ADDITIONS

BY

M. DE LA GRANGE.

ADVERTISEMENT.

The geometricians of the last century paid great attention to the Indeterminate Analysis, or what is commonly called the *Diophantine Algebra;* but Bachet and Fermat alone can properly be said to have added any thing to what Diophantus himself has left us on that subject.

To the former, we particularly owe a complete method of resolving, in integer numbers, all indeterminate problems of the first degree *: the latter is the author of some methods for the resolution of indeterminate equations, which exceed the second degree †; of the singular method, by which we demonstrate that it is impossible for the sum, or the difference of two biquadrates to be a square ‡; of the solution of a great number of very difficult problems; and of several admirable theorems respecting integer numbers, which he left without demonstration, but of which the greater part has since been demonstrated by M. Euler in the Petersburg Commentaries ||.

* See Chap. 3, in these Additions. I do not here mention his Commentary on Diophantus, because that work, properly speaking, though excellent in its way, contains no discovery.

† These are explained in the 8th, 9th, and 10th chapters of the preceding Treatise. Père Billi has collected them from different writings of M. Fermat, and has added them to the new edition of Diophantus, published by M. Fermat, junior.

‡ This method is explained in the 13th chapter of the preceding Treatise; the principles of it are to be found in the *Remarks* of M. Fermat, on the XXVIth Question of the VIth Book of Diophantus.

|| The problems and theorems, to which we allude, are

In the present century, this branch of analysis has been almost entirely neglected; and, except M. Euler, I know no person who has applied to it: but the beautiful and numerous discoveries, which that great mathematician has made in it, sufficiently compensate for the indifference which mathematical authors appear to have hitherto entertained for such researches. The Commentaries of Petersburg are full of the labors of M. Euler on this subject, and the preceding Work is a new service, which he has rendered to the admirers of the Diophantine Algebra. Before the publication of it, there was no work in which this science was treated methodically, and which enumerated and explained the principal rules hitherto known for the solution of indeterminate problems. The preceding Treatise unites both these advantages: but in order to make it still more complete, I have thought it necessary to make several Additions to it, of which I shall now give a short account.

The theory of Continued Fractions is one of the most useful in arithmetic, as it serves to resolve problems with facility, which, without its aid, would be almost unmanageable; but it is of still greater utility in the solution of indeterminate problems, when integer numbers only are sought. This consideration has induced me to explain the theory of them, at sufficient length to make it understood. As it is not to be found in the chief works on arithmetic and algebra, it must be little known to mathematicians; and I shall be happy, if I can contribute to render it more familiar to them. At the end of this theory, which occupies the first Chapter, follow several curious and entirely new problems, depending on the truth of the same theory, but which I have thought proper to treat in a distinct manner, in order that their solution may become more interesting. Among these will particularly be remarked a very simple and easy method of reducing the roots of equations of the second degree to Continued Fractions, and a rigid demonstration, that those fractions must necessarily be always periodical.

The other Additions chiefly relate to the resolution of in-

scattered through the *Remarks* of M. Fermat on the Questions of Diophantus; and through his Letters printed in the *Opera Mathematica*, &c. and in the second volume of the works of Wallis.

There are also to be found, in the Memoirs of the Academy of Berlin, for the year 1770, & seq. the demonstrations of some of this author's theorems, which had not been demonstrated before.

determinate equations of the first and second degree; for these I give new and general methods, both for the case in which the numbers are only required to be rational, and for that in which the numbers sought are required to be integer; and I consider some other important matters relating to the same subject.

The last Chapter contains researches on the functions * which have this property, that the product of two or more similar functions is always a similar function. I give a general method for finding such functions, and shew their use in the resolution of different indeterminate problems, to which the usual methods could not be applied.

Such are the principal objects of these Additions, which might have been made much more extensive, had it not been for exceeding proper bounds; I hope, however, that the subjects here treated will merit the attention of mathematicians, and revive a tase for this branch of algebra, which appears to me very worthy of exercising their skill.

CHAPTER I.

ON

CONTINUED FRACTIONS.

1. As the subject of Continued Fractions is not found in the common books of arithmetic and algebra, and for this reason is but little known to mathematicians, it will be proper to begin these Additions by a short explanation of their theory, which we shall have frequent opportunities to apply in what follows.

In general, we call every expression of this form, a *continued fraction*,

$$\alpha + \cfrac{b}{\beta + \cfrac{c}{\gamma + \cfrac{d}{\delta + , \text{\&c.}}}}$$

* A term used in algebra for any expression containing a certain letter, denoting an unknown quantity, however mixed and compounded with other known quantities or numbers. Thus, $ax + yx$; $2x - a\sqrt{(\frac{a^2 - x^2}{3})}$; $3xy^3 + \sqrt{(\frac{bc+yx}{2})}$, are all functions of x.

in which the quantities α, β, γ, δ, &c. and b, c, d, &c. are integer numbers positive or negative; but at present we shall consider those Continued Fractions only, whose numerators b, c, d, &c. are unity; that is to say, fractions of this form,

$$\alpha + \frac{1}{\beta + \frac{1}{\gamma + \frac{1}{\delta + \text{, \&c.}}}}$$

α, β, γ, &c. being any integer numbers positive or negative; for these are, properly speaking, the only numbers, which are of great utility in analysis, the others being scarcely any thing more than objects of curiosity.

2. Lord Brouncker, I believe, was the first who thought of Continued Fractions; we know that the continued fraction, which he devised to express the ratio of the circumscribed square to the area of the circle was this;

$$1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \text{, \&c.}}}}$$

but we are ignorant of the means which led him to it. We only find in the *Arithmetica infinitorum* some researches on this subject, in which Wallis demonstrates, in an indirect, though ingenious manner, the identity of Brouncker's expression to his, which is, $\dfrac{3 \times 3 \times 5 \times 5 \times 7, \text{ \&c.}}{2 \times 4 \times 4 \times 6 \times 6, \text{ \&c.}}$. He there also gives the general method of reducing all sorts of continued fractions to vulgar fractions; but it does not appear that either of those great mathematicians knew the principal properties and singular advantages of continued fractions; and we shall afterwards see, that the discovery of them is chiefly due to Huygens.

3. Continued fractions naturally present themselves, whenever it is required to express fractional, or imaginary quantities in numbers. In fact, suppose we have to assign the value of any given quantity a, which is not expressible by an integer number; the simplest way is, to begin by seeking the integer number, which will be nearest to the value of a, and which will differ from it only by a fraction less than unity. Let this number be α, and we shall have $a - \alpha$ equal to a fraction less than unity; so that $\dfrac{1}{a-\alpha}$ will, on the contrary, be a number greater than unity: therefore let $\dfrac{1}{a-\alpha} = b$; and, as b must be a number greater than unity,

we may also seek for the integer number, which shall be nearest the value of b; and this number being called β, we shall again have $b - \beta$ equal to a fraction less than unity; and, consequently, $\frac{1}{b-\beta}$ will be equal to a quantity greater than unity, which we may represent by c; so that, to assign the value of c, we have only to seek, in the same manner, for the integer number nearest to c, which being represented by γ, we shall have $c - \gamma$ equal to a quantity less than unity; and, consequently, $\frac{1}{c-\gamma}$ will be equal to a quantity, d, greater than unity, and so on. From which it is evident, that we may gradually exhaust the value of a, and that in the simplest and readiest manner; since we only employ integer numbers, each of which approximates, as nearly as possible, to the value sought.

Now, since $\frac{1}{a-\alpha} = b$, we have $a - \alpha = \frac{1}{b}$, and $a = \alpha + \frac{1}{b}$; likewise, since $\frac{1}{b-\beta} = c$, we have $b = \beta + \frac{1}{c}$; and, since $\frac{1}{c-\gamma} = d$, we have, in the same manner, $c = \gamma + \frac{1}{d}$, &c.; so that by successively substituting these values, we shall have

$$a\begin{cases} = \alpha + \frac{1}{b}, \\ = \alpha + \cfrac{1}{\beta + \cfrac{1}{c}}, \\ = \alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{d}}}; \end{cases}$$

and, in general, $$a = \alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta + \text{, \&c.}}}}$$

It is proper to remark here, that the numbers α, β, γ, &c. which represent, as we have shewn, the approximate integer values of the quantities a, b, c, &c. may be taken each in two different ways; since we may with equal propriety take, for the approximate integer value of a given quantity, either of the two integer numbers between which that quantity lies. There is, however, an essential difference between these two methods of taking the approximate values, with respect to the continued fraction which results from it: for if we always take the approximate values *less* than the true ones, the denominators β, γ, δ, &c. will be all positive; whereas they will be all negative, if we take all the approximate values *greater* than the true ones; and they will be partly positive and partly negative, if the approximate values are taken sometimes too small, and sometimes too great.

In fact, if α be less than a, $a - \alpha$ will be a positive quantity; wherefore b will be positive, and β will be so likewise: on the contrary, $a - \alpha$ will be negative, if α be greater than a; then b will be negative, and β will be so likewise. In the same manner, if β be less than b, $b - \beta$ will always be a positive quantity; therefore c will be positive also, and, consequently, also γ; but if β be greater than b, $b - \beta$ will be a negative quantity; so that c, and consequently also γ, will be negative, and so on.

Farther, when negative quantities are considered, I understand by *less* quantities those which, taken positively, would be *greater*. We shall have occasion, however, sometimes to compare quantities simply in respect of their absolute magnitude; but I shall then take care to premise, that we must pay no attention to the signs.

It must be remarked, also, that if, among the quantities b, c, d, &c. one is found equal to an integer number, then the continued fraction will be terminated; because we shall be able to preserve that quantity in it: for example, if c be an integer number, the continued fraction, which gives the value of a, will be

$$a = \alpha + \cfrac{1}{\beta + \cfrac{1}{c}}.$$

It is evident, indeed, that we must take $\gamma = c$, which gives $d = \dfrac{1}{c - \gamma} = \frac{1}{0} = \infty$; and, consequently, $d = \infty$; so that we shall have

$$a = \alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\infty}}},$$

the following terms vanishing in comparison with the infinite quantity ∞. Now, $\frac{1}{\infty} = 0$, wherefore we shall only have

$$a = \alpha + \frac{1}{\beta + \frac{1}{c}}.$$

This case will happen whenever the quantity a is commensurable; that is to say, expressed by a rational fraction; but when a is an irrational, or transcendental quantity, then the continued fraction will necessarily go on to infinity.

4. Suppose the quantity a to be a vulgar fraction, $\frac{A}{B}$, A and B being given integer numbers; it is evident, that the integer number, α, approaching nearest to $\frac{A}{B}$, will be the quotient of the division of A by B; so that supposing the division performed in the usual manner, and calling α the quotient, and C the remainder, we shall have $\frac{A}{B} - \alpha = \frac{C}{B}$; whence $b = \frac{B}{C}$. Also, in order to have the approximate integer value β of the fraction $\frac{B}{C}$, we have only to divide B by C, and take β for the quotient of this division; then calling the remainder D, we shall have $b - \beta = \frac{D}{C}$, and $c = \frac{C}{D}$. We shall therefore continue to divide C by D, and the quotient will be the value of the number γ, and so on; whence results the following very simple rule for reducing vulgar fractions to continued fractions.

Rule. First, divide the numerator of the given fraction by its denominator, and call the quotient α; then divide the denominator by the remainder, and call the quotient β; then divide the first remainder by the second remainder, and let the quotient be γ. Continue thus, always dividing the last divisor by the last remainder, till you arrive at a division that is performed without any remainder, which must necessarily happen when the remainders are all integer numbers that continually diminish; you will then have the continued fraction

$$\alpha + \frac{1}{\beta + \frac{1}{\gamma + \frac{1}{\delta}}}, \text{\&c.}$$

which will be equal to the given fraction.

5. Let it be proposed, for example, to reduce $\frac{1103}{887}$ to a continued fraction.

First, we divide 1103 by 887, which gives the quotient 1, and the remainder 216; 887 divided by 216, gives the quotient 4, and the remainder 23; 216 divided by 23, gives the quotient 9, and the remainder 9; also dividing 23 by 9, we obtain the quotient 2, and the remainder 5; then 9 by 5, gives the quotient 1, and the remainder 4; 5 by 4, gives the quotient 1, and the remainder 1; lastly, dividing 4 by 1, we obtain the quotient 4, and no remainder; so that the operation is finished: and, collecting all the quotients in order, we have this series 1, 4, 9, 2, 1, 1, 4, whence we form the continued fraction

$$\frac{1103}{887} = 1 + \frac{1}{4 + \frac{1}{9 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4}}}}}}.$$

6. As, in the above division, we took for the quotient the integer number which was equal to, or less than, the fraction proposed, it follows that we shall only obtain from that method continued fractions, of which all the denominators will be positive numbers.

But we may also assume for the quotient the integer number, which is immediately greater than the value of the fraction, when that fraction is not reducible to an integer, and, for this purpose, we have only to increase the value of the quotient found by unity in the usual manner; then the remainder will be negative, and the next quotient will necessarily be negative. So that we may, at pleasure, make the terms of the continued fraction positive, or negative.

In the preceding example, instead of taking 1 for the quotient of 1103 divided by 887, we may take 2; in which case we have the negative remainder -671, by which we must now divide 887; we therefore divide 887 by -671, and obtain either the quotient -1, and the remainder 216, or the quotient -2, and the remainder -455. Let us take the greater quotient -1: then divide the remainder -671 by 216; whence we obtain either the quotient -3, and the remainder -23, or the quotient -4, and the remainder 193. Continuing the division by adopting the greater quotient -3, we have to divide the remainder 216 by the remainder -23, which gives either the quotient -9, and the remainder 9, or the quotient -10, and the remainder -14, and so on.

In this way, we obtain

$$\frac{1103}{887} = 2 + \cfrac{1}{-1 + \cfrac{1}{-3 + \cfrac{1}{-9} +, \&c.}}$$

in which we see that all the denominators are negative.

7. We may also make each negative denominator positive by changing the sign of the numerator; but we must then also change the sign of the succeeding numerator; for it is evident that

$$\left\{\mu + \cfrac{1}{-\nu + \cfrac{1}{\pi} +, \&c.}\right\} = \left\{\mu - \cfrac{1}{\nu - \cfrac{1}{\pi} +, \&c.}\right\}$$

Then we may also, if we choose, remove all the signs $-$ in the continued fraction, and reduce it to another, in which all the terms shall be positive; for we have, in general,

$$\left\{\mu + \cfrac{1}{-\nu} +, \&c.\right\} = \left\{\mu - 1 + \cfrac{1}{1 + \cfrac{1}{\nu - 1} +, \&c.}\right\}$$

as we may easily be convinced of by reducing those two quantities to vulgar fractions *.

We may also, by similar means, introduce negative terms instead of positive; for we have

$$\mu + \cfrac{1}{\nu} +, \&c. \quad = \mu + 1 - \cfrac{1}{1 + \cfrac{1}{\nu - 1} +, \&c.}$$

whence we see, that, by such transformations, we may always simplify a continued fraction, and reduce it to fewer terms: which will take place, whenever there are denominators equal to unity, positive, or negative.

In general, it is evident, that, in order to have the continued fraction approximating as nearly as possible to the

* Thus, the mixed number, $1 + \frac{1}{\nu - 1} = \frac{\nu}{\nu - 1}$; therefore

$$\left.\cfrac{1}{1 + \cfrac{1}{\nu - 1}}\right\} = \frac{\nu - 1}{\nu};$$

and, consequently,

$$\left\{\mu - 1 + \cfrac{1}{1 + \cfrac{1}{\nu - 1}}\right\} = \mu - 1 + \frac{\nu - 1}{\nu} = \mu - \frac{1}{\nu}. \quad \text{B.}$$

value of the given quantity, we must always take α, β, γ, &c. the integer numbers which are nearest the quantities a, b, c, &c. whether they be less, or greater than those quantities. Now, it is easy to perceive that if, for example, we do not take for α the integer number which is nearest to a, either above or below it, the following number β will necessarily be equal to unity; in fact, the difference between a and α will then be greater than $\frac{1}{2}$, consequently, we shall have $b = \dfrac{1}{a-\alpha}$ less than 2; therefore β must be equal to unity.

So that whenever we find the denominators in a continued fraction equal to unity, this will be a proof that we have not taken the preceding denominators as near as we might have done; and, consequently, that the fraction may be simplified by increasing, or diminishing those denominators by unity, which may be done by the preceding formulæ, without the necessity of going through the whole calculation.

8. The method in Art. 4 may also serve for reducing every irrational, or transcendental quantity to a continued fraction, provided it be expressed before in decimals; but as the value in decimals can only be approximate, by augmenting the last figure by unity, we procure two limits, between which the true value of the given quantity must lie; and, in order that we may not pass those limits, we must perform the same calculation with both the fractions in question, and then admit into the continued fraction those quotients only which shall equally result from both operations.

Let it be proposed, for example, to express by a continued fraction the ratio of the circumference of the circle to the diameter.

This ratio expressed in decimals is, by the calculation of Vieta, as 3,1415926535 is to 1; so that we have to reduce the fraction $\dfrac{3,1415926535}{10000000000}$ to a continued fraction by the method above explained. Now, if we take only the fraction $\dfrac{3,14159}{100000}$, we find the quotients 3, 7, 15, 1, &c. and if we take the greater fraction $\dfrac{3,14160}{100000}$, we find the quotients 3, 7, 16, &c. so that the third quotient remains doubtful;

whence we see, that, in order to extend the continued fraction only beyond three terms, we must adopt a value of the circumference, which has more than six figures.

