frac{y^2}{z^2} - 1\right)$ into squares, and it then refers only to the two fractions $\frac{x}{z}$, and $\frac{y}{z}$.

If we now suppose $\frac{x}{z} = \frac{p^2+1}{p^2-1}$, and $\frac{y}{z} = \frac{q^2+1}{q^2-1}$, the last two conditions will be satisfied; for we shall then have $\frac{x^2}{z^2} - 1 = \frac{4p^2}{(p^2-1)^2}$, and $\frac{y^2}{z^2} - 1 = \frac{4q^2}{(q^2-1)^2}$. It only remains, therefore, to consider the first formula

$$\frac{x^2}{z^2} - \frac{y^2}{z^2} = \frac{(p^2+1)^2}{(p^2-1)^2} - \frac{(q^2+1)^2}{(q^2-1)^2} =$$

$$\left(\frac{p^2+1}{p^2-1} + \frac{q^2+1}{q^2-1}\right) \times \left(\frac{p^2+1}{p^2-1} - \frac{q^2+1}{q^2-1}\right).$$

Now, the first factor here is $\frac{2(p^2q^2-1)}{(p^2-1) \times (q^2-1)}$; the second is $\frac{2(q^2-p^2)}{(p^2-1) \times (q^2-1)}$, and the product of these two factors is $= \frac{4(p^2q^2-1) \times (q^2-p^2)}{(p^2-1) \times (q^2-1)^2}$. It is evident that the denominator of this product is already a square, and that the numerator contains the square 4; therefore it is only required to transform into a square the formula $(p^2q^2 - 1) \times (q^2 - p^2)$, or $(p^2q^2 - 1) \times (\frac{q^2}{p^2} - 1)$; and this is done by making $pq = \frac{f^2+g^2}{2fg}$, and $\frac{q}{p} = \frac{h^2+k^2}{2hk}$, because then each factor separately becomes a square. We may also be convinced of this, by remarking that $pq \times \frac{q}{p} = q^2 = \frac{f^2+g^2}{2fg} \times \frac{h^2+k^2}{2hk}$; and, consequently, the product of these two fractions must be a square; as it must also be when multiplied by

$4f^2g^2 \times h^2k^2$, by which means it becomes equal to $fg(f^2 + g^2) \times hk(h^2 + k^2)$. Lastly, this formula becomes precisely the same as that before found, if we make $f = a + b$, $g = a - b$, $h = c + d$, and $k = c - d$; since we have then

$$2(a^4 - b^4) \times 2(c^4 - d^4) = 4 \times (a^4 - b^4) \times (c^4 - d^4),$$

which takes place, as we have seen, when $a^2 = 9$, $b^2 = 4$, $c^2 = 81$; and $d^2 = 49$, or $a = 3$, $b = 2$, $c = 9$, and $d = 7$. Thus, $f = 5$, $g = 1$, $h = 16$, and $k = 2$, whence $pq = \frac{13}{5}$, and $\frac{q}{p} = \frac{260}{64} = \frac{65}{16}$; the product of these two equations gives $q = \frac{65 \times 13}{16 \times 5} = \frac{13 \times 13}{16}$; wherefore $q = \frac{13}{4}$, and it follows that $p = \frac{4}{5}$, by which means we have

$\frac{x}{z} = \frac{p^2+1}{p^2-1} = -\frac{41}{9}$, and $\frac{y}{z} = \frac{q^2+1}{q^2-1} = \frac{185}{153}$; therefore, since $x = -\frac{41z}{9}$, and $y = \frac{185z}{153}$, in order to obtain whole numbers, let us make $z = 153$, and we shall have $x = -697$, and $y = 185$.

Consequently, the three square numbers sought are,

$$\left.\begin{array}{l} x^2 = 485809 \\ y^2 = \ \ 34225 \\ z^2 = \ \ 23409 \end{array}\right\} \text{ and } \left\{\begin{array}{l} x^2 - y^2 = 451584 = (672)^2 \\ y^2 - z^2 = \ \ 10816 = (104)^2 \\ x^2 - z^2 = 462400 = (680)^2. \end{array}\right.$$

It is farther evident, that these squares are much less than those which we should have found, by squaring the three numbers x, y, and z of the preceding solution.

