when divided by x^3, form an equation of the second degree; and x evidently cannot be expressed, except by a new irrational quantity. But if we were to suppose the root to be $k + px + qx^2 + rx^3$, its square would rise to the sixth power; and, consequently, though we should even determine p, q, and r, so as to remove the second, third, and fourth terms, there would still remain the fourth, the fifth, and the sixth powers; and, dividing by x^4, we should again have an equation of the second degree, which we could not resolve without a radical sign. This seems to indicate that we have really exhausted the subject of transforming formulæ into squares: we may now, therefore, proceed to quantities affected by the sign of the cube root.

CHAP. X.

Of the Method *of rendering rational the irrational* Formula $\sqrt[3]{(a + bx + cx^2 + dx^3)}$.

147. It is here required to find such values of x, that the formula $a + bx + cx^2 + dx^3$ may become a cube, and that we may be able to extract its cube root. We see immediately that no such solution could be expected, if the formula exceeded the third degree; and we shall add, that if it were only of the second degree, that is to say, if the term dx^3 disappeared, the solution would not be easier. With regard to the case in which the last two terms disappear, and in which it would be required to reduce the formula $a + bx$ to a cube, it is evidently attended with no difficulty; for we have only to make $a + bx = p^3$, to find at once $x = \frac{p^3 - a}{b}$.

148. Before we proceed farther on this subject, we must again remark, that when neither the first nor the last term is a cube, we must not think of resolving the formula, unless we already know a case in which it becomes a cube, whether that case readily occurs, or whether we are obliged to find it out by trial.

So that we have three kinds of formulæ to consider. One is, when the first term is a cube; and as then the formula is expressed by $f^3 + bx + cx^2 + dx^3$, we immediately perceive the known case to be that of $x = 0$. The second class comprehends the formula $a + bx + cx^2 + g^3x^3$; that is to say, the case in which the last term is a cube. The third class is composed of the two former, and comprehends the cases in which both the first term and the last are cubes.

149. *Case* 1. Let $f^3 + bx + cx^2 + dx^3$ be the proposed formula, which is to be transformed into a cube.

Suppose its root to be $f + px$; and, consequently, that the formula itself is equal to the cube

$$f^3 + 3f^2px + 3fp^2x^2 + p^3x^3;$$

as the first terms disappear of themselves, we shall determine p, so as to make the second terms also disappear; namely, by making $b = 3f^2p$, or $p = \frac{b}{3f^2}$; then the remaining terms being divided by x^2, give $c + dx = 3fp^2 + p^3x$; or $x = \frac{c - 3fp^2}{p^3 - d}$.

If the last term, dx^3, had not been in the formula, we might have simply supposed the cube root to be f, and should have then had $f^3 = f^3 + bx + cx^2$, or $b + cx = 0$, and $x = -\frac{b}{c}$; but this value would not have served to find others.

150. *Case* 2. If, in the second place, the proposed expression has this form, $a + bx + cx^2 + g^3x^3$, we may represent its cube root by $p + gx$, the cube of which is $p^3 + 3p^2gx + 3gp^2x^2 + g^3x^3$; so that the last terms destroy each other. Let us now determine p, so that the last terms but one may likewise disappear: which will be done by supposing $c = 3g^2p$, or $p = \frac{c}{3g^2}$, and the other terms will then give $a + bx = p^3 + 3gp^2x$; whence we find

$$x = \frac{a - p^3}{3gp^2 - b}.$$

If the first term, a, had been wanting, we should have contented ourselves with expressing the cube root by gx, and should have had

$$g^3x^3 = bx + cx^2 + g^3x^3, \text{ or } b + cx = 0,$$

whence $x = -\frac{b}{c}$; but this is of no use for finding other values.

151. *Case 3.* Lastly, let the formula be

$$f^3 + bx + cx^2 + g^3x^3,$$

in which the first and the last terms are both cubes. It is evident that we may consider this as belonging to either of the two preceding cases; and, consequently, that we may obtain two values of x.

But beside this, we may also represent the root by $f+gx$, and then make the formula equal to the cube

$$f^3 + 3f^2gx + 3fg^2x^2 + g^3x^3;$$

and likewise, as the first and last terms destroy each other, the others being divisible by x, we arrive at the equation $b + cx = 3f^2g + 3fg^2x$, which gives

$$x = \frac{b - 3f^2g}{3fg^2 - c}.$$

152. On the contrary, when the given formula belongs not to any of the above three cases, we have no other resource than to try to find such a value for x as will change it into a cube; then, having found such a value, for example, $x = h$, so that $a + bh + ch^2 + dh^3 = k^3$, we suppose $x = h + y$, and find, by substitution,

$$\begin{array}{l} a \\ bh + by \\ ch^2 + 2chy + cy^2 \\ dh^3 + 3dh^2y + 3dhy^2 + dy^3 \\ \hline k^3 + (b + 2ch + 3dh^2)y + (c + 3dh)y^2 + dy^3. \end{array}$$

This new formula belonging to the first case, we know how to determine y, and therefore shall find a new value of x, which may then be employed for finding other values.