If we take the value given by Ludolph to thirty-five decimal places, which is 3,14159, 26535, 89793, 23846, 26433, 83279, 50288; and if we work on with this fraction, as it is, and also with its last figure 8 increased by unity, we shall find the following series of quotients, 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 1; so that we shall have

$$\frac{\textit{Circumference}}{\textit{Diameter}} = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} +, \&c.$$

And as there are here denominators equal to unity, we may simplify the fraction, by introducing negative terms, according to the formulæ of Art. 7, and shall find

$$\frac{\textit{Circumference}}{\textit{Diameter}} = 3 + \frac{1}{7} + \frac{1}{16} - \frac{1}{294} - \frac{1}{3} - \frac{1}{3} +, \&c.$$

$$\frac{\textit{Circumference}}{\textit{Diameter}} = 3 + \frac{1}{7} + \frac{1}{16} + \frac{1}{-294} + \frac{1}{-3} + \frac{1}{-3} +, \&c.$$

9. We have elsewhere shewn how the theory of continued fractions may be applied to the numerical resolution of equations, for which other methods are imperfect and insufficient *. The whole difficulty consists in finding in any equation the nearest integer value, either above, or below the root sought; and for this I first gave some general rules, by which we may not only perceive how many real roots, positive or negative, equal or unequal, the proposed equation contains, but also easily find the limits of each of those roots, and even the limits of the real quantities which compose the imaginary roots. Supposing, therefore, that x is the unknown quantity of the equation proposed, we seek first for the integer number which is nearest to the root sought, and calling that number α, we have only, as in Art. 3, to make

* See the Memoirs of the Academy of Berlin, for the years 1767 and 1768; and Le Gendre's Essai sur la Theorie des Nombres, page 133, first edition.

$x = \alpha + \frac{1}{y}$; x, y, z, &c. representing here what was denoted in that article by a, b, c, &c. and substituting this value instead of x, we shall have, after removing the fractions, an equation of the same degree in y, which must have at least one positive, or negative root greater than unity. After seeking therefore for the approximate integer value of the root, and calling that value β, we shall then make $y = \beta + \frac{1}{z}$, which will give an equation in z, having likewise a root greater than unity, whose approximate integer value we must next seek, and so on. In this manner, the root required will be found expressed by the continued fraction

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta + \text{, \&c.}}}}$$

which will be terminated, if the root is commensurable; but will necessarily go on *ad infinitum*, if it be incommensurable.

In the Memoirs just referred to, there will be found all the principles and details necessary to render this method and its application easy, and even different means of abridging many of the operations which it requires. I believe that I have scarcely left any thing farther to be said on this important subject. With regard to the roots of equations of the second degree, we shall afterwards give (Art. 33 et seq.) a particular and very simple method of changing them into continued fractions.

10. After having thus explained the genesis of continued fractions, we shall proceed to shew their application, and their principal properties.

It is evident, that the more terms we take in a continued fraction, the nearer we approximate to the true value of the quantity which we have expressed by that fraction; so that if we successively stop at each term of the fraction, we shall have a series of quantities converging towards the given quantity.

Thus, having reduced the value of a to the continued fraction,

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta + \text{, \&c.}}}}$$

we shall have the quantities,

$$\alpha, \left\{\alpha + \frac{1}{\beta}\right\}, \left\{\alpha + \frac{1}{\beta} + \frac{1}{\gamma}, \&c.\right\}$$

or, by reduction,

$$\alpha, \frac{\alpha\beta+1}{\beta}, \frac{\alpha\beta\gamma+\alpha+\gamma}{\beta\gamma+1}, \&c.*$$

which approach nearer and nearer to the value of a.

In order to judge better of the law, and of the convergence of these quantities, it must be remarked, that, by the formulæ of Art. 3, we have

$$a = \alpha + \frac{1}{b}, b = \beta + \frac{1}{c}, c = \gamma + \frac{1}{d}, \&c.$$

Whence we immediately perceive, that α is the first approximate value of a; that then, if we take the exact value of a, which is $\frac{\alpha b+1}{b}$, and, in this, substitute for b its approximate value β, we shall have this more approximate value $\frac{\alpha\beta+1}{\beta}$; that we shall, in the same manner, have a third more approximate value of a, by substituting for b its exact value $\frac{\beta c+1}{c}$, which gives $a = \frac{(\alpha\beta+1)c+a}{\beta c+1}$, and then taking for c the approximate value γ; by these means the new approximate value of a will be

$$\frac{(\alpha\beta+1)\gamma+\alpha}{\beta\gamma+1}$$

Continuing the same reasoning, we may approximate nearer, by substituting, in the above expression of a, instead of c, its exact value, $\frac{\gamma d+1}{d}$, which will give

$$a = \frac{((\alpha\beta+1)\gamma+\alpha)d+\alpha\beta+1}{(\beta\gamma+1)d+\beta}$$

and then taking for d its approximate value δ, we shall have, for the fourth approximation, the quantity

$$\frac{((\alpha\beta+1)\gamma+\alpha)\delta+\alpha\beta+1}{(\beta\gamma+1)\delta+\beta}, \text{ and so on.}$$

Hence it is easy to perceive, that, if by means of the numbers α, β, γ, δ, &c. we form the following expressions,

* See note, p. 471.

$$\begin{array}{ll} A = \alpha & A' = 1 \\ B = \beta A + 1 & B' = \beta \\ C = \gamma B + A & C' = \gamma B' + A' \\ D = \delta C + B & D' = \delta C' + B' \\ E = \varepsilon D + C & E' = \varepsilon D' + C' \\ \text{\&c.} & \text{\&c.} \end{array}$$

we shall have this series of fractions converging towards the quantity a, $\frac{A}{A'}$ $\frac{B}{B'}$ $\frac{C}{C'}$ $\frac{D}{D'}$ $\frac{E}{E'}$ $\frac{F}{F'}$, &c.

If the quantity a be rational, and represented by any fraction $\frac{V}{V'}$, it is evident that this fraction will always be the last in the preceding series; since then the continued fraction will be terminated, and the last fraction of the above series must always be equal to the whole continued fraction.

But if the quantity a be irrational, or transcendental, then the continued fraction necessarily going on *ad infinitum*, we may also continue *ad infinitum* the series of converging fractions.

11. Let us now examine the nature of these fractions. 1st, It is evident that the numbers A, B, C, &c. must continually increase, as well as the numbers A', B', C', &c. for 1st, if the numbers α, β, γ, &c. are all positive, the numbers A, B, C, &c. A', B', C', &c. will also be positive, and we shall evidently have $B > A$, $C > B$, $D > C$, &c. and $B' =$, or $> A'$, $C' > B'$, $D' > C'$, &c.

2dly, If the numbers α, β, γ, &c. are all, or partly negative, then amongst the numbers A, B, C, &c. and, A', B', C', there will be some positive, and some negative; but in that case we must consider that we have, by the preceding formulæ,

$$\frac{B}{A} = \beta + \frac{1}{\alpha}, \quad \frac{C}{B} = \gamma + \frac{A}{B}, \quad \frac{D}{C} = \delta + \frac{B}{C}, \text{ \&c.}$$

whence we immediately see, that, if the numbers α, β, γ, &c. are different from unity, whatever their signs be, we shall necessarily have, neglecting the signs, $\frac{B}{A} > 1$; and therefore $\frac{A}{B} < 1$; consequently, $\frac{C}{B} > 1$, and so on: therefore $B > A$, $C > B$, &c.

There is no exception to this but when some of the numbers α, β, γ, &c. are equal to unity. Suppose, for example, that the number γ is the first which is equal to ± 1; we shall then have B 7 A, but C ∠ B, if it happens that the fraction $\frac{A}{B}$ has a different sign from γ; which is evident from the equation $\frac{C}{B} = \gamma + \frac{A}{B}$; because, in that case, $\gamma + \frac{A}{B}$ will be a number less than unity. Now, I say, in this case, we must have D 7 B; for since $\gamma = \pm 1$, we shall have (Art. 10), $c = \pm 1 + \frac{1}{d}$, and $c - \frac{1}{d} = \pm 1$; but as c and d are quantities greater than unity (Art. 3), it is evident, that this equation cannot subsist, unless c and d have the same signs; therefore, since γ and δ are the approximate integer values of c and d, these numbers γ and δ must also have the same sign. Farther, the fraction $\frac{C}{B} = \gamma + \frac{A}{B}$ must have the same sign as γ, because γ is an integer number, and $\frac{A}{B}$ a fraction less than unity; therefore $\frac{C}{B}$, and δ, will be quantities of the same sign; consequently, $\frac{\delta C}{B}$ will be a positive quantity. Now, we have $\frac{D}{C} = \delta + \frac{B}{C}$; and hence, multiplying by $\frac{C}{B}$, we shall have $\frac{D}{B} = \frac{\delta C}{B} + 1$; so that $\frac{\delta C}{B}$ being a positive quantity, it is evident that $\frac{D}{B}$ will be greater than unity; and therefore D 7 B.

Hence we see, that, if in the series A, B, C, &c. there be one term less than the preceding, the following will necessarily be greater; so that putting aside those less terms, the series will always go on increasing.

Besides, if we choose, we may always avoid this inconvenience, either by taking the numbers α, β, γ, &c. positive, or by taking them different from unity, which may always be done.

The same reasonings apply to the series A′, B′, C′, &c. in which we have likewise

$$\frac{B'}{A'} = \beta, \frac{C'}{B'} = \gamma + \frac{A'}{B'}, \ \frac{D'}{C'} = \delta + \frac{B'}{C'}, \text{ \&c.}$$

whence we may form conclusions similar to the preceding.

12. If we now multiply cross-ways the terms of the consecutive fractions, in the series $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. we shall find

$$BA' - AB' = 1,\ CB' - BC' = AB' - BA',$$
$$DC' - CD' = BC' - CB',\ \&c.$$

whence we conclude, in general, that

$$BA' - AB' = 1$$
$$CB' - BC' = -1$$
$$DC' - CD' = 1$$
$$ED' - DE' = -1,\ \&c.$$

This property is very remarkable, and leads to several important consequences.

First, we see that the fractions $\frac{A}{B'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. must be already in their lowest terms; for if, for example, C and C' had any common divisor, the integer numbers $CB' - BC'$ would also be divisible by that same divisor, which cannot be, since $CB' - BC' = -1$.

Next, if we put the preceding equations into this form,

$$\frac{B}{B'} - \frac{A}{A'} = \frac{1}{A'B'}$$
$$\frac{C}{C'} - \frac{B}{B'} = -\frac{1}{C'B'}$$
$$\frac{D}{D'} - \frac{C}{C'} = \frac{1}{C'D'}$$
$$\frac{E}{E'} - \frac{D}{D'} = -\frac{1}{D'E'},\ \&c.$$

it is easy to perceive, that the differences between the adjoining fractions of the series $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, are continually diminishing, so that this is necessarily converging.

Now, I say, that the difference between two consecutive fractions is as small as it is possible for it to be; so that there can be no other fraction whatever between those two fractions, unless it have a denominator greater than the denominators of them.

Let us take, for example, the two fractions $\frac{C}{C'}$, and $\frac{D}{D'}$, the difference of which is $\frac{1}{C'D'}$, and let us suppose, if possible, that there is another fraction, $\frac{m}{n}$, whose value falls between the values of those two fractions, and whose denominator n is less than c', or less than d'. Now, since $\frac{m}{n}$ is between $\frac{c}{c'}$, and $\frac{d}{d'}$, the difference of $\frac{m}{n}$, and $\frac{c}{c'}$, which is $\frac{mc'-nc}{nc'}$, or $\frac{nc-mc'}{nc'}$, must be less than $\frac{1}{c'd'}$, the difference between $\frac{d}{d'}$ and $\frac{c}{c'}$; but it is evident that the former cannot be less than $\frac{1}{nc'}$; and therefore if $n \angle d'$, it will necessarily be greater than $\frac{1}{c'd'}$. Also, as the difference between $\frac{m}{n}$, and $\frac{d}{d'}$ cannot be less than $\frac{1}{nd'}$, it will necessarily be greater than $\frac{1}{c'd'}$, if $n \angle c'$, whereas it must be less.

13. Let us now see how each fraction of the series $\frac{a}{a'}$, $\frac{b}{b'}$, &c. will approximate towards the value of the quantity a. For this purpose, it may be observed that the formulæ of Article 10 give

$$a = \frac{Ab+1}{A'b} \qquad a = \frac{Cd+B}{C'd+B'}$$

$$a = \frac{Bc+a}{B'c+A'} \qquad a = \frac{De+C}{D'e+C'}$$

and so on.

Hence, if we would know how nearly the fraction $\frac{c}{c'}$, for example, approaches to the given quantity, we seek for the difference between $\frac{c}{c'}$ and a; taking for a the quantity $\frac{cd+B}{c'd+B'}$, we shall have

$$a - \frac{C}{C'} = \frac{Cd+B}{C'd+B'} - \frac{C}{C'} = \frac{BC'-CB'}{C'(C'd+B')} = \frac{1}{C'(C'd+B')},$$

because $BC' - CB' = 1$, (Art. 12). Now, as we suppose δ the approximate value of d, so that the difference between d and δ is less than unity (Art. 3), it is evident that the value of d will lie between the two numbers δ and $\delta \pm 1$, (the upper sign being for the case, in which the approximate value δ is less than the true one d, and the lower sign for the case, in which δ is greater than d), and, consequently, that the value of $C'd + B'$, will also be contained between these two, $C'\delta + B'$, and $C'(\delta \pm 1) + B'$, that is to say, between D' and $D' \pm C'$; therefore the difference $a - \frac{C}{C'}$ will be contained between these two limits $\frac{1}{C'D'}$, $\frac{1}{C'(D' \pm C')}$; whence we may judge of the degree of approximation of the fraction $\frac{C}{C'}$.

14. In general, we shall have,

$$a = \frac{A}{A'} + \frac{1}{A'b}$$

$$a = \frac{B}{B'} - \frac{1}{B'(B'c + A')}$$

$$a = \frac{C}{C'} + \frac{1}{C'(C'd + B')}$$

$$a = \frac{D}{D'} - \frac{1}{D'(D'e + C')}, \text{ and so on.}$$

Now, if we suppose that the approximate values, α, β, γ, &c. are always taken less than the real values, these numbers will all be positive, as well as the quantities b, c, d, &c. (Art. 3.) and, consequently, the numbers A', B', C', &c. will be likewise all positive; whence it follows, that the differences between the quantity a, and the fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. will be alternately positive and negative; that is to say, those fractions will be alternately less and greater than the quantity a.

Farther, as $b > \beta$, $c > \gamma$, $d > \delta$, &c. by *hypothesis*, we have $b > B'$, $(B'c + A') > (B'\gamma + A')$, and also $> C'$*, $(C'd + B') > (C'\delta + B')$, and therefore $> D'$, &c. and as $b < (\beta + 1)$, $c < (\gamma + 1)$, $d < (\delta + 1)$, we have $b < (B' + 1)$,

* For since $c > \gamma$, therefore $B'c > B'\gamma$; and, consequently, $(B'c + A') > (B'\gamma + A')$ which is $> C'$, because $B'\gamma + A' = C'$, page 476. And it is exactly the same with the other quantities. B.

$(B'c + A') \angle (B'(\gamma + 1) + A') \angle (C' + B')$, also
$(C'd + B') \angle (C'(\delta + 1) + B') \angle (D' + C')$, &c. so that the errors in taking the fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. for the value of a, would be respectively less than $\frac{1}{A'B'}$, $\frac{1}{B'C'}$, $\frac{1}{C'D'}$, &c. but greater tha $\frac{1}{A'(B'+A')}$, $\frac{1}{B'(C'+B')}$, $\frac{1}{C'(D'+C')}$, &c. which shews how small those errors are, and how they go on diminishing from one fraction to another.

But farther, since the fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. are alternately less and greater than the quantity a, it is evident, that the value of that quantity will always be found between any two consecutive fractions. Now, we have already seen (Art. 12), that it is impossible to find, between two such fractions, any other fraction whatever, which has a denominator less than one of the denominators of those two fractions; whence we may conclude, that each of the fractions in question, express the quantity a more exactly than any other fraction can, whose denominator is less than that of the succeeding fraction; that is to say, the fraction $\frac{C}{C'}$, for example, will express the value of a more exactly than any other fraction $\frac{m}{n}$, in which n would be less than D'.

15. If the approximate values α, β, γ, &c. are all, or partly, greater than the real values, then some of those numbers will necessarily be negative (Art. 3), which will also render negative some terms of the series A, B, C, &c. A′, B′, C′, &c. consequently, the differences between the fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. and the quantity a, will no longer be alternately positive and negative, as in the case of the preceding articles: so that those fractions will no longer have the advantage of giving the limits in *plus* and *minus* of the quantity a; an advantage which appears to me of very great importance, and which must therefore in practice make us always prefer those continued fractions, in which the denominators are all positive. Hence, in what follows, we shall only attempt an investigation of fractions of this kind.

16. Let us, therefore, consider the series $\frac{A}{A'}, \frac{B}{B'}, \frac{C}{C'}, \frac{D}{D'}$, &c. in which the fractions are alternately less and greater than the quantity a, and which it is evident, we may divide into these two series:

$$\frac{A}{A'}, \frac{C}{C'}, \frac{E}{E'}, \text{\&c.}$$

$$\frac{B}{B'}, \frac{D}{D'}, \frac{F}{F'}, \text{\&c.}$$

of which the first will be composed of fractions all less than a, and which go on increasing towards the quantity a; the second will be composed of fractions all greater than a, but which go on diminishing towards that same quantity. Let us therefore examine each of those two series separately; in the first we have (Art. 10, and 12),

$$\frac{C}{C'} - \frac{A}{A'} = \frac{\gamma}{A'C'}$$

$$\frac{E}{E'} - \frac{C}{C'} = \frac{\varepsilon}{C'E'}, \text{\&c.}$$

and in the second we have,

$$\frac{B}{B'} - \frac{D}{D'} = \frac{\delta}{B'D'}$$

$$\frac{D}{D'} - \frac{F}{F'} = \frac{\zeta}{D'F'}, \text{\&c.}$$

Now, if the numbers γ, δ, ε, &c. were all equal to unity, we might prove, as in Art. 12, that between any two consecutive fractions of either of the preceding series, there could never be found any other fraction, whose denominator would be less than the denominators of those two fractions; but it will not be the same, when the numbers γ, δ, ε, &c. are greater than unity; for, in that case, we may insert between the fractions in question as many intermediate fractions as there are units in the numbers $\gamma - 1$, $\delta - 1$, $\varepsilon - 1$, &c. and for this purpose we shall only have to substitute, successively, in the values of C and C', (Art. 10), the numbers $1, 2, 3, \ldots\ldots \gamma$, instead of γ; and, in the values of D and D', the numbers $1, 2, 3, \ldots\ldots \delta$, instead of δ, and so on.