237. Without doubt it will here be objected, that this solution has been found merely by trial, since we have made use of the Table in Article 235. But in reality we have only made use of this, to get the least possible numbers; for if we were indifferent with regard to brevity in the calculation, it would be easy, by means of the rules above given, to find an infinite number of solutions; because, having found $\frac{x}{z} = \frac{p^2+1}{p^2-1}$, and $\frac{y}{z} = \frac{q^2+1}{q^2-1}$, we have reduced the question to that of transforming the product $(p^2q^2 - 1) \times (\frac{q^2}{p^2} - 1)$ into a square. If we therefore make $\frac{q}{p} = m$, or $q = mp$, our formula will become $(m^2p^4 - 1) \times (m^2 - 1)$, which is evidently a square, when $p = 1$; but we shall farther see,

that this value will lead us to others, if we write $p = 1 + s$; in consequence of which supposition, we have to transform the formula

$$(m^2 - 1) \times (m^2 - 1 + 4ms^2 + 6m^2s^2 + 4m^2s^3 + m^2s^4)$$

into a square; it will be no less a square, if we divide it by $(m^2 - 1)^2$; this division gives us

$$1 + \frac{4m^2s}{m^2-1} + \frac{6m^2s^2}{m^2-1} + \frac{4m^2s^3}{m^2-1} + \frac{m^2s^4}{m^2-1};$$

and if to abridge we make $\frac{m^2}{m^2-1} = a$, we shall have to reduce the formula $1 + 4as + 6as^2 + 4as^3 + as^4$ to a square. Let its root be $1 + fs + gs^2$, the square of which is $1 + 2fs + 2gs^2 + f^2s^2 + 2fgs^3 + g^2s^4$, and let us determine f and g in such a manner, that the first three terms may vanish; namely, by making $2f = 4a$, or $f = 2a$, and $6a = 2g + f^2$, or $g = \frac{6a - f^2}{2} = 3a - 2a^2$, the last two terms will furnish the equation $4a + as = 2fg + g^2s$; whence $s = \frac{4a - 2fg}{g^2 - a} = \frac{4a - 12a + 8a^3}{4a^4 - 12a^3 + 9a^2 - a} =$

$\frac{4 - 12a + 8a^2}{4a^3 - 12a^2 + 9a - 1}$, or, dividing by $a - 1$, $s = \frac{4(2a-1)}{4a^2 - 8a + 1}$.

This value is already sufficient to give us an infinite number of answers, because the number m, in the value of a, $= \frac{m^2}{m^2-1}$, may be taken at pleasure. It will be proper to illustrate this by some examples.

1. Let $m = 2$, we shall have $a = \frac{4}{3}$; so that

$s = 4 \times \frac{\frac{5}{3}}{-\frac{23}{9}} = -\frac{60}{23}$; whence $p = -\frac{37}{23}$, and $q = -\frac{74}{23}$;

lastly, $\frac{x}{z} = \frac{949}{420}$, and $\frac{y}{z} = \frac{6005}{4947}$.

2. If $m = \frac{3}{2}$, we shall have $a = \frac{9}{5}$, and

$s = 4 \times \frac{\frac{13}{5}}{-\frac{11}{25}} = -\frac{260}{11}$; consequently, $p = -\frac{249}{11}$, and $q = -\frac{747}{22}$, by which means we may determine the fractions

