CHAP. XI.

Of the Resolution *of the* Formula $ax^2 + bxy + cy^2$ *into its* Factors.

162. The letters x and y shall, in the present formula, represent only integer numbers; for it has been sufficiently seen, from what has been already said, that, even when we were confined to fractional results, the question may always be reduced to integer numbers. For example, if the number sought, x, be a fraction, we have only to make $x = \frac{t}{u}$, and may always assign t and u in integer numbers; and as this fraction may be reduced to its lowest terms, we shall consider the numbers t and u as having no common divisor.

Let us suppose, therefore, in the present formula, that x and y are only integer numbers, and endeavour to determine what values must be given to these letters, in order that the formula may have two or more factors. This preliminary inquiry is very necessary, before we can shew how to transform this formula into a square, a cube, or any higher power.

163. There are three cases to be considered here. The first, when the formula is really decomposed into two rational factors; which happens, as we have already seen, when $b^2 - 4ac$ becomes a square.

The second case is that in which those two factors are equal; and in which, consequently, the formula is a square.

The third case is, when the formula has only irrational factors, whether they be simply irrational, or at the same time imaginary. They will be simply irrational, when $b^2 - 4ac$ is a positive number without being a square; and they will be imaginary, if $b^2 - 4ac$ be negative.

164. If, in order to begin with the first case, we suppose that the formula is resolvible into two rational factors, we may give it this form, $(fx + gy) \times (hx + ky)$, which already contains two factors. If we then wish it to contain, in a general manner, a greater number of factors, we have only to make $fx + gy = pq$, and $hx + ky = rs$; our formula will then become equal to the product $pqrs$; and will thus necessarily contain four factors, and we may increase this number

at pleasure. Now, from these two equations we obtain a double value for x, namely, $x = \frac{pq-gy}{f}$, and $x = \frac{rs-ky}{h}$, which gives $hpq - hgy = frs - fky$; consequently, $y = \frac{frs-hpq}{fk-hg}$, and $x = \frac{kpq-grs}{fk-hg}$: but if we choose to have x and y expressed in integer numbers, we must give such values to the letters p, q, r, and s, that the numerator may be really divisible by the denominator; which happens either when p and r, or q and s, are divisible by that denominator.

165. To render all this more clear, let there be given the formula $x^2 - y^2$, which is composed of the factors $(x+y) \times (x-y)$. Now, if this formula must be resolved into a greater number of factors, we may make $x + y = pq$, and $x - y = rs$; we shall then have $x = \frac{pq+rs}{2}$, and $y = \frac{pq-rs}{2}$; but, in order that these values may become integer numbers, the two products, pq and rs, must be either both even, or both odd.

For example, let $p = 7$, $q = 5$, $r = 3$, and $s = 1$, we shall have $pq = 35$, and $rs = 3$; therefore, $x = 19$, and $y = 16$; and thence $x^2 - y^2 = 105$, which is composed of the factors $7 \times 5 \times 3 \times 1$; so that this case is attended with no difficulty.

166. The second is attended with still less; namely, that in which the formula, containing two equal factors, may be represented thus: $(fx + gy)^2$, that is, by a square, which can have no other factors than those which arise from the root $fx + gy$; for if we make $fx + gy = pqr$, the formula becomes $p^2q^2r^2$, and may consequently have as many factors as we choose. We must farther remark, that one only of the two numbers x and y is determined, and the other may be taken at pleasure; for $x = \frac{pqr-gy}{f}$; and it is easy to give y such a value as will remove the fraction.

The easiest formula to manage of this kind, is x^2; if we make $x = pqr$, the square x^2 will contain three square factors, namely p^2, q^2, and r^2.

167. Several difficulties occur in considering the third case, which is that in which our formula cannot be resolved into two rational factors; and here particular artifices are

necessary, in order to find such values for x and y, that the formula may contain two, or more factors.