17. Suppose, for example, that $\gamma = 4$, we have $C = 4B + A$ and $C' = 4B' + A'$, and we may insert between the fractions $\frac{A}{A'}$ and $\frac{C}{C'}$, three intermediate fractions, which will be

$$\frac{B+A}{B'+A'},\ \frac{2B+A}{2B'+A'},\ \frac{3B+A}{3B'+A'}.$$

Now, it is evident, that the denominators of these fractions form an increasing arithmetical series from A' to C'; and we shall see that the fractions themselves also increase continually from $\frac{A}{A'}$ to $\frac{C}{C'}$; so that it would now be impossible to insert in the series

$$\frac{A}{A'},\ \frac{B+A}{B'+A'},\ \frac{2B+A}{2B'+A'},\ \frac{3B+A}{3B'+A'},\ \frac{4B+A}{4B'+A'},\ \text{or}\ \frac{C}{C'},$$

any fraction, whose value would fall between the values of two consecutive fractions, and whose denominator also would be found between the denominators of the same fractions. For, if we take the differences of the above fractions, since $BA' - AB' = 1$, we have,

$$\frac{B+A}{B'+A'} - \frac{A}{A'} = \frac{1}{A'(B'+A')}$$

$$\frac{2B+A}{2B'+A'} - \frac{B+A}{B'+A'} = \frac{1}{(B'+A') \times (2B'+A')}$$

$$\frac{3B+A}{3B'+A'} - \frac{2B+A}{2B'+A'} = \frac{1}{(2B'+A') \times (3B'+A')}$$

$$\frac{C}{C'} - \frac{3B+A}{3B'+A'} = \frac{1}{(3B'+A')C'};$$

whence we immediately perceive, that the fractions $\frac{A}{A'}$, $\frac{B+A}{B'+A'}$, &c. continually increase, since their differences are all positive; then, as those differences are equal to unity, if divided by the product of the two denominators, we may prove, by a reasoning analogous to that which we employed (Art. 12), that it is impossible for any fraction, $\frac{m}{n}$, to fall between two consecutive fractions of the preceding series, if the denominator n fall between the denominators of those fractions; or, in general, if it be less than the greater of the two denominators.

Farther, as the fractions of which we speak are all greater than the real value of a, and the fraction $\frac{B}{B'}$ is less than it, it is evident that each of those fractions will approximate towards the value of the quantity a, so that the difference will be less than that of the same fraction and the fraction $\frac{B}{B'}$; now, we find

$$\frac{A}{A'} - \frac{B}{B'} = \frac{1}{A'B'}$$
$$\frac{B+A}{B'+A'} - \frac{B}{B'} = \frac{1}{(B'+A')B'}$$
$$\frac{2B+A}{2B'+A'} - \frac{B}{B'} = \frac{1}{(2B'+A')B'}$$
$$\frac{3B+A}{3B'+A'} - \frac{B}{B'} = \frac{1}{(3B'+A')B'}$$
$$\frac{C}{C'} - \frac{B}{B'} = \frac{1}{C'B'}.$$

Therefore, since these differences are also equal to unity divided by the product of the denominators, we may apply to them the reasoning of Article 12, to prove that no fraction, $\frac{m}{n}$, can fall between any one of the fractions

$\frac{A}{A'}$, $\frac{B+A}{B'+A'}$, $\frac{2B+A}{2B'+A'}$, &c. and the fraction $\frac{B}{B'}$, if the denominator n be less than that of the same fraction; whence it follows, that each of those fractions approximates towards the quantity a nearer than any other fraction less than a, and having a less denominator; that is to say, expressed in simpler terms.

18. In the preceding Article, we have only considered the intermediate fractions between $\frac{A}{A'}$, and $\frac{C}{C'}$; but the same will be found true of the *intermediate* fractions between $\frac{C}{C'}$, and $\frac{E}{E'}$, between $\frac{E}{E'}$ and $\frac{G}{G'}$, &c. if ε, η, &c. are numbers greater than unity.

We may also apply what we have just said with respect to the first series $\frac{A}{A'}$, $\frac{C}{C'}$, &c. to the other series $\frac{B}{B'}$, $\frac{D}{D'}$, $\frac{F}{F'}$, &c. so that if the numbers δ, ζ, are greater than unity, we may insert between the fractions $\frac{B}{B'}$ and $\frac{D}{D'}$, $\frac{D}{D'}$ and $\frac{F}{F'}$, &c. different *intermediate* fractions, all greater than a, but which will continually diminish, and will be such as to express the quantity a more exactly than could be done by any other fraction greater than a, and expressed in simpler terms.

Farther, if β is also a number greater than unity, we may likewise place before the fractions $\frac{B}{B'}$ the fractions

$\frac{A+1}{1}$, $\frac{2A+1}{2}$, $\frac{3A+1}{3}$, &c. as far as $\frac{\beta A+1}{\beta}$, that is $\frac{B}{B'}$, and these fractions will have the same properties as the other *intermediate* fractions.

In this manner, we have these two complete series of fractions converging towards the quantity a.

Fractions increasing and less than a.

$$\frac{A}{A'},\ \frac{B+A}{B'+A'},\ \frac{2B+A}{2B'+A'},\ \frac{3B+A}{3B'+A'},\ \&c.\ \frac{\gamma B+A}{\gamma B'+A'},$$
$$\frac{C}{C'},\ \frac{D+C}{D'+C'},\ \frac{2D+C}{2D'+C'},\ \frac{3D+C}{3D'+C'},\ \&c.\ \frac{\varepsilon D+C}{\varepsilon D'+C'},$$
$$\frac{E}{E'},\ \frac{F+E}{F'+E'},\ \frac{2F+E}{2F'+E'},\ \frac{3F+E}{3F'+E'},\ \&c.$$

Fractioms decreasing and greater than a.

$$\frac{A+1}{1},\ \frac{2A+1}{2},\ \frac{3A+1}{3},\ \&c.\ \frac{\beta A+1}{\beta},$$
$$\frac{B}{B'},\ \frac{C+B}{C'+B'},\ \frac{2C+B}{2C'+B'},\ \&c.\ \frac{\delta C+B}{\delta C'+B'},$$
$$\frac{D}{D'},\ \frac{E+D}{E'+D'},\ \frac{2E+D}{2E'+D'},\ \frac{3E+D}{3E'+D'},\ \&c.$$

If the quantity a be irrational, or transcendental, the two preceding series will go on to infinity, since the series of fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. which in future we shall call *principal* fractions, to distinguish them from the *intermediate* fractions, goes on of itself to infinity. (Art. 10.)

But if the quantity a be rational, and equal to any fraction, $\frac{V}{V'}$, we have seen in that article, that the series in question will terminate, and that the last fraction of that series will be the fraction $\frac{V}{V'}$ itself; therefore, this fraction must also terminate one of the above two series, but the other series will go on to infinity.

In fact, suppose that δ is the last denominator of the continued fraction; then $\frac{D}{D'}$ will be the last of the principal fractions, and the series of fractions greater than a will be terminated by this same fraction $\frac{D}{D'}$. Now, the other series of fractions less than a, will naturally stop at the fraction $\frac{C}{C'}$, which precedes $\frac{D}{D'}$; but to continue it, we have only to consider that the denominator ε, which must follow the last denominator δ, will be $= \infty$ (Art. 3); so that the fraction $\frac{E}{E'}$, which would follow $\frac{D}{D'}$ in the series of principal fractions, would be $\frac{\infty D + C}{\infty D' + C'} = \frac{D}{D'}$ *; now, by the law of *intermediate* fractions, it is evident that, since $\varepsilon = \infty$, we might insert between the fractions $\frac{C}{C'}$ and $\frac{E}{E'}$, an infinite number of *intermediate* fractions, which would be

$$\frac{D+C}{D'+C'},\ \frac{2D+C}{2D'+C'},\ \frac{3D+C}{3D'+C'},\ \&c.$$

So that in this case, after the fraction $\frac{C}{C'}$, in the first series of fractions, we may also place the *intermediate* fractions we speak of, and continue them to infinity.

19. *Problem.* A fraction expressed by a great number of figures being given, to find all the fractions, in less terms, which approach so near the truth, that it is impossible to approach nearer without employing greater ones.

* Because an infinite quantity cannot be increased by addition; and therefore $\infty D + C = \infty D$, and $\infty D' + C' = \infty D'$; consequently,

$$\frac{\infty D + C}{\infty D' + C'} = \frac{\infty D}{\infty D'} = \frac{D}{D'}.$$ B.

This problem will be easily resolved by the theory which we have explained.

We shall begin by reducing the fraction proposed to a continued fraction after the method of Art. 4, observing to take all the approximate values less than the real ones, in order that the numbers β, γ, δ, &c. may be all positive; then, by the assistance of the numbers found, α, β, γ, &c. we form, according to the formulæ of Art. 10, the fractions

$\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. the last of which will necessarily be the

same as the fraction proposed; because in that case the continued fraction terminates. Those fractions will alternately be less and greater than the given fraction, and will be successively expressed in greater terms; and farther, they will be such, that each of those fractions will be nearer the given fraction than any other fraction can be, which is expressed in terms less simple. So that by these means we shall have all the fractions, that will satisfy the conditions of the problem, expressed in lower terms than the fraction proposed.

If we wish to consider separately the fractions which are less, and those which are greater, than the given fraction, we may insert between the above fractions as many *intermediate* fractions as we can, and form from them two series of converging fractions, the one all less, and the other all greater than the fraction proposed (Art. 16, 17, and 18); each of which series will have separately the same properties, as the

series of principal fractions $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. for the frac-

tions in each series will be successively expressed in greater terms, and each of them will approximate nearer to the value of the fraction proposed than could be done by any other fraction whether less, or greater, than the given fraction, but expressed in simpler terms.

It may also happen, that one of the *intermediate* fractions of one series does not approximate towards the given fraction so nearly, as one of the fractions of the other series, although expressed in terms less simple than the former; for this reason, it is not proper to employ *intermediate* fractions, except when we wish to have the fractions sought either all less, or all greater, than the given fraction.

20. *Example* 1. According to M. de la Caille, the solar year is $365^d.\ 5^h.\ 48'.\ 49''$, and, consequently, longer by $5^h\ 48'.\ 49''$ than the common year of 365^d. If this difference

were exactly 6 hours, it would make one day at the end of four common years: but if we wish to know, exactly, at the end of how many years this difference will produce a certain number of days, we must seek the ratio between 24^h, and $5^h. 48'. 49''$, which we find to be $\frac{86400}{20929}$; so that at the end of 86400 common years, we must intercalate 20929 days, in order to reduce them to tropical years.

Now, as the ratio of 86400 to 20929 is expressed in very high terms, let it be required to find ratios, in lower terms, as near this as possible.

For this purpose, we must reduce the fraction $\frac{86400}{20929}$ to a continued fraction, by the rule given in Art. 4, which is the same as that by which the greatest common divisor of two given numbers is found. This will give us

```
20929)86400(4 = α
      83716
      -----
       2684)20929(7 = β
            18788
            -----
             2141)2684(1 = γ
                  2141
                  ----
                   543)2141(3 = δ
                       1629
                       ----
                        512)543(1 = ε
                            512
                            ---
                             31)512(16 = ζ
                                496
                                ---
                                 16)31(1 = η
                                    16
                                    --
                                    15)16(1 = θ
                                       15
                                       --
                                        1)15(15 = ι
                                          15
                                          --
                                           0.
```

Now, as we know all the quotients α, β, γ, &c. we easily form from them the series $\frac{A}{A'}$, $\frac{B}{B'}$, &c. in the following manner:

$$\begin{array}{ccccccccc} 4, & 7, & 1, & 3, & 1, & 16, & 1, & 1, & 15. \\ \frac{4}{1}, & \frac{29}{7}, & \frac{33}{8}, & \frac{128}{31}, & \frac{161}{39}, & \frac{2704}{655}, & \frac{2865}{694}, & \frac{5569}{1349}, & \frac{86400}{20929}, \end{array}$$

the last fraction being the same as the one proposed.

In order to facilitate the formation of these fractions, we first write, as is here done, the series of quotients 4, 7, 1, &c. and place under these coefficients the fractions $\frac{4}{1}$, $\frac{29}{7}$, $\frac{33}{8}$, &c. which result from them.

The first fraction will have for its numerator the number which is above it, and for its denominator unity.

The second will have for its numerator the product of the number which is above it by the numerator of the first, plus unity, and for its denominator the number itself which is above it.

The third will have for its numerator the product of the number which is above it by the numerator of the second, plus that of the first; and, in the same manner, for its denominator, the product of the number which is above it by the denominator of the second, plus that of the first.

And, in general, each fraction will have for its numerator the product of the number which is above it by the numerator of the preceding fraction, plus that of the second preceding one; and for its denominator the product of the same number by the denominator of the preceding fraction, plus that of the second preceding one.

So that $29 = 7 \times 4 + 1$, $7 = 7$; $33 = 1 \times 29 + 4$, $8 = 1 \times 7 + 1$; $128 = 3 \times 33 + 29$, $31 = 3 \times 8 + 7$, and so on; which agrees with the formulæ of Art. 10.

Now, we see from the fractions $\frac{4}{1}$, $\frac{29}{7}$, $\frac{33}{8}$, &c. that the simplest intercalation is that of one day in four common years, which is the foundation of the Julian Calendar; but that we should approximate with more exactness by intercalating only 7 days in the space of 29 common years, or eight in the space of 33 years, and so on.

It appears farther, that as the fractions $\frac{4}{1}$, $\frac{29}{7}$, $\frac{33}{8}$, &c. are alternately less and greater than the fraction $\frac{86400}{20929}$, or $\frac{24^h}{5^h.48'.49''}$, the intercalation of one day in four years would be too much, that of seven days in twenty-nine years too little, that of eight days in thirty-three years too much, and so on; but each of these intercalations will be the most exact that it is possible to make in the same space of time.

Now, if we arrange in two separate series the fractions that are less, and those that are greater than the given fraction, we may also insert different secondary fractions to complete the series; and, for this purpose, we shall follow the same process as before, but taking successively, instead of each number of the upper series, all the integer numbers less than that number, when there are any.

So that, considering first the increasing fractions,

$$\begin{array}{ccccc} & 1, & 1, & 1, & 15, \\ \frac{4}{1}, & \frac{33}{8}, & \frac{161}{39}, & \frac{2865}{694}, & \frac{86400}{20929}, \end{array}$$

we see that, since unity is above the second, the third, and the fourth, we cannot place any *intermediate* fraction, either between the first and the second, or between the second and the third, or between the third and the fourth; but as the last fraction stands below the number 15, we may place, between that fraction and the preceding, fourteen *intermediate* fractions, the numerators* of which will form the arithmetical progression $2865 + 5569$, $2865 + 2 \times 5569$, $2865 + 3 \times 5569$, &c. their denominators will also form the arithmetical progression $694 + 1349$, $694 + 2 \times 1349$, $694 + 3 \times 1349$, &c.

So that the complete series of increasing fractions will be

$$\frac{4}{1}, \frac{33}{8}, \frac{161}{39}, \frac{2865}{694}, \frac{8434}{2043}, \frac{14003}{3392}, \frac{19572}{4741}, \frac{25141}{6090},$$
$$\frac{30710}{7439}, \frac{36279}{8788}, \frac{41848}{10137}, \frac{47417}{11486}, \frac{52986}{12835}, \frac{58555}{14184},$$
$$\frac{64124}{15533}, \frac{69693}{16882}, \frac{75262}{18231}, \frac{80831}{19580}, \frac{86400}{20929}.$$

And, as the last fraction is the same as the given fraction, it is evident that this series cannot be carried farther. Hence, if we choose to admit those intercalations only in which the error is too much, the simplest and most exact will be those of one day in four years, or of eight days in thirty-three years, or of thirty-nine in a hundred and sixty-one years, and so on.

Let us now consider the decreasing fractions,

$$\begin{array}{cccc} 7, & 3, & 16, & 1. \\ \frac{29}{7}, & \frac{128}{31}, & \frac{2704}{655}, & \frac{5569}{1349}. \end{array}$$

And first, on account of the number 7, which is above the first fraction, we may place six others before it, the numerators of which will form the arithmetical progression,

$$4 + 1,\ 2 \times 4 + 1,\ 3 \times 4 + 1, \text{\&c.}$$

and the denominators of which will form the progression

* Because $\frac{5569}{1349}$ is the principal fraction between $\frac{2865}{694}$, and $\frac{86400}{20929}$, as is found in the foregoing series. See page 485. B.

1, 2, 3, &c.*; also, on account of the number 3, we may place two *intermediate* fractions between the first and the second; and between the second and the third we may place fifteen, on account of the number 16 which is above the third; but between this and the last we cannot insert any, because the number above it is unity.

Farther, we must remark, that, as the preceding series is not terminated by the given fraction, we may continue it as far as we please, as we have shewn, Art. 18. So that we shall have this series of decreasing fractions,

$$\frac{5}{1}, \frac{9}{2}, \frac{13}{3}, \frac{17}{4}, \frac{21}{5}, \frac{25}{6}, \frac{29}{7}, \frac{62}{15}, \frac{95}{23}, \frac{128}{31},$$
$$\frac{289}{70}, \frac{450}{109}, \frac{611}{148}, \frac{772}{187}, \frac{933}{226}, \frac{1094}{265}, \frac{1255}{304}, \frac{1416}{343},$$
$$\frac{1577}{382}, \frac{1738}{421}, \frac{1899}{460}, \frac{2060}{499}, \frac{2221}{538}, \frac{2382}{577}, \frac{2543}{616},$$
$$\frac{2704}{655}, \frac{5569}{1349}, \frac{91969}{22278}, \frac{178369}{43207}, \frac{264769}{64136}, \frac{351169}{85065},$$
$$\frac{437569}{105994}, \text{ \&c.}$$

which are all less than the fraction proposed, and approach nearer to it than any other fractions expressed in simpler terms.

Hence we may conclude, that if we only attend to the intercalations, in which the error is too small, the simplest and most exact are those of one day in five years, or of two days in nine years, or of three days in thirteen years, &c.

In the Gregorian calendar, only ninety-seven days are intercalated in four hundred years; but it is evident, from the preceding series, that it would be much more exact, to intercalate a hundred and nine days in four hundred and fifty years.