$$\frac{x}{z}, \text{ and } \frac{y}{z}.$$

There is here a particular case that deserves to be attended to; which is that in which a is a square, and takes place, for example, when $m = \frac{5}{3}$; since then $a = \frac{25}{16}$. If here again, in order to abridge, we make $a = b^2$, so that our formula may be $1 + 4b^2s + 6b^2s^2 + 4b^2s^3 + b^2s^4$, we may compare it with the square of $1 + 2b^2s + bs^2$, that is to say, with $1 + 4b^2s + 2bs^2 + 4b^4s^2 + 4b^3s^3 + b^2s^4$; and expunging on both sides the first two terms and the last, and dividing the rest by s^2, we shall have $6b^2 + 4b^2s = 2b + 4b^4 + 4b^3s$, whence $s = \frac{6b^2 - 2b - 4b^4}{4b^3 - 4b^2} = \frac{3b - 1 - 2b^3}{2b^2 - 2b}$; but this fraction being still divisible by $b - 1$, we shall, at last, have $s = \frac{1 - 2b - 2b^2}{2b}$, and $p = \frac{1 - 2b^2}{2b}$.

We might also have taken $1 + 2bs + bs^2$ for the root of our formula; the square of this trinomial being $1 + 4bs + 2bs^2 + 4b^2s^2 + 4b^2s^3 + b^2s^4$, we should have destroyed the first, and the last two terms; and dividing the rest by s, we should have been brought to the equation $4b^2 + 6b^2s = 4b + 2bs + 4b^2s$. But as $b^2 = \frac{25}{16}$, and $b = \frac{5}{4}$, this equation would have given us $s = -2$, and $p = -1$; consequently, $p^2 - 1 = 0$, from which we could not have drawn any conclusion, since we should have had $z = 0$.

To return then to the former solution, which gave $p = \frac{1 - 2b^2}{2b}$; as $b = \frac{5}{4}$, it shews us that if $m = \frac{5}{3}$, we have $p = \frac{17}{20}$, and $q = mp = \frac{17}{12}$; consequently, $\frac{x}{z} = \frac{689}{111}$, and $\frac{y}{x} = \frac{433}{143}$.

238. *Question* 15. Required three square numbers such, that the sum of every two of them may be a square.

Since it is required to transform the three formulæ $x^2 + y^2$, $x^2 + z^2$, and $y^2 + z^2$ into squares, let us divide them by z^2, in order to have the three following,

$$\frac{x^2}{z^2} + \frac{y^2}{z^2} = \square, \quad \frac{x^2}{z^2} + 1 = \square, \quad \frac{y^2}{z^2} + 1 = \square.$$

The last two are answered, by making $\frac{x}{z} = \frac{p^2 - 1}{2p}$, and $\frac{y}{z} = \frac{q^2 - 1}{2q}$, which also changes the first formula into this, $\frac{(p^2 - 1)^2}{4p^2} + \frac{(q^2 - 1)^2}{4q^2}$, which ought also to continue a square after being multiplied by $4p^2q^2$; that is, we must have $q^2(p^2-1)^2+p^2(q^2-1)^2=\square$. Now, this can scarcely be obtained, unless we previously know a case in which this formula becomes a square: and as it is also difficult to find such a case, we must have recourse to other artifices, some of which we shall now explain.

1. As the formula in question may be expressed thus, $q^2(p+1)^2\times(p-1)^2+p^2(q+1)^2\times(q-1)^2=\square$, let us make it divisible by the square $(p+1)^2$; which may be done by making $q-1=p+1$, or $q=p+2$; for then $q+1=p+3$, and the formula becomes
$(p+2)^2\times(p+1)^2\times(p-1)^2+p^2(p+3)^2\times(p+1)^2=\square$; so that dividing by $(p+1)^2$, we have $(p+2)^2\times(p-1)^2+p^2(p+3)^2$, which must be a square, and to which we may give the form $2p^4+8p^3+6p^2-4p+4$. Now, the last term here being a square, let us suppose the root of the formula to be $2+fp+gp^2$, or gp^2+fp+2, the square of which is $g^2p^4+2fgp^3+4gp^2+f^2p^2+4fp+4$, and we shall destroy the last three terms, by making $4f=-4$, or $f=-1$, and $4g+1=6$, or $g=\frac{5}{4}$; also the first terms being divided by p^3, will give $2p+8=g^2p+2fg=\frac{25}{16}p-\frac{5}{2}$;

or $p=-24$, and $q=-22$; whence $\frac{x}{z}=\frac{p^2-1}{2p}=-\frac{575}{48}$;

or $x=-\frac{575}{48}z$, and $\frac{y}{z}=\frac{q^2-1}{2q}=-\frac{483}{44}$, or $y=-\frac{483}{44}z$.