153. Let us endeavour to illustrate this method by some examples.

Suppose it were required to transform into a cube the formula $1 + x + x^2$, which belongs to the first case. We might at once make the cube root 1, and should find $x + x^2 = 0$, that is, $x(1 + x) = 0$, and, consequently, either $x = 0$, or $x = -1$; but from this we can draw no conclusion. Let us therefore represent the cube root by $1 + px$; and as its cube is $1 + 3px + 3p^2x^2 + p^3x^3$, we shall have $3p = 1$, or $p = \frac{1}{3}$; by which means the other terms, being divided by x^2, give $3p^2 + p^3x = 1$, or $x = \frac{1-3p^2}{p^3}$. Now, $p = \frac{1}{3}$, so that $x = \frac{\frac{2}{3}}{\frac{1}{27}} = 18$, and our formula becomes $1 + 18 + 324 = 343$, and the cube root $1 + px = 7$. If now we proceed, by making $x = 18 + y$, our formula will assume the form $343 + 37y + y^2$, and by the first rule we must suppose its cube root to be $7 + py$; comparing it then with the cube

$$343 + 147py + 21p^2y^2 + p^3y^3,$$

it is evident we must make $147p = 37$, or $p = \frac{37}{147}$; the other terms give the equation $21p^2 + p^3y = 1$, whence we obtain the value of

$$y = \frac{1-21p^2}{p^3} = -\frac{147\times(147^2-21\times37^2)}{37^3} = -\frac{1049580}{50653}, \ldots$$

which may lead, in the same manner, to new values.

154. Let it now be required to make the formula $2 + x^2$ equal to a cube. Here, as we easily get the case $x = 5$, we shall immediately make $x = 5 + y$, and shall have $27 + 10y + y^2$; supposing now its cube root to be $3 + py$, so that the formula itself may be $27 + 27py + 9p^2y^2 + p^3y^3$, we shall have to make $27p = 10$, or $p = \frac{10}{27}$; therefore $1 = 9p^2 + p^3y$, and

$$y = \frac{1-9p^2}{p^3} = -\frac{27\times(27^2-9\times10^2)}{1000} = -\frac{4617}{1000}, \text{ and}$$

$x = \frac{383}{1000}$; therefore our formula becomes $2+x^2=\frac{2146689}{1000000}$, the cube root of which must be $3 + py = \frac{129}{100}$.

155. Let us also see whether the formula, $1 + x^3$, can become a cube in any other cases beside the evident ones of $x = 0$, and $x = -1$. We may here remark first, that though this formula belongs to the third class, yet the root $1 + x$ is of no use to us, because its cube, $1 + 3x + 3x^2 + x^3$, being equal to the formula, gives $3x + 3x^2 = 0$, or $3x(1+x) = 0$, that is, again, $x = 0$, or $x = -1$.

If we made $x = -1 + y$, we should have to transform into a cube the formula $3y - 3y^2 + y^3$, which belongs to the second case; so that, supposing its cube root to be $p + y$, or the formula itself equal to the cube $p^3 + 3p^2y + 3py^2 + y^3$, we should have $3p = -3$, or $p = -1$, and thence the equation $3y = p^3 + 3p^2y = -1 + 3y$, which gives $y = \frac{1}{0}$, or infinity; so that we obtain nothing more from this second supposition. In fact, it is in vain to seek for other values of x; for it may be demonstrated, that the sum of two cubes, as $t^3 + x^3$, can never become a cube*; so that, by making $t = 1$, it follows that the formula, $x^3 + 1$, can never become a cube, except in the cases already mentioned.

156. In the same manner, we shall find that the formula, $x^3 + 2$, can only become a cube in the case of $x = -1$. This formula belongs to the second case; but the rule there given cannot be applied to it, because the middle terms are wanting. It is by supposing $x = -1 + y$, which gives $1 + 3y - 3y^2 + y^3$, that the formula may be managed according to all the three cases, and that the truth of what we have advanced may be demonstrated. If, in the first case, we make the root $= 1 + y$, whose cube is $1 + 3y - 3y^2 + y^3$, we have $-3y^2 = 3y^2$, which can only be true when $y = 0$: and if, according to the second case, the root be $-1 + y$, or the formula equal to $-1 + 3y - 3y^2 + y^3$, we have $1 + 3y = -1 + 3y$, and $y = \frac{2}{0}$, or an infinite value; lastly, the third case requires us to suppose the root to be $1 + y$, which has already been done for the first case.