We shall, however, render this inquiry less difficult by observing, that our formula may be easily transformed into another, in which the middle term is wanting; for, in fact, we have only to suppose $x=\frac{z-by}{2a}$, in order to have the following formula:

$$\frac{z^2-2byz+b^2y^2}{4a}+\frac{byz-b^2y^2}{2a}+cy^2=\frac{z^2+(4ac-b^2)y^2}{4a}$$: so that, neglecting the middle term, we shall consider the formula ax^2+cy^2, and shall seek what values we must give to x and y, in order that this formula may be resolved into factors. Here it will be easily perceived, that this depends on the nature of the numbers a and c; so that we shall begin with some determinate formulæ of this kind.

168. Let us, therefore, first propose the formula x^2+y^2, which comprehends all the numbers that are the sum of two squares, the least of which we shall set down; namely, those between 1 and 50:

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50.

Among these numbers there are evidently some prime numbers which have no divisors, namely, the following: 2, 5, 13, 17, 29, 37, 41: but the rest have divisors, and illustrate this question, namely, 'What values are we to adopt for x and y, in order that the formula x^2+y^2 may have divisors, or factors, and that it may have any number of factors?' We shall observe, farther, that we may neglect the cases in which x and y have a common divisor, because then x^2+y^2 would be divisible by the same divisor, and even by its square. For example, if $x=7p$ and $y=7q$, the sum of the squares, or

$$49p^2+49q^2=49(p^2+q^2),$$

will be divisible not only by 7, but also by 49: for which reason, we shall extend the question no farther than the formulæ, in which x and y are prime to each other.

We now easily see where the difficulty lies: for though it is evident, when the two numbers x and y are odd, that the formula x^2+y^2 becomes an even number, and, consequently, divisible by 2; yet it is often difficult to discover whether the formula have divisors or not, when one of the numbers is even and the other odd, because the formula itself in that case is also odd. We do not mention the case in which x and y

are both even, because we have already said, that these numbers must not have a common divisor.

169. The two numbers x and y must therefore be prime to each other, and yet the formula $x^2 + y^2$ must contain two or more factors. The preceding method does not apply here, because the formula is not resolvible into two rational factors; but the irrational factors, which compose the formula, and which may be represented by the product

$$(x + y\sqrt{-1}) \times (x - y\sqrt{-1}),$$

will answer the same purpose. In fact, we are certain, if the formula $x^2 + y^2$ have real factors, that these irrational factors must be composed of other factors; because, if they had not divisors, their product could not have any. Now, as these factors are not only irrational, but imaginary; and farther, as the numbers x and y have no common divisor, and therefore cannot contain rational factors; the factors of these quantities must also be irrational, and even imaginary.

170. If, therefore, we wish the formula $x^2 + y^2$ to have two rational factors, we must resolve each of the two irrational factors into two other factors; for which reason, let us first suppose

$$x + y\sqrt{-1} = (p + q\sqrt{-1}) \times)r + s\sqrt{-1});$$

and since $\sqrt{-1}$ may be taken *minus*, as well as *plus*, we shall also have

$$x - y\sqrt{-1} = (p - q\sqrt{-1}) \times (r - s\sqrt{-1}).$$

Let us now take the product of these two quantities, and we shall find our formula $x^2 + y^2 = (p^2 + q^2) \times (r^2 + s^2)$; that is, it contains the two rational factors $p^2 + q^2$, and $r^2 + s^2$.

It remains, therefore, to determine the values of x and y, which must likewise be rational. Now, the supposition we have made, gives

$$x + y\sqrt{-1} = pr - qs + ps\sqrt{-1} + qr\sqrt{-1}, \text{ and}$$
$$x - y\sqrt{-1} = pr - qs - ps\sqrt{-1} - qr\sqrt{-1}.$$

If we add these formulæ together, we shall have $x = pr - qs$; if we subtract them from each other, we find

$$2y\sqrt{-1} = 2ps\sqrt{-1} + 2qr\sqrt{-1}, \text{ or } y = ps + qr.$$

Hence it follows, if we make $x = pr - qs$, and $y = ps + qr$, that our formula $x^2 + y^2$ must have two factors, since we find $x^2 + y^2 = (p^2 + q^2) \times (r^2 + s^2)$. If, after this, a greater number of factors be required, we have only to assign, in the same manner, such values to p and q, that $p^2 + q^2$ may have two factors; we shall then have three factors in all, and the number might be augmented by this method to any length.