Let us now make $z=16\times3\times11$; we shall then have $x=575\times11$, and $y=483\times12$; and, consequently, the roots of the three squares sought will be:

$$x=6325=11\times23\times25;$$
$$y=5796=12\times21\times23;$$
$$\text{and } z=528=3\times11\times16;$$

for from these result,

$$x^2+y^2=23^2(275^2+252^2)=23^2\times373^2.$$
$$x^2+z^2=11^2(575^2+48^2)=11^2\times577^2.$$
$$\text{and } y^2+z^2=12^2(483^2+44^2)=12^2\times485^2.$$

2. We may also make our formula divisible by a square, in an infinite number of ways; for example, if we suppose $(q+1)^2=4(p+1)^2$, or $q+1=2(p+1)$, that is to say, $q=2p+1$, and $q-1=2p$, the formula will become $(2p+1)^2\times(p+1)^2\times(p-1)^2+p^2\times4(p+1)^2\times4p^2=\square$; which may be divided by $(p+1)^2$, by which means we have $(2p+1)^2\times(p-1)^2+16p^4=\square$, or
$20p^4-4p^3-3p^2+2p+1=\square$; but from this we derive nothing.

3. Let us then rather make $(q-1)^2 = 4(p+1)^2$, or $q-1 = 2(p+1)$; we shall then have $q = 2p+3$, and $q+1 = 2p+4$, or $q+1 = 2(p+2)$, and after having divided our formula by $(p+1)^2$, we shall obtain the following; $(2p+3)^2 \times (p-1)^2 + 16p^2(p+2)^2$, or $9 - 6p + 53p^2 + 68p^3 + 20p^4$. Let its root be $3-p+gp^2$, the square of which is $9 - 6p + 6gp^2 + p^2 - 2gp^3 + g^2p^4$; the first two terms vanish, and we may destroy the third by making $6g + 1 = 53$, or $g = \frac{26}{3}$; so that the other terms are divisible by p, and give $20p + 68 = g^2p - 2g$, or $\frac{496}{9}p = \frac{256}{3}$; therefore $p = \frac{48}{31}$, and $q = \frac{189}{31}$, by which means we obtain a new solution.

4. If we make $q - 1 = \frac{4}{3}(p-1)$, we have $q = \frac{4}{3}p - \frac{1}{3}$, and $q + 1 = \frac{4}{3}p + \frac{2}{3} = \frac{2}{3}(2p+1)$, and the formula, after being divided by $(p-1)^2$, becomes

$\left(\frac{4p-1}{9}\right)^2 \times (p+1)^2 + \frac{64}{81}p^2(2p+1)^2$; multiplying by 81, we have $9(4p-1)^2 \times (p+1)^2 + 64p^2(2p+1)^2 =$

$$400p^4 + 472p^3 + 73p^2 - 54p + 9,$$

in which the first and last terms are both squares. If, therefore, we suppose the root to be $20p^2 - 9p + 3$, the square of which is $400p^4 - 360p^3 + 120p^2 + 81p^2 - 54p + 9$, we shall have $472p + 73 = -360p + 201$; wherefore $p = \frac{2}{13}$, and $q = \frac{8}{39} - \frac{1}{3} = -\frac{5}{39}$.

We might likewise have taken for the root $20p^2+9p-3$, the square of which is $400p^4+360p^3-120p^2+81p^2-54p+9$; but comparing this square with our formula, we should have found $472p + 73 = 360p - 39$, and consequently $p = -1$, a value which can be of no use to us.