157. Let the formula $3x^3 + 3$ be also required to be transformed into a cube. This may be done, in the first place, if $x = -1$, but from that we can conclude nothing: then also, when $x = 2$; and if, in this second case, we suppose $x = 2 + y$, we shall have the formula $27 + 36y + 18y^2 + 3y^3$; and as this belongs to the first case, we shall represent its root by $3 + py$, the cube of which is
$27 + 27py + 9p^2y^2 + p^3y^3$; then, by comparison, we find $27p = 36$, or $p = \frac{4}{3}$; and thence results the equation

$$18 + 3y = 9p^2 + p^3y = 16 + \tfrac{64}{27}y,$$

which gives $y = \dfrac{-54}{17}$, and, consequently, $x = \dfrac{-20}{17}$: therefore our formula $3 + 3x^3 = -\frac{9261}{4913}$, and its cube root $3 + py = \frac{21}{17}$; which solution would furnish new values, if we chose to proceed.

158. Let us also consider the formula $4 + x^2$, which becomes a cube in two cases that may be considered as known; namely, $x = 2$, and $x = 11$. If now we first make $x = 2 + y$, the formula $8 + 4y + y^2$ will be required to become a cube, having for its root $2 + \frac{1}{3}y$, and this cubed being $8 + 4y + \frac{2}{3}y^2 + \frac{1}{27}y^3$, we find $1 = \frac{2}{3} + \frac{1}{27}y$; therefore $y = 9$, and $x = 11$; which is the second given case.

If we here suppose $x = 11 + y$, we shall have $125 + 22y + y^2$, which, being made equal to the cube of $5 + py$, or to $125 + 75py + 15p^2y^2 + p^3y^3$, gives $p = \frac{22}{75}$;

* See Article 247 of this Part.

and thence $15p^2 + p^3y = 1$, or $p^3y = 1 - 15p^2 = -\frac{100}{375}$; consequently, $y = -\frac{122625}{10648}$, and $x = -\frac{5497}{10648}$.

And since x may either be negative or positive, x^2 being found alone in the given formula, let us suppose

$x = \dfrac{2+2y}{1-y}$, and our formula will become $\dfrac{8+8y^2}{(1-y)^2}$, which must be a cube; let us therefore multiply both terms by $1 - y$, in order that the denominator may become a cube; and this will give $\dfrac{8-8y+8y^2-8y^3}{(1-y)^3}$: then we shall only have the numerator $8 - 8y + 8y^2 - 8y^3$, or if we divide by 8, only the formula $1 - y + y^2 - y^3$, to transform into a cube; which formula belongs to all the three cases. Let us, according to the first, take for the root $1 - \frac{1}{3}y$; the cube of which is $1 - y + \frac{1}{3}y^2 - \frac{1}{27}y^3$; so that we have $1 - y = \frac{1}{3} - \frac{1}{27}y$, or $27 - 27y = 9 - y$; therefore $y = \frac{9}{13}$; also, $1 + y = \frac{22}{13}$, and $1 - y = \frac{4}{13}$; whence $x = 11$, as before.

We should have exactly the same result, if we considered the formula as coming under the second case.

Lastly, if we apply the third, and take $1 - y$ for the root, the cube of which is $1 - 3y + 3y^2 - y^3$, we shall have $-1 + y = -3 + 3y$, and $y = 1$; so that $x = \frac{1}{0}$, or infinity; and, consequently, a result which is of no use.

159. But since we already know the two cases, $x = 2$, and $x = 11$, we may also make $x = \dfrac{2+11y}{1+y}$; for, by these means, if $y = 0$, we have $x = 2$; and if $y = \infty$, or infinity, we have $x = 11$.

Therefore, let $x = \dfrac{2+11y}{1-y}$, and our formula becomes

$4 + \dfrac{4+44y+121y^2}{1+2y+y^2}$, or $\dfrac{8+52y+125y^2}{(1+y)^2}$. Multiply both terms by $1 + y$, in order that the denominator may become a cube, and we shall only have the numerator $8 + 60y + 177y^2 + 125y^3$, to transform into a cube. And if, for this purpose, we suppose the root to be $2 + 5y$, we shall not only have the first terms disappear, but also the last. We may, therefore, refer our formula to the second case, taking $p + 5y$ for the root, the cube of which is $p^3 + 15p^2y + 75py^2 + 125y^3$; so that we must make $75p = 177$, or $p = \frac{59}{25}$; and there will result $8 + 60y = p^3 + 15p^2y$, or $-\frac{2943}{125}y = \frac{80379}{15625}$, and $y = \frac{80379}{367875}$, whence we might obtain a value of x.