171. As in this solution we have found only the second powers of p, q, r, and s, we may also take these letters *minus*. If q, for example, be negative, we shall have $x = pr + qs$, and $y = ps - qr$; but the sum of the squares will be the same as before; which shews, that when a number is equal to a product, such as $(p^2 + q^2) \times (r^2 + s^2)$, we may resolve it into two squares in two ways; for we have first found $x = pr - qs$, and $y = ps - qr$, and then also

$$x = pr + qs, \text{ and } y = ps - qr.$$

For example, let $p = 3$, $q = 2$, $r = 2$, and $s = 1$: then we shall have the product $13 \times 5 = 65 = x^2 + y^2$; in which $x = 4$, and $y = 7$; or $x = 8$, and $y = 1$; since in both cases $x^2 + y^2 = 65$. If we multiply several numbers of this class, we shall also have a product, which may be the sum of two squares in a greater number of ways. For example, if we multiply together $2^2 + 1^2 = 5$, $3^2 + 2^2 = 13$, and $4^2 + 1^2 = 17$, we shall find 1105, which may be resolved into two squares in four ways, as follows:

$$1.\ 33^2 + 4^2, \quad 2.\ 32^2 + 9^2,$$
$$3.\ 31^2 + 12^2, \quad 4.\ 24^2 + 23^2.$$

172. So that among the numbers that are contained in the formula $x^2 + y^2$, are found, in the first place, those which are, by multiplication, the product of two or more numbers, prime to each other; and, secondly, those of a different class. We shall call the latter *simple factors* of the formula $x^2 + y^2$, and the former *compound factors;* then the simple factors will be such numbers as the following:

1, 2, 5, 9, 13, 17, 29, 37, 41, 49, &c.

and in this series we shall distinguish two kinds of numbers; one are prime numbers, as 2, 5, 13, 17, 29, 37, 41, which have no divisor, and are all (except the number 2), such, that if we subtract 1 from them, the remainder will be divisible by 4; so that all these numbers are contained in the expression $4n + 1$. The second kind comprehends the square numbers 9, 49, &c. and it may be observed, that the roots of these squares, namely, 3, 7, &c. are not found in the series, and that their roots are contained in the formulæ $4n - 1$. It is also evident, that no number of the form $4n - 1$ can be the sum of two squares; for since all numbers of this form are odd, one of the two squares must be even, and the other odd. Now, we have already seen, that all even squares are divisible by 4, and that the odd squares are contained in the formula $4n + 1$: if we therefore add together an even and an odd square, the sum will always have the form of $4n + 1$, and never of $4n - 1$. Farther, every prime number, which belongs to the formula $4n + 1$, is the sum of two squares; this is undoubtedly true, but it is not easy to demonstrate it*.

173. Let us proceed farther, and consider the formula $x^2 + 2y^2$, that we may see what values we must give to x and y, in order that it may have factors. As this formula is expressed by the imaginary factors $(x + y\sqrt{-2}) \times (x - y\sqrt{-2})$, it is evident, as before, that, if it have divisors, these imaginary factors must likewise have divisors. Suppose, therefore,

$$x + y\sqrt{-2} = (p + q\sqrt{-2}) \times (r + s\sqrt{-2}),$$

whence it immediately follows, that

$$x - y\sqrt{-2} = (p - q\sqrt{-2}) \times (r - s\sqrt{-2}),$$

and we shall have

$$x^2 + 2y^2 = (p^2 + 2q^2) \times (r^2 + 2s^2);$$

so that this formula has two factors, both of which have the same form. But it remains to determine the values of x and y, which produce this transformation. For this purpose, we shall consider that, since