5. We may also make our formula divisible by the two squares, $(p+1)^2$, and $(p-1)^2$, at the same time. For this purpose, let us make $q = \frac{pt+1}{p+t}$; so that

$$q+1 = \frac{pt+p+t+1}{p+t} = \frac{(p+1)\times(t+1)}{p+t}, \text{ and}$$

$$q-1 = \frac{pt-p-t+1}{p+t} = \frac{(p-1)\times(t-1)}{p+t}:$$

this formula will be divisible by $(p+1)^2 \times (p-1)^2$, and will be reduced to $\frac{(pt+1)^2}{(p+t)^2} + \frac{(t+1)^2\times(t-1)^2}{(p+t)^4} \times p^2$. If we multiply by $(p+t)^4$, the formula, as before, must be transformable into a square, and we shall have

$$(pt+1)^2 \times (p+t)^2 + p^2(t+1)^2 \times (t-1)^2, \text{ or}$$

$t^2p^4+2t(t^2+1)p^3+2t^2p^2+(t^2+1)^2p^2+(t^2-1)^2p^2+2t(t^2+1)p+t^2$ in which the first and the last terms are squares. Let us therefore take for the root $tp^2+(t^2+1)p-t$, the square of which is

$t^2p^4+2t(t^2+1)p^3-2t^2p^2+(t^2+1)^2p^2-2t(t^2+1)p+t^2$, and we shall have, by comparing,

$$2t^2p+(t^2+1)^2p+2t(t^2+1)+(t^2-1)^2p=$$
$$-2t^2p+(t^2+1)^2p-2t(t^2+1)\text{, or, by subtraction,}$$
$$4t^2p+4t(t^2+1)+(t^2-1)^2p=0\text{, or}$$
$$(t^2+1)^2p^*+4t(t^2+1)=0,$$

that is to say, $t^2+1=\frac{-4t}{p}$; whence $p=\frac{-4t}{t^2+1}$; consequently, $pt+1=\frac{-3t^2+1}{t^2+1}$, and $p+t=\frac{t^3-3t}{t^2+1}$; lastly, $q=\frac{-3t^2+1}{t^3-3t}$, where the value of the letter t is arbitrary.

For example, let $t=2$; we shall then have $p=\frac{-8}{5}$ and $q=\frac{-11}{2}$; so that $\frac{x}{z}=\frac{p^2-1}{2p}=+\frac{39}{80}$, and $\frac{y}{z}=\frac{q^2-1}{2q}=\frac{117}{44}$, or $x=\frac{3\times13}{4\times4\times5}z$, and $y=\frac{9\times13}{4\times11}z$.

Farther, if $x=3\times11\times13$, we have

$$y=4\times5\times9\times13\text{, and}$$
$$z=4\times4\times5\times11,$$

and the roots of the three squares sought are

$$x=3\times11\times13=429,$$
$$y=4\times5\times9\times13=2340\text{, and}$$
$$z=4\times4\times5\times11=880:$$

where it is evident that these are still less than those found above, from which we derive

$$x^2+y^2=3^2\times13^2(121+3600)=3^2\times13^2\times61^2,$$
$$x^2+z^2=11^2\times(1521+6400)=11^2\times89^2,$$
$$y^2+z^2=20^2\times(13689+1936)=20^2\times125^2.$$

6. The last remark we shall make on this question is, that each answer easily furnishes a new one; for when we have

* Thus, $(t^2-1)^2=t^4-2t^2+1$, which multiplied by p becomes pt^4-2pt^2+p,

Then adding $4pt^2$

We have $pt^4+2pt^2+p=(t^2+1)^2p$, as above.

found three values, $x = a, y = b$, and $z = c$, so that $a^2 + b^2 = \square$, $a^2 + c^2 = \square$, and $b^2 + c^2 = \square$, the three following values will likewise be satisfactory, namely, $x = ab$, $y = bc$, and $z = ac$. Then we must have

$$x^2 + y^2 = a^2b^2 + b^2c^2 = b^2(a^2 + c^2) = \square,$$
$$x^2 + z^2 = a^2b^2 + a^2c^2 = a^2(b^2 + c^2) = \square,$$
$$y^2 + z^2 = a^2c^2 + b^2c^2 = c^2(a^2 + b^2) = \square.$$