But it must be observed, that in the Gregorian reformation, the determination of the year given by Copernicus was made use of, which is $365^d.\ 5^h.\ 49'.\ 20''$: and substituting this, instead of the fraction $\frac{86400}{20929}$, we shall have $\frac{86400}{20960}$, or rather $\frac{540}{131}$; whence we may find, by the preceding method, the quotients 4, 8, 5, 3, and from them the principal fractions,

$$\begin{matrix} 4, & 8, & 5, & 3. \\ \frac{4}{1}, & \frac{33}{8}, & \frac{169}{41}, & \frac{540}{131}, \end{matrix}$$

which, except the first two, are quite different from the fractions found before. However, we do not perceive among them the fraction $\frac{400}{97}$ adopted in the Gregorian calendar; and this fraction cannot even be found among the *intermediate* fractions, which may be inserted in

* See page 485.

the two series $\frac{4}{1}$, $\frac{169}{41}$, and $\frac{33}{8}$, $\frac{540}{131}$; for it is evident, that it could fall only between those last fractions, between which, on account of the number 3, which is above the fraction $\frac{540}{131}$, there may be inserted two intermediate fractions, which will be $\frac{202}{49}$, and $\frac{371}{90}$; whence it appears, that it would have been more exact, if in the Gregorian reformation they had only intercalated ninety days in the space of three hundred and seventy-one years.

If we reduce the fraction $\frac{400}{97}$, so as to have for its numerator the number 86400, it will become $\frac{86400}{20952}$, which estimates the tropical year at $365^d. 5^h. 49'. 12''$.

In this case, the Gregorian intercalation would be quite exact; but as observations make the year to be shorter by more than $20''$, it is evident that, at the end of a certain period of time, we must introduce a new intercalation.

If we keep to the determination of M. de la Caille, as the denominator 97 of the fraction $\frac{400}{97}$ lies between the denominators of the fifth and sixth principal fractions already found, it follows, from what we have demonstrated (Art. 14), that the fraction $\frac{161}{39}$ will be nearer the truth than the fraction $\frac{400}{97}$; but as astronomers are still divided with regard to the real length of the year, we shall refrain from giving a decisive opinion on this subject; our only object in the above detail is to facilitate the means of understanding continued fractions and their application: with this view, we shall also add the following example.

21. *Example* 2. We have already given, in Art. 8, the continued fraction, which expresses the ratio of the circumference of the circle to the diameter, as it results from the fraction of Ludolph; so that we have only to calculate, according to the manner taught in the preceding example, the series of fractions, converging towards that ratio, which will be

$$\begin{array}{ccccccc} 3, & 7, & 15, & 1, & 292, & 1, & 1, \\ \frac{3}{1}, & \frac{22}{7}, & \frac{333}{106}, & \frac{355}{113}, & \frac{103993}{33102}, & \frac{104348}{33215}, & \frac{208341}{66317}, \end{array}$$

$$\begin{array}{ccccc} 1, & 2, & 1, & 3, & 1, \\ \frac{312689}{99532}, & \frac{833719}{265381}, & \frac{1146408}{364913}, & \frac{4272943}{1360120}, & \frac{5419351}{1725033}, \end{array}$$

$$\begin{array}{cccc} 14, & 2, & 1, & 1, \\ \frac{80143857}{25510582}, & \frac{165707065}{52746197}, & \frac{245850922}{78256779}, & \frac{411557987}{131002976}, \end{array}$$

$$\begin{array}{ccc} 2, & 2, & 2 \\ \frac{1068966896}{340262731}, & \frac{2549491779}{811528438}, & \frac{6167950454}{1963319607}, \end{array}$$

$$\begin{array}{ccc} 2, & 1, & 84, \\ \frac{14885392687}{4738167652}, & \frac{21053343141}{6701487259}, & \frac{1783366216531}{567663097408}, \end{array}$$

2, $\frac{3587785776203}{1142027682075}$

1, $\frac{8958937768937}{2851718461558}$,

3, $\frac{428224593349304}{136308121570117}$,

1, $\frac{6134899525417045}{1952799169684491}$,

2, $\frac{66627445592888887}{21208174623389167}$,

6, $\frac{2646693125139304345}{842468587426513207}$,

1, $\frac{5371151992734}{1709690779483}$,

15, $\frac{139755218526789}{44485467702853}$,

13, $\frac{5706674932067741}{1816491048114374}$,

4, $\frac{30246273033735921}{9627687726852338}$,

6, $\frac{430010946591069243}{136876735467187340}$,

1, $\frac{3076704071730373588}{979345322893700547}$.

These fractions will therefore be alternately less and greater than the real ratio of the circumference to the diameter; that is to say, the first $\frac{3}{1}$ will be less, the second $\frac{22}{7}$ greater, and so on; and each of them will approach nearer the truth than can be done by any other fraction expressed in simpler terms; or, in general, having a denominator less than that of the succeeding fraction: so that we may be assured that the fraction $\frac{3}{1}$ approaches nearer the truth than any other fraction whose denominator is less than 7; also the fraction $\frac{22}{7}$ approaches nearer the truth than any other fraction whose denominator is less than 106; and so of others.

With regard to the error of each fraction, it will always be less than unity divided by the product of the denominator of that fraction, by the denominator of the following fraction. Thus, the error of the fraction $\frac{3}{1}$ will be less than $\frac{1}{7}$, that of the fraction $\frac{22}{7}$ will be less than $\frac{1}{7 \times 106}$, and so on. But, at the same time, the error of each fraction will be greater than unity divided by the product of the denominator of that fraction, into the sum of this denominator, and of the denominator of the succeeding fraction; so that the error of the fraction $\frac{3}{1}$ will be greater than $\frac{1}{8}$, that of the fraction $\frac{22}{7}$ greater than $\frac{1}{7 \times 113}$, and so on (Art. 14).

If we now wish to separate the fractions that are less than the ratio of the circumference to the diameter, from those which are greater, by inserting the proper *intermediate* fractions, we may form two series of fractions, the one increasing, and the other decreasing, towards the true ratio in question; in this manner we shall have

Fractions less than the ratio of the circumference to the diameter.

$$\frac{3}{1},\ \frac{25}{8},\ \frac{47}{15},\ \frac{69}{22},\ \frac{91}{29},\ \frac{113}{36},\ \frac{135}{43},\ \frac{157}{50},\ \frac{173}{57},$$
$$\frac{201}{64},\ \frac{223}{71},\ \frac{245}{78},\ \frac{267}{85},\ \frac{289}{92},\ \frac{311}{99},\ \frac{333}{106},\ \frac{688}{219},$$
$$\frac{1043}{332},\ \frac{1398}{445},\ \frac{1753}{558},\ \frac{2108}{671},\ \frac{2463}{784},\ \&c.$$

Fractions greater than the ratio of the circumference to the diameter.

$$\frac{4}{1},\ \frac{7}{2},\ \frac{10}{3},\ \frac{13}{4},\ \frac{16}{5},\ \frac{19}{6},\ \frac{22}{7},\ \frac{355}{113},\ \frac{104348}{33215},$$
$$\frac{312689}{99532},\ \frac{1146408}{364913},\ \frac{5419351}{1725033},\ \frac{85563208}{27235615},\ \frac{165707065}{52746197},$$
$$\frac{411557987}{131002976},\ \frac{1480524883}{471265707},\ \&c.$$

Each fraction of the first series approaches nearer the truth than any other fraction whatever, expressed in simpler terms, and the error of which consists in being too small; and each fraction of the second series likewise approaches nearer the truth than any other fraction, which is expressed in simpler terms, and the error of which consists in its being too large.

These series would become very long, if we were to continue them as far as we have done that of the principal fractions before given. The limits of this work do not permit us to insert them at full length; but they may be found, if wanted, in Chap. XI. of Wallis's Algebra. (*Oper. Mathemat.*).

SCHOLIUM.

22. The first solution of this problem was given by Wallis in a small treatise, which he added to the posthumous works of Horrox, and it is to be found in his Algebra as quoted above; but the method of this author is indirect, and very laborious. That which we have given belongs to Huygens, and is to be considered as one of the principal discoveries of that great mathematician. The construction of his planetary automaton appears to have led him to it: for, it is evident, that, in order to represent the motions and periods of the planets exactly, we should employ wheels, in which the teeth are precisely in the same ratios, with respect to number, as the periods in question; but as teeth cannot be multiplied beyond a certain limit, depending on the size of

the wheel, and, besides, as the periods of the planets are incommensurable, or, at least, cannot be represented, with any exactness, but by very large numbers, we must content ourselves with an approximation; and the difficulty is reduced to finding ratios expressed in smaller numbers, which approach the truth as nearly as possible, and nearer than any other ratios can, that are not expressed in greater numbers.

Huygens resolves this question by means of continued fractions as we have done; and explains the manner of forming those fractions by continual divisions, and then demonstrates the principal properties of the converging fractions, which result from them, without forgetting even the *intermediate* fractions. See, in his *Opera Posthuma*, the Treatise entitled *Descriptio Automati Planetarii.*

Other celebrated mathematicians have since considered continued fractions in a more general manner. We find particularly in the Commentaries of Petersburgh (Vol. IX. and XI. of the old, and Vol. IX. and XI. of the new), Memoirs by M. Euler, full of the most profound and ingenious researches on this subject; but the theory of these fractions, considered in an arithmetical view, which is the most curious, has not yet, I think, been cultivated so much as it deserves; which was my inducement for composing this small Treatise, in order to render it more familiar to mathematicians. See, also, the Memoirs of Berlin for the years 1767, and 1768.

I have only to observe farther, that this theory has a most extensive application through the whole of arithmetic; and there are few problems in that science, at least among those for which the common rules are insufficient, which do not, directly or indirectly, depend on it.

John Bernoulli has made a happy and useful application of it in a new species of calculation, which he devised for facilitating the construction of Tables of proportional parts. See Vol. I. of his *Recueil pour les Astronomes.*

CHAP. II.

Solution *of some curious and new* Arithmetical Problems.

Although the problems, which we are now to consider, are immediately connected with the preceding, and depend on the same principles, it will be proper to treat of them in a direct manner, without supposing any thing of what has been before demonstrated: by which means we shall have the satisfaction of seeing how necessarily these subjects lead to the theory of Continued Fractions. Besides, this theory will be rendered much more evident, and receive from it a greater degree of perfection.

23. *Problem* 1. A positive quantity a, whether rational or not, being given, to find two integer positive numbers, p and q, prime to each other; such, that $p - aq$ (abstracting from the sign), may be less than it would be, if we assigned to p and q any less values whatever.

In order to resolve this problem directly, we shall begin by supposing that we have already found values of p and q, which have the requisite conditions; wherefore, assuming for r and s, any integer positive numbers less than p and q, the value of $p - aq$ must be less than that of $r - as$, abstracting from the signs of these two quantities; that is to say, taking them both positive: now, if the numbers r and s be such, that $ps - qr = \pm 1$, (the upper sign applying when $p - aq$ is a positive number, and the under, when $p - aq$ is a negative number) we may conclude, in general, that the value of the expression $y - az$ will always be greater (abstracting from the sign) than that of $p - aq$, as long as we give to z and y only integer values, less than those of p and q, we may hence draw the following conclusion.

First, it is evident, that we may suppose, in general, $y = pt + ru$, and $z = qt + ru$, t and u being two unknown quantities. Now, by the resolution of these equations, we have $t = \frac{sy - rz}{ps - qr}$, $u = \frac{qy - pz}{qr - ps}$; and therefore, since $ps - qr = \pm 1$, $t = \pm (sy - rz)$, and $u = \pm (qy - pz)$; whence it is evident, that t and u will always be integer numbers, since p, q, r, s, y, and z are supposed to be integers.

Therefore, since t and u are integer numbers, and p, q, r, s integer positive numbers, it is evident, in order that the values of y and z may be less than those of p and q, that the numbers t and u must necessarily have different signs.

Now, I say, that the value of $r - as$ will also have a different sign from that of $p - aq$; for, making $p - aq = \text{P}$, and $r - as = \text{R}$, we shall have $\frac{p}{q} = a + \frac{\text{P}}{q}$, $\frac{r}{s} = a + \frac{\text{R}}{s}$; but the equation, $ps - qr = \pm 1$, gives $\frac{p}{q} - \frac{r}{s} = \pm \frac{1}{qs}$;

wherefore $\frac{P}{q} - \frac{R}{s} = \pm \frac{1}{qs}$; and, since we suppose the doubtful sign to be taken conformably to that of the quantity $p - aq$, or P, the quantity $\frac{P}{q} - \frac{R}{s}$ must be positive, if P be positive; and negative, if P be negative: now, as $s < q$, and $R > P$ (*hyp.*), it is evident that $\frac{R}{s} > \frac{P}{q}$, (abstracting from the sign); therefore, the quantity $\frac{P}{q} - \frac{R}{s}$ will always have its sign different from that of $\frac{R}{s}$; that is to say, from that of R, since s is positive; and, consequently, P and R will necessarily have different signs.

This being laid down, we shall have, by substituting the above values of y and z,

$$y - az = (p - aq)t + (r - as)u = Pt + Ru.$$

Now t and u having different signs, as well as P and R, it is evident, that Pt and Ru will be quantities of like signs; therefore, since t and u are integer numbers, it is clear that the value of $y - az$ will always be greater than P; that is to say, than the value of $p - aq$, abstracting from the signs.

But it remains to know whether, when the numbers p and q are given, we can always find numbers r and s less than those, and such that $ps - qr = \pm 1$, the doubtful signs being arbitrary; now, this follows evidently from the theory of continued fractions; but it may be demonstrated directly, and independently of that theory. For the difficulty is reduced to proving, that there necessarily exists an integer and positive number less than p, which being assumed for r, will make $qr \pm 1$ divisible by p. Now, suppose we successively substitute for r the natural numbers 1, 2, 3, &c. as far as p, and that we divide the numbers $q \pm 1$, $2q \pm 1$, $3q \pm 1$, &c. $pq \pm 1$ by p, we shall then have p remainders less than p, which will necessarily be all different from one another; since, for example, if $mq \pm 1$, and $nq \pm 1$ (m and n being distinct integer numbers not exceeding p), when divided by p, give the same remainder, it is evident that their difference $(m - n)q$, must be divisible by p; now, this is impossible, because q is prime to p, and $m - n$ is a number less than p.

Therefore, since all the remainders in question are integer, positive numbers less than p, and different from each other, and are p in number, it is evident that 0 must be among those remainders, and, consequently, that there is one of the numbers $q \pm 1$, $2q \pm 1$, $3q \pm 1$, &c. $pq \pm 1$, which is divisible by p. Now, it is evident that this cannot be the last; so that there is certainly a value of r less than p, which will make $rq \pm 1$ divisible by p; and it is evident, at the same time, that the quotient will be less than q; therefore there will always be an integer and positive value of r less than p, and another similar value of s, and less than q, which will satisfy the equation $s = \frac{qr \pm 1}{p}$, or $ps - qr = \pm 1$.

24. The question is therefore now reduced to this; to find four positive whole numbers, p, q, r, s, the last two of which may be less than the first two; that is, $r \angle p$, and $s \angle q$, and such, that $ps - qr = \pm 1$; farther, that the quantities $p - aq$, and $r - as$, may have different signs, and, at the same time, that $r - as$ may be a quantity greater than $p - aq$, abstracting from the signs.

In order to simplify, let us denote r by p', and s by q', so that we have $pq' - qp' = \pm 1$; and as $q \; 7 \; q$ (*hyp.*), let μ be the quotient that would be produced by the division of q by q', and let the remainder be q'', which will consequently be $\angle q'$; also, let μ' be the quotient of the division of q' by q'', and q''' the remainder, which will be $\angle q''$; in like manner, let μ'' be the quotient of the division of q'' by q''', and q^{iv} the remainder $\angle q'''$, and so on, till there is no remainder; in this way, we shall have

$$\begin{aligned} q &= \mu q' + q'' \\ q' &= \mu' q'' + q''' \\ q'' &= \mu'' q''' + q^{iv} \\ q''' &= \mu''' q^{iv} + q^{v}, \text{ \&c.} \end{aligned}$$

where the numbers μ, μ', μ'', &c. will all be integer and positive, and the numbers p, q', q'', q''', &c. will also be integer and positive, and will form a series decreasing to nothing.

In like manner, let us suppose

$$\begin{aligned} p &= \mu p' + p'' \\ p' &= \mu' p'' + p''' \\ p'' &= \mu'' p''' + p^{iv} \\ p''' &= \mu''' p^{iv} + p^{v}, \text{ \&c.} \end{aligned}$$

And as the numbers p and p' are considered here as given, as well as the numbers μ, μ', μ'', &c. we may determine from these equations the numbers p'', p''', p^{iv}, &c. which will evidently be all integer.

Now, as we must have $pq' - qp' = \pm 1$, we shall also have, by substituting the preceding values of p and q, and effacing what is destroyed, $p''q' - q''p' = \pm 1$. Again, substituting in this equation the values of p' and q', there will result $p''q''' - q''p''' = \pm 1$, and so on; so that we shall have, generally,

$$\begin{aligned} p\,q' - q\,p' &= \pm 1 \\ p'\,q'' - q'\,p'' &= \mp 1 \\ p''q''' - q''p''' &= \pm 1 \\ p'''q^{iv} - q'''p^{iv} &= \mp 1, \text{ \&c.} \end{aligned}$$

So that, if q''', for example, were $= 0$, we should have $- q''p''' = \pm 1$; also, $q'' = 1$, and $p''' = \mp 1$: but if q^{iv} were $= 0$, we should have $- q'''p^{iv} = \mp 1$; therefore $q''' = 1$, and $p^{iv} = \pm 1$; so that, in general, if $q^{\varrho} = 0$, we shall have $q^{\varrho-1} = 1$; and then $p^{\varrho} = \pm 1$, if ϱ is even, and $p^{\varrho} = \mp 1$, if ϱ is odd.

Now, as we do not previously know whether the upper, or the under sign is to take place, we must successively suppose $p^{\varrho} = 1$, and $= -1$: but I say that one of these cases may at all times be reduced to the other; and, for this purpose, it is evidently sufficient to prove, that we can always make the ϱ of the term q^{ϱ}, which must be nothing, either even, or odd, at pleasure.

For example, let us suppose that $q^{iv} = 0$, we shall then have $q''' = 1$, and $q'' > 1$, that is $q'' = 2$, or > 2, because the numbers q, q', q'', &c. naturally form a decreasing series; therefore, since $q'' = \mu''q''' + q^{iv}$; we shall have $q'' = \mu''$, so that $\mu'' =$ or > 2; thus, if we choose, we may diminish μ'' by unity, without that number being reduced to nothing, and then q^{iv}, which was 0, will become 1, and $q^{v} = 0$; for putting $\mu'' - 1$, instead of μ'', we shall have $q'' = (\mu'' - 1)q''' + q^{iv}$; but $q'' = \mu''$, $q''' = 1$; wherefore, $q^{iv} = 1$; then having $q''' = \mu'''q^{iv} + q^{v}$, that is, $1 = \mu''' + q^{v}$, we shall necessarily have $\mu''' = 1$, and $q^{v} = 0$.