$$x + y\sqrt{-2} = pr - 2qs + qr\sqrt{-2} + ps\sqrt{-2}, \text{ and}$$
$$x - y\sqrt{-2} = pr - 2qs - qr\sqrt{-2} - ps\sqrt{-2},$$

we have the sum $2x = 2pr - 4qs$; and, consequently, $x = pr - 2qs$: also the difference

$$2y\sqrt{-2} = 2qr\sqrt{-2} + 2ps\sqrt{-2};$$

so that $y = qr + ps$. When, therefore, our formula $x^2 + 2y^2$ has factors, they will always be numbers of the same kind as the formula; that is to say, one will have the form $p^2 + 2q^2$, and the other the form $r^2 + 2s^2$; and, in order that this may be the case, x and y may also be determined in two different ways, because q may be either positive or negative; for we shall first have $x = pr - 2qs$, and $y = ps + qr$; and, in the second place, $x = pr + 2qs$, and $y = ps - qr$.

174. This formula $x^2 + 2y^2$ comprehends therefore all the numbers which result from adding together a square and twice another square. The following is an enumeration of these numbers as far as 50;

1, 2, 3, 4, 6, 8, 9, 11, 12, 16, 17, 18, 19, 22, 24, 25, 27, 32, 33, 34, 36, 38, 41, 43, 44, 49, 50.

* The curious reader may see it demonstrated by Gauss, in his "Disquisitiones Arithmeticæ;" and by De la Grange, in the Memoirs of Berlin, 1768.

We shall divide these numbers, as before, into simple and compound; the simple, or those which are not compounded of the preceding numbers, are these: 1, 2, 3, 11, 17, 19, 25, 41, 43, 49, all which, except the squares 25 and 49, are prime numbers; and we may remark, in general, that, if a number is prime, and is not found in this series, we are sure to find its square in it. It may be observed, also, that all prime numbers contained in our formula, either belong to the expression $8n+1$, or $8n+3$; while all the other prime numbers, namely, those which are contained in the expressions $8n+5$, and $8n+7$, can never form the sum of a square and twice a square: it is farther certain, that all the prime numbers which are contained in one of the other formulæ, $8n+1$, and $8n+3$, are always resolvible into a square added to twice a square.

175. Let us proceed to the examination of the general formula x^2+cy^2, and consider by what values of x and y we may transform it into a product of factors.

We shall proceed as before; that is, we shall represent the formula by the product

$$(x+y\sqrt{-c}) \times (x-y\sqrt{-c}),$$

and shall likewise express each of these factors by two factors of the same kind; that is, we shall make

$x+y\sqrt{-c}=(p+q\sqrt{-c}) \times (r+s\sqrt{-c})$, and
$x-y\sqrt{-c}=(p-q\sqrt{-c}) \times (r-s\sqrt{-c})$; whence
$x^2+cy^2=(p^2+cq^2) \times (r^2+cs^2)$.

We see, therefore, that the factors are again of the same kind with the formula. With regard to the values of x and y, we shall readily find $x=pr+cqs$, and $y=qr-ps$; or $x=pr-cqs$, and $y=ps+qr$; and it is easy to perceive how the formula may be resolved into a greater number of factors.