Now, as we have just found

$$x = a = 3 \times 11 \times 13,$$
$$y = b = 4 \times 5 \times 9 \times 13, \text{ and}$$
$$z = c = 4 \times 4 \times 5 \times 11,$$

we have, therefore, according to the new solution,

$$x = ab = 3 \times 4 \times 5 \times 9 \times 11 \times 13 \times 13,$$
$$y = bc = 4 \times 4 \times 4 \times 5 \times 5 \times 9 \times 11 \times 13,$$
$$z = ac = 3 \times 4 \times 4 \times 5 \times 11 \times 11 \times 13.$$

And all these three values being divisible by

$$3 \times 4 \times 5 \times 11 \times 13,$$

are reducible to the following,

$x = 9 \times 13$, $y = 3 \times 4 \times 4 \times 5$, and $z = 4 \times 11$; or $x = 117$, $y = 240$, and $z = 44$,

which are still less than those which the preceding solution gave, and from them we deduce

$$x^2 + y^2 = 71289 = 267^2,$$
$$x^2 + z^2 = 15625 = 125^2,$$
$$y^2 + z^2 = 59536 = 244^2.$$

239. *Question* 16. Required two such numbers, x and y, that each being added to the square of the other, may make a square; that is, that $x^2 + y = \square$, and $y^2 + x = \square$.

If we begin with supposing $x^2 + y = p^2$, and from that deduce $y = p^2 - x^2$, we shall have for the other formula $p^4 - 2p^2x^2 + x^4 + x = \square$, which it would be difficult to resolve.

Let us, therefore, suppose one of the formulæ $x^2 + y = (p - x)^2 = p^2 - 2px + x^2$; and, at the same time, the other $y^2 + x = (q - y)^2 = q^2 - 2qy + y^2$, and we shall thus obtain the two following equations,

$$y + 2px = p^2, \text{ and } x + 2py = q^2,$$

from which we easily deduce

$$x = \frac{2qp^2 - q^2}{4pq - 1}, \text{ and } y = \frac{2pq^2 - q^2}{4pq - 1},$$

in which p and q are indeterminate. Let us, therefore, suppose, for example, $p = 2$, and $y = 3$, then we shall have for the two numbers sought $x = \frac{15}{23}$, and $y = \frac{32}{23}$, by which means $x^2 + y = \frac{225}{529} + \frac{32}{23} = \frac{961}{529} = (\frac{31}{23})^2$, and $y^2 + x = \frac{1024}{529} + \frac{15}{23} = \frac{1369}{529} = (\frac{37}{23})^2$. If we made $p = 1$, and $q = 3$, we should have $x = -\frac{3}{11}$, and $y = \frac{17}{11}$, an answer which is inadmissible, since one of the numbers sought is negative.

But let $p = 1$, and $q = \frac{3}{2}$, we shall then have $x = \frac{3}{20}$, and $y = \frac{7}{10}$, whence we derive

$$x^2 + y = \frac{9}{400} + \frac{7}{10} = \frac{289}{400} = (\tfrac{17}{20})^2, \text{ and}$$
$$y^2 + x = \frac{49}{100} + \frac{3}{20} = \frac{64}{100} = (\tfrac{8}{10})^2.$$

240. *Question* 17. To find two numbers, whose sum may be a square, and whose squares added together may make a biquadrate.