Hence we may conclude, in general, that if $q^{\varrho} = 0$, we shall have $q^{\varrho-1} = 1$, and $p^{\varrho} = \pm 1$, the doubtful sign being arbitrary.

Now, if we substitute the values of p and q, given by the preceding formulæ, in $p - aq$, those of p' and q', in $p' - aq'$, and so of others, we shall have

$$\begin{aligned} p - aq &= \mu\,(p' - aq') + p'' - aq'' \\ p' - aq' &= \mu'\,(p'' - aq'') + p''' - aq''' \\ p'' - aq'' &= \mu''(p''' - aq''') + p^{iv} - aq^{iv} \\ p''' - aq''' &= \mu'''(p^{iv} - aq^{iv}) + p^{v} - aq^{v}, \text{ \&c.} \end{aligned}$$

whence we find

$$\mu = \frac{aq'' - p''}{p' - aq'} + \frac{p - aq}{p' - aq'}$$
$$\mu' = \frac{aq''' - p'''}{p'' - aq''} + \frac{p' - aq'}{p'' - aq''}$$
$$\mu'' = \frac{aq^{\text{iv}} - p^{\text{iv}}}{p''' - aq'''} + \frac{p'' - aq''}{p''' - aq'''}$$
$$\mu''' = \frac{aq^{\text{v}} - p^{\text{v}}}{p^{\text{iv}} - aq^{\text{iv}}} + \frac{p''' - aq'''}{p^{\text{iv}} - aq^{\text{iv}}}, \text{ \&c.}$$

Now, as by hypothesis the quantities $p - aq$, and $p' - aq'$, are of different signs; and farther, as $p' - aq'$ (abstracting from the signs) must be greater than $p - aq$, it follows that $\frac{p - aq}{p' - aq'}$, will be a negative quantity, and less than unity. Therefore, in order that μ may be an integer, positive number, as it must, it is evident, that $\frac{aq'' - p''}{p' - aq'}$ must be a positive quantity greater than unity; and it is obvious, at the same time, that μ can only be the integer number, that is immediately less than $\frac{aq'' - p''}{p' - aq'}$; that is to say, contained between the limits $\frac{aq'' - p''}{p' - aq'}$, and $\frac{aq'' - p''}{p' - aq'} - 1$; for since $-\frac{p - aq}{p' - aq'} > 0$, and < 1, we shall have $\mu < \frac{aq'' - p''}{p' - ap'}$ and

$$> \frac{aq'' - p''}{p' - aq'} - 1.$$

Also, since we have seen, that $\frac{aq'' - p''}{p' - aq'}$ must be a positive quantity greater than unity, it follows that $\frac{p' - aq'}{p'' - aq''}$ will be a negative quantity less than unity, (I say less than unity, abstracting from the sign). Wherefore, in order that μ' may be an integer, positive number, $\frac{aq''' - p'''}{p'' - aq''}$ must be a positive quantity greater than unity, and consequently the number μ' can only be the integer number, which will be immediately below the quantity $\frac{aq''' - p'''}{p'' - aq''}$.

In the same manner, and from the consideration, that μ''

must be an integer, positive number, we may prove, that the quantity $\frac{aq^{iv} - p^{iv}}{p''' - aq'''}$ will necessarily be positive, and greater than unity, and that μ'' can only be the integer number immediately below the same quantity; and so on.

It follows, 1st, that the quantities $p - aq$, $p' - aq'$, $p'' - aq''$, &c. will successively have different signs; that is, alternately positive and negative, and will form a series continually increasing. 2dly, that if we denote by the sign $\angle$ the integer number which is immediately less than the value of the quantity placed after that sign, we shall have, for the determination of the numbers μ, μ', μ'', &c.

$$\mu \angle \frac{aq'' - p''}{p' - aq'}$$
$$\mu' \angle \frac{aq''' - p'''}{p'' - aq''}$$
$$\mu'' \angle \frac{aq^{iv} - p^{iv}}{p''' - aq'''}$$

Now, we have already seen, that the series q, q', q'', &c. must terminate in 0; and that then the preceding term will be 1, and the term corresponding to 0 in the other series p, p', p'', &c. will be $= \pm 1$ at pleasure.

For example, let us suppose that $q^{iv} = 0$, we shall then have $q''' = 1$, and $p^{iv} = 1$; therefore

$$p''' - aq''' = p''' - a, \text{ and}$$
$$p^{iv} - aq^{iv} = 1;$$

therefore $p''' - a$ must be a negative quantity, and less than 1, abstracting from the sign; that is, $a - p'''$ must be $\urcorner 0$, and $\angle 1$; so that p''' must be the integer number immediately below a; we shall therefore know the values of these four terms,

$$p^{iv} = 1 \qquad q^{iv} = 0$$
$$p''' \angle a \qquad q''' = 1$$

by means of which, going back through the former formulæ, we may find all the preceding terms. We shall first have the value of μ'', then we shall have p'' and q'', by the formulæ,

$$p'' = \mu'' p''' + p^{iv}, \text{ and}$$
$$q'' = \mu''' q''' + q^{iv};$$

from which we shall get μ', and then p' and q'; and so of the rest.

In general, let $q^{\varrho} = 0$, then we shall have $q^{\varrho - 1}$, and $p^{\varrho} = 1$; and shall prove, as before, that $p^{\varrho - 1}$ can only be the integer number immediately below a; so that we shall have these four terms,

$$p^{\varrho} = 1 \qquad\qquad q^{\varrho} = 0$$
$$p^{\varrho-1} \angle a \qquad\qquad q^{\varrho-1} = 1;$$

we shall then have

$$\mu^{\varrho-2} \angle \frac{aq^{\varrho} - p^{\varrho}}{p^{\varrho-1} - aq^{\varrho-1}} \angle \frac{1}{a - p^{\varrho-1}}.$$

$$p^{\varrho-2} = \mu^{\varrho-2}p^{\varrho-1} + p^{\varrho},\ q^{\varrho-2} = \mu^{\varrho-2}q^{\varrho-1} + q^{\varrho}$$

$$\mu^{\varrho-3} \angle \frac{aq^{\varrho-1} - p^{\varrho-1}}{p^{\varrho-2} - aq^{\varrho-2}}$$

$$p^{\varrho-3} = \mu^{\varrho-3}p^{\varrho-2} + p^{\varrho-1},\quad q^{\varrho-3} = \mu^{\varrho-3}q^{\varrho-2} + q^{\varrho-1},$$

and so on.

In this manner, therefore, we may go back to the first terms, p and q; but it must be observed, that all the succeeding terms, p', q', p'', q'', &c. possess the same properties, and serve equally to resolve the problem proposed. For it is evident, in the preceding formulæ, that the numbers p, p', p'', &c. and q, q', q'', &c. are all integer and positive, and form two series continually decreasing; the first of which is terminated by unity, and the second by 0.

Farther, we have seen that these numbers are such, that $pq' - qp' = \pm 1$, $p'q'' - q'p'' = \mp 1$, &c. and that the quantities $p - aq$, $p' - aq'$, $p'' - aq''$, &c. are alternately positive and negative, and at the same time form a series continually increasing. Whence it follows, that the same conditions which exist among the four numbers p, q, r, s, or p, q, p', q', and on which, as we have seen, the solution of the problem depends, equally exist among the numbers p', q', p'', q'', and among these, p'', q'', p''', q''', and so on.

Therefore, beginning with the last terms p^{ϱ} and q^{ϱ}, and going back always by the formulæ we have just found, we shall successively have all the values of p and q that can resolve the question proposed.

25. As the values of the terms p^{ϱ}, $p^{\varrho-1}$, &c. q^{ϱ}, $q^{\varrho-1}$, &c. are independent of the exponent, ϱ, we may abstract from it, and denote the terms of these two increasing series thus,

$$p^{0}, p', p'', p''', p^{iv}, \text{\&c. } q^{0}, q', q'', q''', q^{iv}, \text{\&c.}$$

so that we shall have the following results.

$$\begin{array}{ll}
p^{0} = 1 & q^{0} = 0 \\
p' = \mu & q' = 1 \\
p' = \mu' p' + 1 & q'' = \mu' \\
p''' = \mu'' p'' + p' & q''' = \mu'' q'' + q' \\
p^{iv} = \mu''' p''' + p'' & q^{iv} = \mu''' q''' + q'' \\
\text{\&c.} & \text{\&c.}
\end{array}$$

Then

$$\mu \angle a$$

$$\mu' \angle \frac{p^0 - aq^0}{aq' - p'} \angle \frac{1}{a - \mu}$$

$$\mu'' \angle \frac{aq' - p'}{p'' - aq''}$$

$$\mu''' \angle \frac{p'' - aq''}{aq''' - p'''}$$

$$\mu^{\text{iv}} \angle \frac{aq''' - p'''}{p^{\text{iv}} - aq^{\text{iv}}}, \text{ \&c.}$$

Where the sign $\angle$ denotes the integer number immediately less than the value of the quantity placed after that sign.

Thus, we shall successively find all the values of p and q that can satisfy the problem; these values being only the correspondent terms of the two series p^0, p', p'', p''', &c. and q^0, q', q'', q''', &c.

26. *Corollary* 1. If we make

$$b = \frac{p^0 - ap^0}{aq' - p'}$$

$$c = \frac{aq' - p'}{p'' - aq''}$$

$$d = \frac{p'' - aq''}{aq''' - p'''}, \text{ \&c.}$$

we shall have, as it is easy to perceive,

$$b = \frac{1}{a - \mu}$$

$$c = \frac{1}{b - \mu'}$$

$$d = \frac{1}{c - \mu''}, \text{ \&c.}$$

and $\mu \angle a$, $\mu' \angle b$, $\mu'' \angle c$, $\mu''' \angle d$, &c. therefore the numbers μ, μ', μ'', &c. will be no other than those which we have denoted by α, β, γ, &c. in Art. 3; that is to say, these numbers will be the terms of the continued fraction, which represents the value of a; so that we shall have here

$$a = \mu + \cfrac{1}{\mu' + \cfrac{1}{\mu''} + , \text{ \&c.}}$$

Consequently, the numbers p', p'', p''', &c. will be the numerators, and q', q'', q''', &c. the denominators of the fractions converging to a, fractions which we have already denoted by $\frac{A}{A'}$, $\frac{B}{B'}$, $\frac{C}{C'}$, &c. (Art. 10).

So that the whole is reduced to converting the value of a into a continued fraction, having all its terms positive; which may be done by the methods already explained, provided we are always careful to take the approximated values too small; then we shall only have to form the series of *principal* fractions converging towards a, and the terms of each of these fractions will give the values of p and q, which will resolve the problem proposed; so that $\frac{p}{q}$ can only be one of these fractions.

27. *Corollary* 2. Hence results a new property of the fractions we speak of; calling $\frac{p}{q}$ one of the *principal* fractions converging towards a, (provided they are deduced from a continued fraction, all the terms of which are positive), the quantity $p - aq$ will always have a less value (abstracting from the sign), than it would have, were we to substitute in the room of p and q any other smaller numbers.

28. *Problem* 2. The quantity

$$Ap^m + Bp^{m-1}q + Cp^{m-2}q^2 +, \&c. + Vq^m,$$

being proposed, in which A, B, C, &c. are given integers, positive or negative, and p and q unknown numbers, which must be integer and positive; it is required to determine what values we must give to p and q, in order that the quantity proposed may become the least possible.

Let α, β, γ, &c. be the real roots, and $\mu \pm \nu\sqrt{-1}$, $\pi \pm \varrho\sqrt{-1}$, &c. the imaginary roots of the equation

$$Ax^m + Bx^{m-1} + Cx^{m-2} +, \&c. + V = 0,$$

then we shall have, by the theory of equations,

$$\begin{aligned}
&Ap^m + Bp^{m-1}q + Cp^{m-2}q^2 +, \&c. + Vq^m = \\
&A(p - aq) \times (p - \beta q) \times (p - \gamma q) \ldots \times \\
&\quad (p - (\mu + \nu\sqrt{-1})q) \times (p - (\mu - \nu\sqrt{-1})q) \times \\
&\quad (p - (\pi + \varrho\sqrt{-1})q) \times (p - (\pi - \varrho\sqrt{-1})q) \ldots = \\
&A(p - \alpha q) \times (p - \beta q) \times (p - \gamma q) \ldots \times \\
&((p - \mu q)^2 + \nu^2 q^2) \times ((p - nq)^2 + \varrho^2 q^2) * \ldots
\end{aligned}$$

* Because $(p - (\mu + \nu\sqrt{-1})q) \times (p - (\mu - \nu\sqrt{-1})q) = p^2 - 2p\mu q + \mu^2 q^2 + \nu^2 q^2 = (p - \mu q)^2 + \nu^2 q^2$, and the same with the others. B.

Therefore the question is reduced to making the product of the quantities $p - \alpha q$, $p - \beta q$, $p - \gamma q$, &c. and

$$(p - \mu q)^2 + \nu^2 q^2,\ (p - \pi q)^2 + \varrho^2 q^2,\ \&c.$$

the least possible, when p and q are integer, positive numbers.

Now, suppose we have found the values of p and q which answer to the *minimum;* and if we substitute other smaller numbers for p and q, the product in question must acquire a greater value. It will therefore be necessary for each of the factors to increase in value. Now, it is evident, that if α, for example, were negative, the factor $p - \alpha q$ would always diminish, when p and q decreased; the same thing would happen to the factor $(p - \mu q)^2 + \nu^2 q^2$, if μ were negative, and so of the others; whence it follows, that among the simple real factors none but those where the roots are positive, can increase in value; and among the double imaginary factors, those only, in which the real part of the imaginary root is positive, can increase. Farther, it must be remarked, with regard to these last, that in order that $(p - \mu q)^2 + \nu^2 q^2$ may increase, whilst p and q diminish, the part $(p - \mu q)^2$ must necessarily increase, because the other term $\nu^2 q^2$ necessarily diminishes; so that the increase of this factor will depend on the quantity $p - \mu q$; and so of the others.

Therefore, the values of p and q, which answer to the *minimum*, must be such, that the quantity $p - aq$ may increase, by giving less values to p and q, and taking for a one of the real positive roots of the equation,

$$\text{A}x^m + \text{B}x^{m-1} + \text{C}x^{m-2} +, \&c. + \text{V} = 0,$$

or one of the real positive parts of the imaginary roots of the same equation, if there be any.

Let r and s be two integer, positive numbers less than p and q; then $r - as$ must be $> (p - aq)$, abstracting from the sign of the two quantities. Let us therefore suppose, as in Art. 23, that these numbers are such, that $ps - qr = \pm 1$, the upper sign taking place, when $p - aq$ is positive; and the under, when $p - aq$ is negative; so that the two quantities $p - aq$, and $r - as$, become of different signs, and we shall exactly have the case to which we reduced the preceding problem, Art. 24, and of which we have already given the solution.

Hence, by Art. 26, the values of p and q will necessarily be found among the terms of the *principal* fractions converging towards a; that is, towards any one of the quantities, which we have said may be taken for a. So that we must reduce all these quantities to continued fractions; which may easily be done by the methods elsewhere taught, and then deduce the converging fractions required: after which, we must successively make p equal to all the numerators of these fractions, and q equal to the corresponding denominators, and of these suppositions, that which shall give the least value of the proposed function will necessarily answer likewise to the *minimum* required.

29. *Scholium* 1. We have supposed that the numbers p and q must both be positive; it is evident that if we were to take them both negative, no change would result in the absolute value of the formula proposed; it would only change its sign in the case of the exponent m being odd; and it would remain quite the same, in the case of the exponent m being even: so that it is of no consequence what signs we give the numbers p and q, when we suppose them both of the same kind.

But it will not be the same, if we give different signs to p and q; for then the alternate terms of the equation proposed will change their signs, which will also change the signs of the roots α, β, γ, &c. $\mu \pm \nu \sqrt{-1}$, $\pi \pm \varrho \sqrt{-1}$, &c. so that those of the quantities α, β, γ, &c. μ, π, &c. which were negative, and consequently useless in the first case, will become positive in this, and must be employed instead of the other.

Hence, I conclude, generally, that when we investigate the *minimum* of the proposed formula, without any other restriction, than that of p and q being whole numbers, we must successively take for a all the real roots α, β, γ, &c. and all the real parts μ, π, &c. of the imaginary roots of the equation $\text{A}x^m + \text{B}x^{m-1} + \text{C}x^{m-2} +$, &c. $+ \text{V} = 0$; abstracting from the signs of these quantities; but then we must give the same signs, or different signs, to p and q, according as the quantity we have taken for a, had originally the positive, or the negative sign.

30. *Scholium* 2. When among the real roots α, β, γ, &c. there are some commensurable, then it is evident that the quantity proposed will become nothing, by making $\frac{p}{q}$ equal to one of these roots; so that in this case, properly speaking, there will be no *minimum*. In all the other cases, it will be impossible for the quantity in question to become 0, whilst p and q are whole numbers. Now, as the coefficients A, B, C, &c. are also whole numbers, *by hypothesis*, this quantity will always be equal to a whole number; and, consequently, it can never be less than unity.

If we had, therefore, to resolve the equation

$$Ap^m + Bp^{m-1}q + Cp^{m-2}q^2 +, \&c. + Vq^m = \mp 1,$$

in whole numbers, we must seek for the values of p and q by the method of the preceding problem, except in the case where the equation

$$Ax^m + Bx^{m-1} + Cx^{m-2} +, \&c. + V = 0,$$

had roots, or any divisors commensurable; for then, it is evident, that the quantity

$$Ap^m + Bp^{m-1}q + Cp^{m-2}q^2 +, \&c.$$

might be decomposed into two or more similar quantities of less degrees; so that it would be necessary for each of these partial formulæ to be separately equal to unity, which would give at least two equations that would serve to determine p and q.

We have elsewhere given a solution of this last problem (*Memoires pour l'Academie de Berlin pour l'Année* 1768); but the one we are going to explain is much more simple and direct, although both depend on the same theory of continued fractions *.