176. It will not now be difficult to obtain factors for the formula x^2-cy^2; for, in the first place, we have only to write $-c$, instead of $+c$; but, farther, we may find them immediately in the following manner. As our formula is equal to the product

$$(x+y\sqrt{c}) \times (x-y\sqrt{c}),$$

let us make $x+y\sqrt{c}=(p+q\sqrt{c}) \times (rs+\sqrt{c})$, and
$x-y\sqrt{c}=(p-q\sqrt{c}) \times (r-s\sqrt{c})$, and we shall immediately have $x^2-cy^2=(p^2-cq^2) \times (r^2-cs^2)$; so that this formula, as well as the preceding, is equal to a product whose factors resemble it in form. With regard to

the values of x and y, they will likewise be found to be double; that is to say, we shall have

$x = pr + cqs$, and $y = qr + ps$; we shall also have
$x = pr - cqs$, and $y = ps - qr$. If we chose to make trial, and see whether we obtain from these values the product already found, we should have, by trying the first,

$$x^2 = p^2r^2 + 2cpqrs + c^2q^2s^2, \text{ and}$$
$$y^2 = p^2s^2 + 2pqrs + q^2r^2, \text{ or}$$
$$cy^2 = cp^2s^2 + 2cpqrs + cq^2r^2; \text{ so that}$$
$x^2 - cy^2 = p^2r^2 - cp^2s^2 + c^2q^2s^2 - cq^2r^2$, which is just the product already found, $(p^2 - cq^2) \times (r^2 - cs^2)$.

177. Hitherto we have considered the first term as without a coefficient; but we shall now suppose that term to be multiplied also by another letter, and shall seek what factors the formula $ax^2 + cy^2$ may contain.

Here it is evident that our formula is equal to the product $(x\sqrt{a} + y\sqrt{-c}) \times (x\sqrt{a} - y\sqrt{-c})$, and, consequently, that it is required to give factors also to these two factors. Now, in this a difficulty occurs; for if, according to the second method, we make

$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c}) \times (r\sqrt{a} + s\sqrt{-c}) =$
$apr - cqs + ps\sqrt{-ac} + qr\sqrt{-ac}$, and
$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c}) \times (r\sqrt{a} - s\sqrt{-c}) =$
$apr - cqs - ps\sqrt{-ac} - qr\sqrt{-ac}$, we shall have
$2x\sqrt{a} = 2apr - 2cqs$, and
$2y\sqrt{-c} = 2ps\sqrt{-ac} + 2qr\sqrt{-ac}$; that is to say, we have found both for x and for y irrational values, which cannot here be admitted.

178. But this difficulty may be removed thus: let us make

$x\sqrt{a} + y\sqrt{-c} = (p\sqrt{a} + q\sqrt{-c}) \times (r + s\sqrt{-ac}) =$
$pr\sqrt{a} - cqs\sqrt{a} + qr\sqrt{-c} + aps\sqrt{-c}$, and
$x\sqrt{a} - y\sqrt{-c} = (p\sqrt{a} - q\sqrt{-c}) \times (r - s\sqrt{-ac}) =$
$pr\sqrt{a} - cqs\sqrt{a} - qr\sqrt{-c} - aps\sqrt{-c}$. This supposition will give the following values for x and y; namely, $x = pr - cqs$, and $y = qr + aps$; and our formula, $ax^2 + cy^2$, will have the factors $(ap^2 + cq^2) \times (r^2 + acs^2)$, one of which only is of the same form with the formula, the other being different.

179. There is still, however, a great affinity between these two formulæ, or factors; since all the numbers contained in the first, if multiplied by a number contained in the second, revert again to the first. We have already seen, that two numbers of the second form $x^2 + acy^2$, which

returns to the formula $x^2 + cy^2$, and which we have already considered, if multiplied together, will produce a number of the same form.

It only remains, therefore, to examine to what formula we are to refer the product of two numbers of the first kind, or of the form $ax^2 + cy^2$.

For this purpose, let us multiply the two formulæ $(ap^2 + cq^2) \times (ar^2 + cs^2)$, which are of the first kind. It is easy to see that this product may be represented in the following manner: $(apr + cqs)^2 + ac(ps - qr)^2$. If, therefore, we suppose

$$apr + cqs = x, \text{ and } ps - qr = y,$$

we shall have the formula $x^2 + acy^2$, which is of the last kind. Whence it follows, that if two numbers of the first kind, $ax^2 + cy^2$, be multiplied together, the product will be a number of the second kind. If we represent the numbers of the first kind by I, and those of the second by II, we may represent the conclusion to which we have been led, abridged as follows:

I × I gives II; I × II gives I; II × II gives II.