But we may also suppose $x = \frac{2+11y}{1-y}$; and, in this case, our formula becomes

$$4 + \frac{4+44y+121y^2}{1-2y+y^2} = \frac{8+36y+125y^2}{(1-y)^2};$$

so that multiplying both terms by $1-y$, we have $8+28y+89y^2-125y^3$ to transform into a cube. If we therefore suppose, according to the first case, the root to be $2+\frac{7}{3}y$, the cube of which is $8+28y+\frac{98}{3}y^2+\frac{343}{27}y^3$, we have $89-125y = \frac{98}{3}+\frac{343}{27}y$, or $\frac{3718}{27}y = \frac{169}{3}$; and, consequently, $y = \frac{1521}{3718} = \frac{9}{22}$; whence we get $x = 11$; that is, one of the values already known.

But let us rather consider our formula with reference to the third case, and suppose its root to be $2-5y$; the cube of this binomial being $8-60y+150y^2-125y^3$, we shall have $28+89y = -60+150y$; therefore $y = \frac{88}{61}$, whence we get $x = -\frac{1090}{27}$; so that our formula becomes $\frac{1191016}{729}$, or the cube of $\frac{106}{9}$.

160. The foregoing are the methods which we at present know, for reducing such formulæ as we have considered, either to squares, or to cubes, provided the highest power of the unknown quantity does not exceed the fourth power in the former case, nor the third in the latter.

We might also add the problem for transforming a given formula into a biquadrate, in the case of the unknown quantity not exceeding the second degree. But it will be perceived, that, if such a formula as $a+bx+cx^2$ were proposed to be transformed into a biquadrate, it must in the first place be a square; after which it will only remain to transform the root of that square into a new square, by the rules which we have given.

If x^2+7, for example, is to be made a biquadrate, we first make it a square, by supposing

$$x = \frac{7p^2-q^2}{2pq}, \text{ or } x = \frac{q^2-7p^2}{2pq};$$

the formula then becomes equal to the square

$$\frac{q^4-14q^2p^2+49p^4}{4p^2q^2}+7 = \frac{q^4+14q^2p^2+49p^4}{4p^2q^2},$$

the root of which $\frac{7p^2+q^2}{2pq}$ must likewise be transformed into a square; for this purpose, let us multiply the two terms by $2pq$, in order that the denominator becoming a square, we may have only to consider the numerator $2pq(7p^2+q^2)$. Now, we cannot make a square of this formula, without having previously found a satisfactory case; so that supposing $q = pz$, we must have the formula

$$2p^2z(7p^2 + p^2z^2) = 2p^4z(7 + z^2),$$

and, consequently, if we divide by p^4, the formula $2z(7+z^2)$ must become a square. The known case is here $z = 1$, for which reason we shall make $z = 1 + y$, and we shall thus have

$$(2 + 2y) \times (8 + 2y + y^2) = 16 + 20y + 6y^2 + 2y^3,$$

the root of which we shall suppose to be $4 + \frac{5}{2}y$; then its square will be $16 + 20y + \frac{25}{4}y^2$, which, being made equal to the formula, gives $6 + 2y = \frac{25}{4}$; therefore $y = \frac{1}{8}$, and $z = \frac{9}{8}$: also, $z = \frac{q}{p}$; so that $q = 9$, and $p = 8$, which makes $x = \frac{367}{144}$, and the formula $7 + x^2 = \frac{279841}{20736}$. If we now extract the square root of this fraction, we find $\frac{529}{144}$; and taking the square root of this also, we find $\frac{23}{12}$; consequently, the given formula is the biquadrate of $\frac{23}{12}$.

161. Before we conclude this chapter, we must observe, that there are some formulæ, which may be transformed into cubes in a general manner; for example, if cx^2 must be a cube, we have only to make its root $= px$, and we find $cx^2 = p^3x^3$, or $c = p^3x$, that is, $x = \frac{c}{p^3}$, or $x = cq^3$, if we write $\frac{1}{q}$ instead of p.

The reason of this evidently is, that the formula contains a square, on which account, all such formulæ, as $a(b+cx)^2$, or $ab^2 + 2abcx + ac^2x^2$, may very easily be transformed into cubes. In fact, if we suppose its cube root to be $\frac{b+cx}{q}$, we shall have the equation $a(b + cx)^2 = \frac{(b + cx)^3}{q^3}$, which, divided by $(b + cx)^2$, gives $a = \frac{b + cx}{q^3}$, whence we get $x = \frac{aq^3 - b}{c}$, a value in which q is arbitrary.

This shews how useful it is to resolve the given formulæ into their factors, whenever it is possible: on this subject, therefore, we think it will be proper to dwell at some length in the following chapter.