Let us call these numbers x and y; and since $x^2 + y^2$ must become a biquadrate, let us begin with making it a square: in order to which, let us suppose $x = p^2 - q^2$, and $y = 2pq$, by which means, $x^2 + y^2 = (p^2 + q^2)^2$. But, in order that this square may become a biquadrate, $p^2 + q^2$ must be a square; let us therefore make $p = r^2 - s^2$, and $q = 2rs$, in order that $p^2 + q^2 = (r^2 + s^2)^2$; and we immediately have $x^2 + y^2 = (r^2 + s^2)^4$, which is a biquadrate. Now, according to these suppositions, we have $x = r^4 - 6r^2s^2 + s^4$, and $y = 4r^3s - 4rs^3$; it therefore remains to transform into a square the formula

$$x + y = r^4 + 4r^3s - 6r^2s^2 - 4rs^3 + s^4.$$

Supposing its root to be $r^2 + 2rs + s^2$, or the formula equal to the square of this, $r^4 + 4r^3s + 6r^2s^2 + 4rs^3 + s^4$, we may expunge from both the first two terms and also s^4, and divide the rest by rs^2, so that we shall have

$$6r + 4s = -6r - 4s, \text{ or } 12r + 8s = 0; \text{ so that}$$

$s = -\dfrac{12r}{8} = -\frac{3}{2}r$. We might also suppose the root to be $r^2 - 2rs + s^2$, and make the formula equal to its square $r^4 - 4r^3s + 6r^2s^2 - 4rs^3 + s^4$; the first and the last two terms being thus destroyed on both sides, we should have, by dividing the other terms by r^2s, $4r - 6s = -4r + 6s$, or $8r = 12s$; consequently, $r = \frac{3}{2}s$; so that by this second supposition, if $r = 3$, and $s = 2$, we shall find $x = -119$, or a negative value.

But let us make $r = \frac{3}{2}s + t$, and we shall have for our formula

$$r^2 = \tfrac{9}{4}s^2 + 3st + t^2;\; r^3 = \tfrac{27}{8}s^3 + \tfrac{27}{4}s^2t + \tfrac{9}{2}st^2 + t^3.$$

Therefore $r^4 = \frac{81}{16}s^4 + \frac{27}{2}s^3t + \frac{27}{2}s^2t^2 + 6st^3 + t^4$
$+ 4r^3s = \frac{27}{2}s^4 + 27s^3t + 18s^2t^2 + 4st^3$
$- 6r^2s^2 = - \frac{27}{2}s^4 - 18s^3t - 6s^2t^2$
$- 4rs^3 = - 6s^4 - 4s^3t$
$+ s^4 = + s^4$; and, consequently, the formula will be $\frac{1}{16}s^4 + \frac{37}{2}s^3t + \frac{51}{2}s^2t^2 + 10st^3 + t^4$.

This formula ought also to be a square, if multiplied by 16, by which means it becomes

$$s^4 + 296s^3t + 408s^2t^2 + 160st^3 + 16t^4.$$

Let us make this equal to the square of $s^2 + 148st - 4t^2$, that is, to $s^4 + 296s^3t + 21896s^2t^2 - 1184st^3 + 16t^4$; the first two terms, and the last, are destroyed on both sides, and we thus obtain the equation

$$21896s - 1184t = 408s + 160t, \text{ which gives}$$

$$\frac{s}{t} = \frac{1344}{21488} = \frac{336}{5372} = \frac{84}{1343}.$$

Therefore, since $s = 84$, and $t = 1343$, we shall have $r = \frac{3}{2}s + t = 1469$, and, consequently,

$$x = r^4 - 6r^2s^2 + s^4 = 4565486027761, \text{ and}$$
$$y = 4r^3s - 4rs^3 = 1061652293520.$$

CHAP. XV.

Solutions *of some* Questions, *in which* Cubes *are required.*

241. In the preceding chapter, we have considered some questions, in which it was required to transform certain formulæ into squares, and they afforded an opportunity of explaining several artifices requisite in the application of the rules which have been given. It now remains, to consider questions, which relate to the transformation of certain formulæ into cubes; and the following solutions will throw some light on the rules, which have been already explained for transformations of this kind.

242. *Question* 1. It is required to find two cubes, x^3, and y^3, whose sum may be a cube.

Since $x^3 + y^3$ must be a cube, if we divide this formula by y^3, the quotient ought likewise to be a cube, or $\frac{x^3}{y^3} + 1 = $ c. If, therefore, $\frac{x}{y} = z - 1$, we shall have