And this shews much better what the result ought to be, if we multiply together more than two of these numbers; namely, that I × I × I gives I; that I × I × II gives II; that I × II × II gives I; and lastly, that II × II × II gives II.

180. In order to illustrate the preceding Article, let $a = 2$, and $c = 3$; there will result two kinds of numbers, one contained in the formula $2x^2 + 3y^2$, the other contained in the formula $x^2 + 6y^2$. Now, the numbers of the first kind, as far as 50, are

1st, 2, 3, 5, 8, 11, 12, 14, 18, 20, 21, 27, 29, 30, 32, 35, 44, 45, 48, 50;

and the numbers of the second kind, as far as 50, are

2d, 1, 4, 6, 7, 9, 10, 15, 16, 22, 24, 25, 28, 31, 33, 36, 40, 42, 49.

If, therefore, we multiply a number of the first kind, for example, 35, by a number of the second, suppose 31, the product 1085 will undoubtedly be contained in the formula $2x^2 + 3y^2$; that is, we may find such a number for y, that $1085 - 3y^2$ may be the double of a square, or $= 2x^2$: now, this happens, first, when $y = 3$, in which case $x = 23$; in the second place, when $y = 11$, so that $x = 19$; in the third place, when $y = 13$, which gives $x = 17$; and, in the fourth place, when $y = 19$, whence $x = 1$.

We may divide these two kinds of numbers, like the others, into *simple* and *compound* numbers: we shall apply this latter term to such as are composed of two or more of the smallest numbers of either kind; so that the simple numbers of the first kind will be 2, 3, 5, 11, 29; and the compound numbers of the same class will be 8, 12, 14, 18, 20, 27, 30, 32, 35, 40, 45, 48, 50, &c.

The simple numbers of the second class will be 1, 7, 31 and all the rest of this class will be compound numbers. namely, 4, 6, 9, 10, 15, 16, 22, 24, 25, 28, 33, 36, 40, 42, 49.

CHAP. XII.

Of the Transformation *of the* Formula $ax^2 + cy^2$ *into* Squares, *and higher* Powers.

181. We have seen that it is frequently impossible to reduce numbers of the form $ax^2 + cy^2$ to squares; but whenever it is possible, we may transform this formula into another, in which $a = 1$.

For example, the formula $2p^2 - q^2$ may become a square; for, as it may be represented by

$$(2p + q)^2 - 2(p + q)^2,$$

we have only to make $2p + q = x$, and $p + q = y$, and we shall get the formula $x^2 - 2y^2$, in which $a = 1$, and $c = 2$. A similar transformation always takes place, whenever such formulæ can be made squares. Thus, when it is required to transform the formula $ax^2 + cy^2$ into a square, or into a higher power, (provided it be even) we may, without hesitation, suppose $a = 1$, and consider the other cases as impossible.

182. Let, therefore, the formula $x^2 + cy^2$ be proposed, and let it be required to make it a square. As it is composed of the factors $(x + y\sqrt{-c}) \times (x - y\sqrt{-c})$, these factors must either be squares, or squares multiplied by the same number. For, if the product of two numbers, for example, pq, must be a square, we must have $p = r^2$, and $q = s^2$; that is to say, each factor is of itself a square; or $p = mr^2$, and $q = ms^2$; and therefore these factors are squares multiplied both by the same number. For which reason, let us make $x + y\sqrt{-c} = m(p + q\sqrt{-c})^2$; it will follow that $x - y\sqrt{-c} = m(p - q\sqrt{-c})^2$, and we shall have $x^2 + cy^2 = m^2(p^2 + cq^2)^2$, which is a square.