31. *Problem* 3. Required the values of p and q, which will render the quantity $Ap^2 + Bpq + Cq^2$ the least possible, supposing that whole numbers only are admitted for p and q.

This problem evidently is only a particular case of the preceding; but it may be proper to consider it separately, because it is capable of a very simple and elegant solution; and, besides, we shall have occasion afterwards to make use of it, in resolving quadratic equations for two unknown quantities in whole numbers.

According to the general method, we must begin, therefore, by seeking the roots of the equation $Ax^2 + Bx + C = 0$, which we know to be, $\frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}$.

1st, If $B^2 - 4AC$ be a square number, the two roots will be commensurable, and there will properly be no *minimum*, because the quantity $Ap^2 + Bpq + Cq^2$ will become 0.

2d, If $B^2 - 4AC$ be not a square, then the two roots will be irrational, or imaginary, according as $B^2 - 4AC$ will be $>$, or $<$ 0, which makes two cases that must be considered separately; we shall begin with the latter, which it is most easy to resolve.

First case, when $B^2 - 4AC < 0$.

32. The two roots being in this case imaginary, we shall

* See also Le Gendre's Essai sur la Theorie des Nombres, page 169.

have $\frac{-\text{B}}{2\text{A}}$ for the whole real part of these roots, which must consequently be taken for a. So that we shall only have to reduce the fraction $\frac{-\text{B}}{2\text{A}}$, abstracting from the sign it may have, to a continued fraction, by the method of Art. 4, and then deduce from it the series of converging fractions (Art. 10), which will necessarily terminate. This being done, we shall successively try for p the numerators of these fractions, and the corresponding denominators for q, taking care to give p and q the same, or different signs, according as $\frac{-\text{B}}{2\text{A}}$ is a positive, or negative number. In this manner, we shall find the values of p and q, that may render the formula proposed a *minimum*.

Example. Let there be proposed, for example, the quantity

$$49p^2 - 238pq + 290q^2.$$

Here, we shall have $\text{A} = 49$, $\text{B} = -238$, $\text{C} = 290$; wherefore $\text{B}^2 - 4\text{AC} = -196$, and $\frac{-\text{B}}{2\text{A}} = \frac{238}{98} = \frac{17}{7}$. Working with this fraction according to the method of Art. 4, we shall find the quotients 2, 2, 3; by means of which, we shall form these fractions (see Art. 20),

$$2,\ 2,\ 3.$$
$$\frac{1}{0},\ \frac{2}{1},\ \frac{5}{2},\ \frac{17}{7}.$$

So that the numbers to try with will be 1, 2, 5, 17, for p, and 0, 1, 2, 7, for q. Now, denoting the quantity proposed by P, we shall have

p	q	P
1	0	49
2	1	10
5	2	5
17	7	49;

whence we perceive, that the least value of p is 5, which results from these suppositions $p = 5$, and $q = 2$; so that we may conclude, in general, that the given formula can never become less than 5, while p and q are whole numbers; so that the *minimum* will take place, when $p = 5$, and $q = 2$.

Second case, when $\text{B}^2 - 4\text{AC} > 0$.

33. As, in the present case, the equation $\text{A}x^2 + \text{B}x + \text{C} = 0$,

has two real irrational roots, they must both be reduced to continued fractions. This operation may be performed with the greatest ease by a method which we have elsewhere explained, and which it may be proper to repeat here, since it is naturally deduced from the formulæ of Art. 25, and likewise contains all the principles necessary for the complete and general solution of the problem proposed.

Let us, therefore, denote the root which is to be thrown into a continued fraction by a, which we shall suppose to be always positive; at the same time, let b be the other root, and we shall evidently have $a + b = -\frac{B}{A}$, and $ab = \frac{C}{A}$; whence $a - b = \frac{\sqrt{(B^2 - 4AC)}}{A}$; or, for the sake of abridgment, making $B^2 - 4AC = E$, $a - b = \frac{\sqrt{E}}{A}$, where the radical $\sqrt{E}$ may be positive, or negative: it will be positive, when the root a is the greater of the two, and negative, when that root is the less; therefore

$$a = \frac{-B + \sqrt{E}}{2A},\ b = \frac{-B - \sqrt{E}}{2A}.$$

Now, if we preserve the denominations of Art. 25, we shall only have to substitute for a the preceding value, and the difficulty will only consist in determining the integer, approximate values, μ', μ'', μ''', &c.

To facilitate these determinations, I multiply the numerator and the denominator of the fractions,

$\frac{p^0 - aq^0}{aq' - p'}$, $\frac{aq' - p'}{p'' - aq''}$, $\frac{p'' - aq''}{aq''' - p'''}$, &c. respectively by

$$A(bq' - p'),\ A(p'' - bq''),\ A(bq''' - p'''),\ \&c.$$

and as we have

$$A(p^0 - aq^0) \times (p^0 - bq^0) = A$$

$$A(aq' - p') \times (bq' - p') = Ap'^2 - A(a + b)p'q' + Aabq'^2 =$$

$$Ap'^2 + Bp'q' + Cq'^2,$$

$$A(p'' - aq'') \times (p'' - bq'') = Ap''^2 - A(a + b)p''q'' + Aabq''^2 =$$

$$Ap''^2 + Bp''q'' + Cq''^2,\ \&c.$$

$$A(p^0 - aq^0) \times (bq' - p') = -\mu A - \tfrac{1}{2}B - \tfrac{1}{2}\sqrt{E},$$

$$A(aq' - p') \times (p'' - bq'') =$$
$$- Ap'p'' + Aap''q' + Abp'q'' - Aabq'q'' =$$
$$- Ap'p'' - Cq'q'' - \tfrac{1}{2}B(p'q'' + q'p'') + \tfrac{1}{2}\sqrt{E}(p''q' - q''p'),$$
$$A(p'' - aq'') \times (bq''' - p''') =$$
$$- Ap''p''' + Aap'''p'' + Abp''q''' - Aabq''q''' =$$
$$- Ap''p''' - Cq''q''' - \tfrac{1}{2}B(p''q''' + q''p''') + \tfrac{1}{2}\sqrt{E}(p'''q'' - q'''p''),$$

and so on. Now, in order to abridge, let us make

$$P^0 = A$$
$$P' = A\overset{\prime}{p}{}^2 + Bp'q' + C\overset{\prime}{q}{}^2$$
$$P'' = A\overset{\prime\prime}{p}{}^2 + Bp''q'' + C\overset{\prime\prime}{q}{}^2$$
$$P''' = A\overset{\prime\prime\prime}{p}{}^2 + Bp'''q''' + C\overset{\prime\prime\prime}{q}{}^2, \ \&c.$$

$$Q^0 = \tfrac{1}{2}B$$
$$Q' = A\mu + \tfrac{1}{2}B$$
$$Q'' = Ap'p'' + \tfrac{1}{2}B(p'q'' + q'p'') + Cq'q''$$
$$Q''' = Ap''p''' + \tfrac{1}{2}B(p''q''' + q''p''') + Cq''q''', \ \&c.$$

Because

$$p''q' - q''p' = 1,\ p'''q'' - q'''p'' = -1,\ p^{iv}q''' - q^{iv}p''' = 1, \&c.$$

we shall have the following values,

$$\mu \angle \frac{-Q^0 + \tfrac{1}{2}\sqrt{E}}{P^0}$$
$$\mu' \angle \frac{-Q' - \tfrac{1}{2}\sqrt{E}}{P'}$$
$$\mu'' \angle \frac{-Q'' + \tfrac{1}{2}\sqrt{E}}{P''}$$
$$\mu''' \angle \frac{-Q''' - \tfrac{1}{2}\sqrt{E}}{P'''}, \ \&c.$$

Now, if in the expression of Q'' we put, for p'' and q'', their values, $\mu'p' + 1$, and μ'', it will become $\mu'P' + Q'$; also, if we substitute in the expression of Q''', for p''' and q''', their values $\mu''p'' + p'$, and $\mu''q'' + q'$, it will be changed into $\mu''P'' + Q''$, and so on; so that we shall have

$$Q' = \mu\ P^0 + Q^0$$
$$Q'' = \mu'\ P' + Q'$$
$$Q''' = \mu''\ P'' + Q''$$
$$Q^{iv} = \mu'''P''' + Q''', \ \&c.$$

Likewise, if we substitute the values of p'', and q'', in the expression of P'', it will become $\overset{\prime}{\mu}{}^2P' + 2\mu'Q' + A$; and if we substitute the values of p''', and Q''', in the expression of P''', it will become $\overset{\prime\prime}{\mu}{}^2 P'' + 2\mu'' Q'' + P'$, and so on; so that we shall have

$$P' = \mu^2 P^0 + 2\mu\, Q^0 + C$$

$$P'' = \overset{\prime}{\mu}{}^2 P' + 2\mu'\, Q' + P^0$$

$$P''' = \overset{\prime\prime}{\mu}{}^2 P'' + 2\mu''\, Q'' + P'$$

$$P^{iv} = \overset{\prime\prime\prime}{\mu}{}^2 P''' + 2\mu''' Q''' + P'', \&c.$$

By means of this formulæ, therefore, we may continue the several series of numbers, μ, μ', μ''; Q^0, Q', Q'', and P^0, P', P'', &c. to any length, which, as we see, mutually depend on each other, without its being necessary, at the same time, to calculate the numbers p^0, p', p'', &c. and q^0, q', q'', &c.

We may also find the values of P', P'', P''', &c. by more simple formulæ than the preceding, observing that we have

$$\overset{\prime}{Q}{}^2 - P' = (\mu' A + \tfrac{1}{2}B)^2 - A(\overset{\prime}{\mu}{}^2 A + \overset{\prime}{\mu} B + C) = \tfrac{1}{4}B^2 - AC,$$

$$\overset{\prime\prime}{Q}{}^2 - P'P'' = (\mu' P' + Q')^2 - P'(\overset{\prime}{\mu}{}^2 P' + 2\mu' Q' + A) = \overset{\prime}{Q}{}^2 - AP',$$

and so on; that is to say,

$$\overset{\prime}{Q}{}^2 - P^0 P' = \tfrac{1}{4}E$$

$$\overset{\prime\prime}{Q}{}^2 - P'P'' = \tfrac{1}{4}E$$

$$\overset{\prime\prime\prime}{Q}{}^2 - P''P''' = \tfrac{1}{4}E, \&c.$$

Whence we get

$$P' = \frac{\overset{\prime}{Q}{}^2 - \frac{1}{4}E}{P^0}$$

$$P'' = \frac{\overset{\prime\prime}{Q}{}^2 - \frac{1}{4}E}{P'}$$

$$P''' = \frac{\overset{\prime\prime\prime}{Q}{}^2 - \frac{1}{4}E}{P''}, \&c.$$

The numbers μ, μ', μ'', &c. having thus been found, we have (Art. 26), the continued fraction,

$$\alpha = \mu + \cfrac{1}{\mu' + \cfrac{1}{\mu'' + , \&c.}}$$

and, in order to find the *minimum* of the formula

$Ap^2 + Bpq + cq^2$, we shall only have to calculate the numbers p^0, p', p'', p''', &c. and q^0, q', q'', q''', &c. (Art. 25), and then to try them instead of p and q; but this operation may likewise be dispensed with, if we consider, that the quantities P^0, P', P'', &c. are nothing but the values of the formula in question, when we successively make $p = p^0, p', p''$, &c. and $q = q^0, q', q''$, &c. We have, therefore, only to consider which is the least term of the series P^0, P', P'', &c. which we calculate at the same time with the series, μ, μ', μ'', &c. and that will be the *minimum* required; we shall then find the corresponding values of p and q by means of the formulæ above quoted.

34. Now I say, that continuing the series, P^0, P', P'', &c. we must necessarily arrive at two consecutive terms with different signs; and that then the succeeding terms, also, will all have different signs two by two. For, by the preceding Article, we have

$$P^0 = A(p^0 - aq^0) \times (p^0 - bq^0),$$
$$P' = A(p' - aq') \times (p' - bq'), \text{ \&c.}$$

And, from what we demonstrated in Problem 2, it follows, that the quantities $p^0 - aq^0$, $p' - aq'$, $p'' - aq''$, &c. must have alternate signs, and go on diminishing; therefore, 1st, if b is a negative quantity, the quantities $p^0 - bq^0$, $p' - bq'$, &c. will all be positive; consequently, the numbers P^0, P', P'', will all have alternate signs; 2dly, if b is a positive quantity, as the quantities $p' - aq'$, $p'' - aq''$, &c. and much more the quantities $\frac{p'}{q'} - a$, $\frac{p''}{q''} - a$, form a series, decreasing to infinity, we shall necessarily arrive at one of these last quantities, as $\frac{p'''}{q'''} - a$, which will be $\angle\ (a - b)$, abstracting from the sign, and then all the following, $\frac{p^{iv}}{q^{iv}} - a$, $\frac{p^{v}}{q^{v}} - a$, will be so likewise; so that all the quantities

$a - b + \frac{p'''}{q'''} - a$, $a - b + \frac{p^{iv}}{q^{iv}} - a$, &c. will necessarily have the same sign as the quantity $a - b$; consequently, the quantities $\frac{p'''}{q'''} - b$, $\frac{p^{iv}}{q^{iv}}$, $- b$, &c. and these $p''' - bq'''$, $p^{iv} - bq^{iv}$, &c. to infinity, will all have the same sign; therefore, all the numbers P''', P^{iv}, will have alternate signs.

Suppose now, in general, that we have arrived at terms, with alternate signs, in the series P', P'', P''', &c. and that

P_λ is the first of those terms, so that all the terms P_λ, $P^{\lambda+1}$, $P^{\lambda+2}$, &c. to infinity, are alternately positive and negative; I say that none of those terms can be greater than E. If, for example, P''', P^{iv}, P^{v}, &c. have all alternate signs, it is evident that the products, two by two, $P'''P^{iv}$, $P^{iv}P^{v}$, &c. will necessarily be negative; but (by the preceding Article), we have $\overset{iv}{Q}{}^2 - P'''P^{iv} = E$, $\overset{v}{Q}{}^2 - P^{iv}P^{v} = E$, &c. wherefore the positive numbers, $-P'''P^{iv}$, $-P^{iv}P^{v}$, will all be less than E, or at least not greater than E; so that, as the numbers P', P'', P''', &c. must be integers, the numbers P''', P^{iv}, &c. and, in general, the numbers P^λ, $P^{\lambda+1}$, &c. abstracting from their signs, can never exceed the number E.

Hence it follows, also, that the terms Q^{iv}, Q^{v}, &c. and, in general, $Q^{\lambda+1}$, $Q^{\lambda+2}$, &c. can never be greater than $\sqrt{E}$.

Whence it is easy to conclude, that the two series P^λ, $P^{\lambda+1}$, $P^{\lambda+2}$, &c. and $Q^{\lambda+1}$, $Q^{\lambda+2}$, &c. though carried to infinity, can never be composed but of a certain number of different terms, those terms being, for the first, only the natural numbers as far as E, taken positively, or negatively; and for the second, the natural numbers as far as $\sqrt{E}$, with the intermediate fractions $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. likewise taken positively, or negatively; for it is evident, from the formulæ of the preceding Article, that the numbers Q', Q'', Q''', &c. will always be integer, when B is even; but that they will each contain the fraction $\frac{1}{2}$, when B is odd.

Therefore, continuing the two series P', P'', P''', &c. and Q', Q'', Q''', &c. it will necessarily happen, that two corresponding terms, as P^π and Q^π, will return after a certain interval of terms, the number of which may always be supposed even; for, as the same terms P^π and Q^π, must return together an infinite number of times, because the number of different terms in both series is limited, and consequently also the number of their different combinations, it is evident, that if these two terms always returned, after the interval of an odd number of terms, we should only have to consider their returns alternately, and then the intervals would all be composed of an even number of terms.

Denoting, therefore, the number of intermediate terms by 2ρ, we shall have $P^{\pi+2\rho} = P^\pi$, and $Q^{\pi+2\rho} = Q^\pi$, and then all the terms P^π, $P^{\pi+1}$, $P^{\pi+2}$, &c. Q^π, $Q^{\pi+1}$, $Q^{\pi+2}$, and μ^π, $\mu^{\pi+1}$, $\mu^{\pi+2}$, &c. will also return at the end of each interval of 2ρ terms. For it is evident, from the formulæ given in the preceding Article, for the determination of the numbers, μ', μ'', μ''', &c. Q', Q'', Q''', &c. and P', P'', P''', &c. that, since we shall have $P^{\pi+2\rho} = P^\pi$, and $Q^{\pi+2\rho} = Q^\pi$, we shall also have

$\mu^{\pi+2\varrho} = \mu^{\pi}$, then $Q^{\pi+2\varrho+1} = Q^{\pi+1}$, and $P^{\pi+2\varrho+1} = P^{\pi+1}$; whence, also, $\mu^{\pi+2\varrho+1} = \mu^{\pi+2\varrho}$, and so on.

So that, if Π is any number equal, or greater than π, and m denotes any integer positive number, we shall have, in general,

$$P^{\Pi+2m\varrho} = P^{\Pi},\ Q^{\Pi+2m\varrho} = Q^{\Pi},\ \mu^{\Pi+2m\varrho} = \mu^{\Pi};$$

therefore, by knowing the $\pi + 2\varrho$ leading terms of each of the three series, we shall likewise know all the succeeding, which will be only the 2ϱ last terms repeated, in the same order, to infinity.

From all this it follows, that, in order to find the least value of $P = Ap^2 + Bpq + Cq^2$, it is sufficient to continue the series P^{0}, P', P'', &c. and Q^{0}, Q', Q'', &c. until two corresponding terms, as P^{π} and Q^{π} appear again together, after an even number of intermediate terms, so that we may have $P^{\pi+2\varrho} = P^{\pi}$, and $Q^{\pi+2\varrho} = Q^{\pi}$; then the least term of the series P^{0}, P', P'', &c. $P^{\pi+2\varrho}$ will be the *minimum* required.

35. *Corollary* 1. If the least term of the series P^{0}, P', P'', &c. $P^{\pi+2\varrho}$ is not found before the term P^{π}, then that term will be repeated an infinite number of times in the same series infinitely prolonged; so that we shall then have an infinite number of values of p and q answering to the *minimum*, and all discoverable by the formulæ of Art. 25, by continuing the series of the numbers μ', μ'', μ''', &c. beyond the term $\mu^{2\varrho+\pi}$ by the repetition of the same terms $\mu^{\pi+1}$, $\mu^{\pi+2}$, as we have already said.

In this case we may likewise have general formulæ representing all the values of p and q in question; but an explanation of the method for arriving at this, would carry me too far; for the present, I shall only refer to the *Memoires de Berlin* already quoted, ann. 1768, page 123, &c. where will be found a general and new theory of periodical continued fractions.

36. *Corollary* 2. We have demonstrated (Art. 34), that, by continuing the series P', P'', P''', &c. we ought to find consecutive terms with different signs. Let us suppose, therefore, for example, that P''' and P^{iv} are the first two terms, with this property. We shall necessarily have the two quantities $p''' - bq'''$, and $p^{iv} - bq^{iv}$, with the same signs, because the quantities $p''' - aq'''$, and $p^{iv} - aq^{iv}$, have from their nature, different signs. Now, by putting in the quantities $p^{v} - bq^{v}$, $p^{vi} - bq^{vi}$, &c. the values of p^{v}, p^{vi}, &c. q^{v}, q^{vi}, &c. (Art. 25), we shall have

$$p^{v} - bq^{v} = \mu^{iv}(p^{iv} - bq^{iv}) + p''' - bq'''$$
$$p^{vi} - bq^{vi} = \mu^{v}(p^{v} - bq^{v}) + p^{iv} - bq^{iv}, \text{ \&c.}$$

Whence, because μ^{iv}, μ^{v}, &c. are positive numbers, it is evident that all the quantities $p^{v} - bq^{v}$, $p^{vi} - bq^{vi}$, &c. to infinity, will have the same signs as the quantities $p''' - bq'''$, and $p^{iv} - bq^{iv}$; consequently, all the terms P''', P^{iv}, P^{v}, &c. to infinity, will alternately have the signs *plus* and *minus*.

From the preceding equations, we shall now have

$$\mu^{iv} = \frac{p^{v} - bq^{v}}{p^{iv} - bq^{iv}} - \frac{p''' - bq'''}{p^{iv} - bq^{iv}}$$

$$\mu^{v} = \frac{p^{vi} - bq^{vi}}{p^{v} - bq^{v}} - \frac{p^{iv} - bq^{iv}}{p^{v} - bq^{v}}$$

$$\mu^{vi} = \frac{p^{vii} - bq^{vii}}{p^{vi} - bq^{vi}} - \frac{p^{v} - bq^{v}}{p^{vi} - bq^{vi}}, \text{ \&c.}$$

where the quantities, $\frac{p''' - bq'''}{p^{iv} - bq^{iv}}$, $\frac{p^{iv} - bq^{iv}}{p^{v} - bq^{v}}$, &c. are all positive.

Wherefore, since the numbers μ^{iv}, μ^{v}, μ^{vi}, &c. must be all positive integers, by hypothesis, the quantity $\frac{p^{v} - bq^{v}}{p^{iv} - bq^{iv}}$ must be positive, and > 1; so also must the quantities $\frac{p^{vi} - bq^{vi}}{p^{v} - bq^{v}}$, $\frac{p^{vii} - bq^{vii}}{p^{vi} - bq^{vi}}$, &c. wherefore the quantities $\frac{p^{iv} - bq^{iv}}{p^{v} - bq^{v}}$, $\frac{p^{v} - bq^{v}}{p^{vi} - bq^{vi}}$, &c. will be positive, and less than unity; so that the numbers μ^{v}, μ^{vi}, &c. can only be the integer numbers, which are immediately less than the values of $\frac{p^{vi} - bq^{vi}}{p^{v} - bq^{v}}$, $\frac{p^{vii} - bq^{vii}}{p^{vi} - bq^{vi}}$, &c. As to the number μ^{iv}, it will also be equal to the integer number, which is immediately less than the value of $\frac{p^{v} - bq^{v}}{p^{vi} - bq^{vi}}$, whenever we have

$$\frac{p''' - bq'''}{p^{iv} - bq^{iv}} < 1.$$

Thus, we shall have

$$\mu^{iv} < \frac{p^{v} - bq^{v}}{p^{iv} - bq^{iv}}, \text{ if } \frac{p''' - bq'''}{p^{iv} - bq^{iv}} < 1.$$

$$\mu^{v} < \frac{p^{vi} - bq^{vi}}{p^{v} - bq^{v}}.$$

$$\mu^{vi} < \frac{p^{vii} - bq^{vii}}{p^{vi} - bq^{vi}}, \text{ \&c.}$$

the sign $\angle$ placed after the numbers μ''', μ^{iv}, μ^{v}, &c. denoting as before, the integer numbers which are immediately under the quantities which follow that same sign.

Now, by reductions similar to those of Art. 33, it is easy to transform the quantities $\frac{p^{v}-bq^{v}}{p^{iv}-bq^{iv}}$, $\frac{p^{vi}-bq^{vi}}{p^{v}-bq^{v}}$, &c. into these, $\frac{Q^{v}+\frac{1}{2}\sqrt{E}}{P^{iv}}$, $\frac{Q^{vi}-\frac{1}{2}\sqrt{E}}{P^{v}}$, &c. Farther, the condition of $\frac{pq'''-bq'''}{p^{iv}-bq^{iv}} \angle 1$ may be reduced to this, $\frac{-P'''}{P^{iv}} \angle \frac{aq'''-p'''}{p^{iv}-aq^{iv}}$; which, because $\frac{aq'''-p'''}{p^{iv}-a^{iv}} 7 1$, will certainly take place, when $\frac{-P'''}{P^{iv}} =$ or $\angle$ 1; wherefore we shall have

$$\mu^{iv} \angle \frac{Q^{v}+\frac{1}{2}\sqrt{E}}{P^{iv}}, \text{ if } \frac{-P'''}{P^{iv}} = \text{ or } \angle 1.$$

$$\mu^{v} \angle \frac{Q^{vi}-\frac{1}{2}\sqrt{E}}{P^{v}},$$

$$\mu^{vi} \angle \frac{Q^{vii}+\frac{1}{2}\sqrt{E}}{P^{vi}}, \text{ \&c.}$$

Combining now these formulæ with those of Art. 33, which contain the law of the series P', P'', P''', &c. and Q', Q'', Q''', &c. we shall easily see, that, if two corresponding terms of these two series be supposed to be given, the rank of which is higher than 3, we may go back to the preceding terms, as far as P^{iv} and Q^{v}, and even to the terms P''' and Q^{iv}, if the condition of $\frac{-P'''}{P^{iv}} =$ or $\angle$ 1 takes place; so that all these terms will be absolutely determined by those which we have supposed to be given.

For example, knowing P^{vi}, and Q^{vi}, we shall immediately know P^{v} from the equation $\overset{vi}{Q}{}^{2} - P^{v}P^{vi} = \frac{1}{4}E$; then, having Q^{vi} and P^{v}, we shall find the value of μ^{v}; by means of which, we shall next find the value of Q^{v} from the equation $Q^{vi} = \mu^{v}P^{v} + Q^{v}$. Now, the equation $\overset{v}{Q}{}^{2} - P^{iv}P^{v} = \frac{1}{4}E$, will give P^{iv}; and if we previously know, that $\frac{-P'''}{P^{iv}}$ must be $=$ or $\angle$ 1, we shall find μ^{iv}; after which, we shall have Q^{iv} from the equation $Q^{v} = \mu^{iv}P^{iv} + Q^{iv}$, and then P''' from this, $Q^{iv\,2} - P'''P^{iv} = \frac{1}{4}E$.

Whence it is easy to draw this general conclusion, that if P^{λ} and $P^{\lambda+1}$ are the first terms of the series P', P'', P''', &c. which are successively found with different signs, the term $P^{\lambda+1}$, and the following, will all return, after a certain number of intermediate terms, and that it will be the same with the term P^{λ}, if we have $\frac{\pm P}{P^{\lambda+1}} =$ or $\angle 1$.

For let us imagine, as in Art. 34, that we have found $P^{\pi+2\varrho} = P^{\pi}$, and $Q^{\pi+2\varrho} = Q^{\pi}$, and suppose that π is $7\ \lambda$, that is to say, $\pi = \lambda + \nu$; wherefore we may go back, on the one hand, from the term P^{π} to the term $P^{\lambda+1}$, or P^{λ}, and on the other, from the term $P^{\pi+2\varrho}$ to the term $P^{\lambda+2\varrho+1}$, or $P^{\lambda+2\varrho}$; and, as the terms from which we set out are equal on both sides, all the terms derived from them will likewise be respectively equal; so that we shall have $P^{\lambda+2\varrho+1} = P^{\lambda+1}$, or even $P^{\lambda+\varrho} = P^{\lambda}$, if $\frac{\pm P^{\lambda}}{P^{\lambda+1}} =$ or $\angle 1$.

We may, therefore, judge beforehand of the beginning of the periods in the series P°, P', P'', P''', &c, and consequently in the other series also, Q°, Q', Q'', Q''', &c. μ, μ', μ'', μ''', &c. but as to the length of the periods, that depends on the nature of the number E, and entirely on the value of that number, as I could demonstrate, were I not afraid of being led into too long a detail.

37. *Corollary* 3. What we have demonstrated in the preceding corollary, may serve to prove the following theorem: *Every equation of the form* $p^2 - Kq^2 = 1$, (*in which* K *is a positive integer number, but not a square, and* p *and* q *two indeterminate numbers*) *is resolvible in integer numbers.*

For, by comparing the formula $p^2 - Kq^2$ with the general formula, $Ap^2 + Bpq + Cq^2$, we have $A = 1$, $B = 0$, $C = -K$; wherefore $E = B^2 - 4AC = 4K$, and $\frac{1}{2}\sqrt{E} = \sqrt{K}$ (Art. 33). Wherefore, $P^{\circ} = 1$, $Q^{0} = 0$; likewise $\mu \angle \sqrt{K}$, $Q' = \mu$, and $P' = \mu^2 - K$; whence we see *first*, that P' is negative, and consequently has a different sign from P°; *secondly*, that $-P'$ is $=$ or $7\ 1$, because K and μ are integer numbers; so that we shall have $\frac{P^{\circ}}{-P'} =$ or $\angle 1$; whence we shall find, from the preceding Article,

$$\lambda = 0, \text{ and } P^{2\varrho} = P^{\circ} = 1;$$

so that by continuing the series P^0, P', P'', &c. the term, $P^0 = 1$, will necessarily return after a certain interval of terms; consequently, we may always find an infinite number of values for p and q, which will render the formula $p^2 - Kq^2$ equal to unity.

38. *Corollary* 4. We may likewise demonstrate this theorem: *If the equation* $P^2 - Kq^2 = \pm H$ *be resolvible in integer numbers, by supposing* K *a positive number, not square, and* H *a positive number, less than* $\sqrt{K}$, *the numbers* P *and* Q *must be such, that* $\frac{P}{Q}$ *may be one of the* principal *fractions converging to the value of* $\sqrt{K}$.

Let us suppose that the upper sign must take place, so that $p^2 - Kq^2 = H$; wherefore, we shall have

$$p - q\sqrt{K} = \frac{H}{p+q\sqrt{K}}, \text{ and } \frac{p}{q} - \sqrt{K} = \frac{H}{q^2(\frac{p}{q} + \sqrt{K})}.$$

Now, let us seek two integer positive numbers, r and s, less than p and q, and such, that $ps - qr = 1$, which is always possible, as we have demonstrated (Art. 23), and we shall have $\frac{p}{q} - \frac{r}{s} = \frac{1}{qs}$; subtracting this equation from the preceding, we shall have

$$\frac{r}{s} - \sqrt{K} = \frac{H}{q^2(\frac{p}{q} + \sqrt{K})} - \frac{1}{qs}; \text{ so that we have}$$

$$p - q\sqrt{K} = \frac{H}{q(\frac{p}{q} + \sqrt{K})},$$

$$r - s\sqrt{K} = \frac{1}{q}\left(\frac{sH}{q(\frac{p}{q} + \sqrt{K})} - 1\right).$$

Now, as $\frac{p}{q} > \sqrt{K}$, and $H < \sqrt{K}$, it is evident, that $\frac{H}{\frac{p}{q} + \sqrt{K}}$ will be $< \frac{1}{2}$; whence $p - q\sqrt{K}$ will be $< \frac{1}{2q}$;

wherefore, $\frac{sH}{q(\frac{p}{q} + \sqrt{K})}$ will much more be $< \frac{1}{2}$, since $s < q$; so that $r - s\sqrt{K}$ will be a negative quantity, which taken positively, will be $> \frac{1}{2q}$, because that $1 - \frac{s\text{H}}{q(\frac{p}{q} + \text{K})} > \frac{1}{2}$.

So that we shall have the two quantities, $p - q\sqrt{\text{K}}$, and $r - s\sqrt{\text{K}}$; or rather, making $a = \sqrt{\text{K}}$, $p - aq$, and $r - as$: which will be subject to the same conditions as we have supposed in Art. 24, and from which we shall draw similar conclusions: therefore, &c. (Art. 26), if we had $p^2 - \text{K}q^2 = -\text{H}$, then it would be necessary to seek the numbers r and s such, that $ps - qr = -1$, and we should have these two equations,

$$q\sqrt{\text{K}} - p = \frac{\text{H}}{q(\sqrt{\text{K}} + \frac{p}{q})}$$

$$s\sqrt{\text{K}} - r = \frac{1}{q}\left(\frac{s\text{H}}{q(\sqrt{\text{K}} + \frac{p}{q})} - 1\right).$$

As $\text{H} < \sqrt{\text{K}}$, and $s < q$, it is evident, that $\frac{s\text{H}}{q(\sqrt{\text{K}} + \frac{p}{q})}$ will be < 1; so that the quantity $s\sqrt{\text{K}} - r$ will be negative. Now, I say that this quantity, taken positively, will be greater than $q\sqrt{\text{K}} - p$; to prove which, it must be demonstrated, that $\frac{1}{q}\left(1 - \frac{s\text{H}}{q(\sqrt{\text{K}} + \frac{p}{q})}\right) > \frac{\text{H}}{q(\sqrt{\text{K}} + \frac{p}{q})}$,

or rather, that $1 > \frac{\text{H}(1 + \frac{s}{q})}{\sqrt{\text{K}} + \frac{p}{q}}$; that is to say,

$\sqrt{\text{K}} + \frac{p}{q} > \text{H} + \frac{s\text{H}}{q}$; but $\text{H} < \sqrt{\text{K}}$ (*hyp.*); it is therefore sufficient to prove, that $\frac{p}{q} > \frac{s\sqrt{\text{K}}}{q}$, or that $p > s\sqrt{\text{K}}$; which is evident, because the quantity $s\sqrt{\text{K}} - r$ being negative, we must have $r > s\sqrt{\text{K}}$, and much more $p > s\sqrt{\text{K}}$, since $p > r$.

Thus, the two quantities, $p - q\sqrt{\text{K}}$, and $r - s\sqrt{\text{K}}$, will have different signs, and the second will be greater than the first (abstracting from the signs), as in the preceding case; therefore, &c.

So that when we have to resolve, in integer numbers, an equation, of the form, $p^2 - \text{K}q^2 = \pm \text{H}$, where $\text{H} \angle \sqrt{\text{K}}$, we have only to follow the same process as in Art. 33, making $\text{A} = 1$, $\text{B} = 0$, and $\text{C} = -\text{K}$; and, if in the series P^0, P', P'', P''', &c. $\text{P}^{\varkappa+2\varrho}$, we find a term $= \pm \text{H}$, we shall have the solution required; if not, we may be certain that the given equation admits of no solution in integer numbers.

39. *Scholium.* We have considered (Art. 33) only one root of the equation $\text{A}x^2 + \text{B}x + \text{C} = 0$, which we have supposed positive; if this equation have both its roots positive, we must take them successively for a, and perform the same operation with both; but if one of the two roots, or both, were negative, then we should first change them into positive, by only changing the sign of B, and should proceed as before: but then we should take the values of p and q with contrary signs; that is to say, the one positive, and the other negative (Art. 29).

In general, therefore, we shall give the ambiguous sign $\pm$ to the value of B, as well as to $\sqrt{\text{E}}$; that is to say, we shall make $\text{Q}' = \mp \frac{1}{2}\text{B}$, and let us put $\pm$ before $\sqrt{\text{E}}$, and we must take these signs, so that the root

$$a = \frac{\mp \frac{1}{2}\text{B} \pm \frac{1}{2}\sqrt{\text{E}}}{\text{A}}$$

may be positive, which may always be done in two different ways: the upper sign of B will indicate a positive root; in which case, we must take both p and q with the same signs; on the contrary, the lower sign of B will indicate a negative root; in which case, the values of p and q must be taken with contrary signs.

40. *Example.* Required what integer numbers must be taken for p and q, in order that the quantity,

$$9p^2 - 118pq + 378q^2$$

may become the least possible.

Comparing this quantity with the general formula of Problem 3, we shall have $\text{A} = 9$, $\text{B} = -118$, $\text{C} = 378$; wherefore, $\text{B}^2 - 4\text{AC} = 316$; whence we see that this case belongs to that of Art. 33. We shall therefore make $\text{E} = 316$, and $\frac{1}{2}\sqrt{\text{E}} = \sqrt{79}$, where we at once observe, that $\sqrt{79} \, 7 \, 8$, and $\angle 9$; so that in the formulæ of which we shall only have to find the approximate integer value, we may immediately take, instead of $\sqrt{79}$, the number 8, or 9, according as that radical shall be added, or subtracted, from the other numbers of the same formula.

We shall now give the ambiguous sign $\pm$ to B, as well as to $\sqrt{E}$, and shall then take these signs such, that

$$a = \frac{\pm 59 \pm \sqrt{79}}{9}$$

may be a positive quantity (Art. 39); whence we see, that we must always take the upper sign for the number 59; and, that for the radical $\sqrt{79}$, we may either take the upper, or the under. So that we shall always make $Q^0 = -\frac{1}{2}B$, and $\sqrt{E}$ may be taken, successively, *plus* and *minus*.

First, therefore, if $\frac{1}{2}\sqrt{E} = \sqrt{79}$ with the positive sign, we shall make (Art. 33), the following calculation:

$$\begin{array}{lll}
Q^0 = -59, & P^0 = 9, & \mu < \frac{59 + \sqrt{79}}{9} = 7, \\
Q' = 9 \times 7 - 59 = 4, & P' = \frac{16 - 79}{9} = -7, & \mu' < \frac{-4 - \sqrt{79}}{-7} = 1, \\
Q'' = -7 \times 1 + 4 = -3, & P'' = \frac{9 - 79}{-7} = 10, & \mu'' < \frac{3 + \sqrt{79}}{10} = 1, \\
Q''' = 10 \times 1 - 3 = 7, & P''' = \frac{49 - 79}{10} = -3, & \mu''' < \frac{-7 - \sqrt{79}}{-3} = 5, \\
Q^{iv} = -3 \times 5 + 7 = -8, & P^{iv} = \frac{64 - 79}{-3} = 5, & \mu^{iv} < \frac{8 + \sqrt{79}}{5} = 3, \\
Q^{v} = 5 \times 3 - 8 = 7, & P^{v} = \frac{49 - 79}{5} = -6, & \mu^{v} < \frac{-7 - \sqrt{79}}{-6} = 2, \\
Q^{vi} = -6 \times 2 + 7 = -5, & P^{vi} = \frac{25 - 79}{-6} = 9, & \mu^{vi} < \frac{5 + \sqrt{79}}{9} = 1, \\
Q^{vii} = 9 \times 1 - 5 = 4, & P^{vii} = \frac{16 - 79}{9} = -7, & \mu^{vii} < \frac{-4 - \sqrt{79}}{-7} = 1,
\end{array}$$

&c. &c. &c.

Here I stop, because I perceive that $Q^{vii} = Q'$, and

$P^{vii} = P'$, and that the difference between the two indices, 1 and 7, is even; whence it follows, that all the succeeding terms will likewise be the same as the preceding; so that we shall have $Q^{vii} = 4$, $Q^{viii} = -3$, $Q^{ix} = 7$, &c. $P^{vii} = -7$, $P^{viii} = 10$, &c. so that, if we choose, we may continue the above series to infinity, only by repeating the same terms.

Secondly, let us take the radical $\sqrt{79}$ with a negative sign, and the calculation will be as follows:

$$Q^0 = -59, \quad P^0 = 9, \quad \mu < \frac{59-\sqrt{79}}{9} = 5,$$

$$Q' = 9 \times 5 - 59 = -14, \quad P' = \frac{196-79}{9} = 13, \quad \mu' < \frac{14+\sqrt{79}}{13} = 1,$$

$$Q'' = 13 \times 1 - 14 = -1, \quad P'' = \frac{1-79}{13} = -6, \quad \mu'' < \frac{1-\sqrt{79}}{-6} = 1,$$

$$Q''' = -6 \times 1 - 1 = -7, \quad P''' = \frac{49-79}{-6} = 5, \quad \mu''' < \frac{7+\sqrt{79}}{5} = 3,$$

$$Q^{iv} = 5 \times 3 - 7 = 8, \quad P^{iv} = \frac{64-79}{5} = -3, \quad \mu^{iv} < \frac{-8-\sqrt{79}}{-3} = 5,$$

$$Q^{v} = -3 \times 5 + 8 = -7, \quad P^{v} = \frac{49-79}{-3} = 10, \quad \mu^{v} < \frac{7+\sqrt{79}}{10} = 1,$$

$$Q^{vi} = 10 \times 1 - 7 = 3, \quad P^{vi} = \frac{9-79}{10} = -7, \quad \mu^{vi} < \frac{-3-\sqrt{79}}{-7} = 1,$$

$$Q^{vii} = -7 \times 1 + 3 = -4, \quad P^{vii} = \frac{16-79}{-7} = 9, \quad \mu^{vii} < \frac{4+\sqrt{79}}{9} = 1,$$

$$Q^{viii} = 9 \times 1 - 4 = 5, \quad P^{viii} = \frac{25-79}{9} = -6, \quad \mu^{viii} < \frac{-5-\sqrt{79}}{-6} = 2,$$

$$Q^{ix} = -6 \times 2 + 5 = -7, \quad P^{ix} = \frac{49-79}{-6} = 5, \quad \mu^{ix} < \frac{7+\sqrt{79}}{5} = 3,$$

&c. &c. &c.

We may stop here, since we have found $Q^{ix} = Q'''$, and $P^{ix} = P'''$, the difference of the indices 9 and 3 being even; for, by continuing the series, we should only find the same terms that we have found already.

Now, if we consider the values of the terms P^0, P', P'', P''', &c. found in the two cases, we shall perceive that the least of these terms is equal to -3; in the first case, it is the term P''', to which the values p''' and q''' answer; and, in the second case, it is the term P^{iv}, to which the values p^{iv} and q^{iv} answer.

Whence it follows, that the least value, which the given quantity can receive, is -3; and, in order to have the values of p and q, which answer to it, we shall take, in the first case, the numbers μ, μ', μ'', namely, 7, 1, and 1, and shall form with them the *principal* converging fractions $\frac{7}{1}$, $\frac{8}{1}$, $\frac{15}{2}$;

the third fraction will, therefore, be $\frac{p'''}{q'''}$, so that we shall have

$p''' = 15$, and $q''' = 2$; that is to say, the values required will be $p = 15$, and $q = 2$. In the second case, we shall take the numbers μ, μ', μ'', μ''', namely, 5, 1, 1, 3, which will give these fractions, $\frac{5}{1}$, $\frac{6}{1}$, $\frac{11}{2}$, $\frac{39}{7}$; so that we shall have $p^{iv} = 39$, and $q^{iv} = 7$; therefore, $p = 39$, and $q = 7$.

The values which we have just found for p and q, in the case of the *minimum*, are also the least possible; but if we choose, we may likewise successively find others greater: for it is evident, that the same term, -3, will always return at the end of every interval of six terms; so that, in the first case, we shall have $P''' = -3$, $P^{ix} = -3$, $P^{xv} = -3$, &c. and, in the second, $P^{iv} = -3$, $P^{x} = -3$, $P^{xvi} = -3$, &c.

Therefore, in the first case, the satisfactory values of p and q will be these; p''', q''', p^{ix}, q^{ix}, p^{xv}, q^{xv}, &c.; and, in the second case, p^{iv}, q^{iv}, p^{x}, q^{x}, p^{xvi}, q^{xvi}, &c. Now, the values of μ, μ', μ'', &c. are in the first case 7, 1, 1, 5, 3, 2, 1; 1, 1, 5, 3, 2, 1; 1, 1, 5, 3, &c. to infinity, because $\mu^{vii} = \mu'$, and $\mu^{viii} = \mu''$, &c. so that we shall only have to form, by the method of Art. 20, the fractions,

$$\begin{array}{cccccccccc} 7, & 1, & 1, & 5, & 3, & 2, & 1, & 1, & 1, & 5, \\ \frac{7}{1}, & \frac{8}{1}, & \frac{15}{2}, & \frac{83}{11}, & \frac{264}{35}, & \frac{611}{81}, & \frac{875}{116}, & \frac{1486}{197}, & \frac{2361}{313}, & \frac{13291}{1762}, \text{ \&c.} \end{array}$$

And we may take for p the numerators of the third, ninth, &c. and for q the corresponding denominators: we shall therefore have $p = 15$, $q = 2$, or $p = 2361$, $q = 313$, &c.

In the second case, the values of μ', μ'', μ''', &c. will be 5, 1, 1, 3, 5, 1, 1, 1, 2; 3, 5, 1, 1, 1, 2, &c. because μ^{ix}, μ''', $\mu^{x} = \mu^{iv}$, &c. We shall, therefore, form these fractions,

$$\begin{array}{cccccccccc} 5, & 1, & 1, & 3, & 5, & 1, & 1, & 1, & 2, & 3, \\ \frac{5}{1}, & \frac{6}{1}, & \frac{11}{2}, & \frac{39}{7}, & \frac{206}{37}, & \frac{245}{44}, & \frac{451}{81}, & \frac{696}{125}, & \frac{1843}{331}, & \frac{6225}{1118}, \text{ \&c.} \end{array}$$

And the fourth fraction, the tenth, &c. will give the values of p and q; which will therefore be

$$p = 39,\ q = 7, \text{ or } p = 6225,\ q = 1118, \text{ \&c.}$$

In this manner, therefore, we may regularly find all the values of p and q, that will make the given formula $= -3$, the least value it can receive. We might even have a general value, which would comprehend all these values of p and q. Any person who has the curiosity may find it by a method which we have elsewhere explained, and which has been already noticed (Art. 35).

We have just found, that the *minimum* of the quantity proposed is -3, and consequently negative; now, it might be proposed to find the least positive value, that the same quantity can receive: we should then only have to examine the series P^0, P', P'', P''', &c. in the two cases, and we should see that the least positive term is 5 in both cases; and as in the first case it is P^{iv}, and in the second P''', which is 5, the values of p and q, that will give the least positive value of the quantity proposed, will be p^{iv}, q^{iv}, or p^{x}, q^{x}, or &c. in the first case, and p''', q''', or p^{xi}, q^{xi}, &c. in the second; so that we shall have, from the above fractions, $p = 83$, $q = 11$; or $p = 13291$, $q = 1762$, &c. or $p = 11$, $q = 2$; $p = 1843$, $q = 331$, &c.

We must not forget to observe, that the numbers μ, μ', μ'', &c. found in the above two cases, are no other than the terms of the continued fractions, which represent the two roots of the equation $9x^2 - 118x + 378 = 0$.

So that these roots will be,

$$7 + \frac{1}{1} + \frac{1}{1} + \frac{1}{5} + \frac{1}{3} +, \text{ \&c.}$$

$$5 + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} +, \text{ \&c.}$$

expressions which we might continue to infinity merely by repeating the same numbers.

Thus, we perceive how we are to set about reducing to continued fractions the roots of every equation of the second degree.

41. *Scholium.* In volume XI. of the New Commentaries of Petersburg, M. EULER has given a method similar to the preceding; but deduced from principles somewhat different, for reducing to a continued fraction the root of any integer number, not a square, and has added a Table, in which the continued fractions are calculated for all the

natural numbers, that are not squares, as far as 100. This Table being useful on various occasions, and particularly for the solution of indeterminate numbers of the second degree, as we shall afterwards find (Chap. 7), we shall here present it to our readers. It will be observed, that there are two series of integers answering to each radical number; the upper is that of the numbers P°, $-P'$, P'', $-P'''$, &c. and the under that of the numbers, μ, μ', μ'', μ''', &c.

$\sqrt{2}$	1 1 1 1 &c. 1 2 2 2 &c.
$\sqrt{3}$	1 2 1 2 1 2 1 &c. 1 1 2 1 2 1 2 &c.
$\sqrt{5}$	1 1 1 1 &c. 2 4 4 4 &c.
$\sqrt{6}$	1 2 1 2 1 2 1 &c. 2 2 4 2 4 2 4 &c.
$\sqrt{7}$	1 3 2 3 1 3 2 3 1 &c. 2 1 1 1 4 1 1 1 4 &c.
$\sqrt{8}$	1 4 1 4 1 4 1 &c. 2 1 4 1 4 1 4 &c.
$\sqrt{10}$	1 1 1 1 &c. 3 6 6 6 &c.
$\sqrt{11}$	1 2 1 2 1 2 1 &c. 3 3 6 3 6 3 6 &c.
$\sqrt{12}$	1 3 1 3 1 3 1 &c. 3 2 6 2 6 2 6 &c.
$\sqrt{13}$	1 4 3 3 4 1 4 3 3 4 1 &c. 3 1 1 1 1 6 1 1 1 1 6 &c
$\sqrt{14}$	1 5 2 5 1 5 2 5 1 &c. 3 1 2 1 6 1 2 1 6 &c.
$\sqrt{15}$	1 6 1 6 1 6 1 &c. 3 1 6 1 6 1 6 &c.
$\sqrt{17}$	1 1 1 1 1 &c. 4 8 8 8 8 &c.
$\sqrt{18}$	1 2 1 2 1 2 1 2 1 &c. 4 4 8 4 8 4 8 4 8 &c.
$\sqrt{19}$	1 3 5 2 5 3 1 3 5 2 5 3 1 &c. 4 2 1 3 1 2 8 2 1 3 1 2 8 &c.
$\sqrt{20}$	1 4 1 4 1 4 1 4 1 &c. 4 2 8 2 8 2 8 2 8 &c.
$\sqrt{21}$	1 5 4 3 4 5 1 5 4 3 4 5 1 &c. 4 1 1 2 1 1 8 1 1 2 1 1 8 &c.

$\sqrt{22}$	1 6 3 2 3 6 1 6 3 2 3 6 1 &c. 4 1 2 4 2 1 8 1 2 4 2 1 8 &c.
$\sqrt{23}$	1 7 2 7 1 7 2 7 1 &c. 4 1 3 1 8 1 3 1 8 &c.
$\sqrt{24}$	1 8 1 8 1 8 1 &c. 4 1 8 1 8 1 8 &c.
$\sqrt{26}$	1 1 1 1 &c. 5 10 10 10 &c.
$\sqrt{27}$	1 2 1 2 1 2 1 &c. 5 5 10 5 10 5 10 &c.
$\sqrt{28}$	1 3 4 3 1 3 4 3 1 &c. 5 3 2 3 10 3 2 3 10 &c.
$\sqrt{29}$	1 4 5 5 4 1 4 5 5 4 1 &c. 5 2 1 1 2 10 2 1 1 2 10 &c.
$\sqrt{30}$	1 5 1 5 1 5 1 5 1 &c. 5 2 10 2 10 2 10 2 10 &c.
$\sqrt{31}$	1 6 5 3 2 3 5 6 1 6 5 &c. 5 1 1 3 5 3 1 1 10 1 1 &c.
$\sqrt{32}$	1 7 4 7 1 7 4 7 1 &c. 5 1 1 1 10 1 1 1 10 &c.
$\sqrt{33}$	1 8 3 8 1 8 3 8 1 &c. 5 1 2 1 10 1 2 1 10 &c.
$\sqrt{34}$	1 9 2 9 1 9 2 9 1 &c. 5 1 4 1 10 1 4 1 10 &c.
$\sqrt{35}$	1 10 1 10 1 10 1 10 &c. 5 1 10 1 10 1 10 1 &c.
$\sqrt{37}$	1 1 1 1 1 &c. 6 12 12 12 12 &c.
$\sqrt{38}$	1 2 1 2 1 2 1 &c. 6 6 12 6 12 6 12 &c.
$\sqrt{39}$	1 3 1 3 1 3 1 &c. 6 4 12 4 12 4 12 &c.
$\sqrt{40}$	1 4 1 4 1 4 1 &c. 6 3 12 3 12 3 12 &c.
$\sqrt{41}$	1 5 5 1 5 5 1 &c. 6 2 2 12 2 2 12 &c.
$\sqrt{42}$	1 6 1 6 1 6 1 &c. 6 2 12 2 12 2 12 &c.
$\sqrt{43}$	1 7 6 3 9 2 9 3 6 7 1 7 6 &c. 6 1 1 3 1 5 1 3 1 1 12 1 1 &c.
$\sqrt{44}$	1 8 5 7 4 7 5 8 1 8 5 &c. 6 1 1 1 2 1 1 1 12 1 1 &c.
$\sqrt{45}$	1 9 4 5 4 9 1 9 4 5 4 9 1 9 4 &c. 6 1 2 2 2 1 12 1 2 2 2 1 12 1 2 &c.

$\sqrt{46}$	1 10 3 7 6 5 2 5 6 7 3 10 1 10 3 &c.
	6 1 3 1 1 2 6 2 1 1 3 1 12 1 3 &c.
$\sqrt{47}$	1 11 2 11 1 11 2 11 1 &c.
	6 1 5 1 12 1 5 1 12 &c.
$\sqrt{48}$	1 12 1 12 1 12 &c.
	6 1 12 1 12 1 &c.
$\sqrt{50}$	1 1 1 1 &c.
	7 14 14 14 &c.
$\sqrt{51}$	1 2 1 2 1 2 &c.
	7 7 14 7 14 7 &c.
$\sqrt{52}$	1 3 9 4 9 3 1 3 9 4 9 3 1 3 &c.
	7 4 1 2 1 4 14 4 1 2 1 4 14 4 &c.
$\sqrt{53}$	1 4 7 7 4 1 4 7 7 4 1 4 7 &c.
	7 3 1 1 3 14 3 1 1 3 14 3 1 &c.
$\sqrt{54}$	1 5 9 2 9 5 1 5 9 2 9 5 1 5 &c.
	7 2 1 6 1 2 14 2 1 6 1 2 14 2 &c.
$\sqrt{55}$	1 6 5 6 1 1 6 5 6 1 &c.
	7 2 2 2 14 2 2 2 14 2 &c.
$\sqrt{56}$	1 7 1 7 1 7 1 &c.
	7 2 14 2 14 2 14 &c.
$\sqrt{57}$	1 8 7 3 7 8 1 8 7 &c.
	7 1 1 4 1 1 14 1 1 &c.
$\sqrt{58}$	1 9 6 7 7 6 9 1 9 6 &c.
	7 1 1 1 1 1 1 14 1 1 &c.
$\sqrt{59}$	1 10 5 2 5 10 1 10 5 &c.
	7 1 2 7 2 1 14 1 2 &c.
$\sqrt{60}$	1 11 4 11 1 11 4 &c.
	7 1 2 1 14 1 2 &c.
$\sqrt{61}$	1 12 3 4 9 5 5 9 4 3 12 1 12 3 &c.
	7 1 4 3 1 2 2 1 3 4 1 14 1 4 &c.
$\sqrt{62}$	1 13 2 13 1 13 2 &c.
	7 1 6 1 14 1 6 &c.
$\sqrt{63}$	1 14 1 14 1 14 &c.
	7 1 14 1 14 1 &c.
$\sqrt{65}$	1 1 1 1 &c.
	8 16 16 16 &c.
$\sqrt{66}$	1 2 1 2 1 &c.
	8 8 16 8 16 &c.
$\sqrt{67}$	1 3 6 7 9 2 9 7 6 3 1 3 6 &c.
	8 5 2 1 1 7 1 1 2 5 16 5 2 &c.
$\sqrt{68}$	1 4 1 4 1 4 &c.
	8 4 16 4 16 4 &c.
$\sqrt{69}$	1 5 4 11 3 11 4 5 1 5 4 &c.
	8 3 3 1 4 1 3 3 16 3 3 &c.

$\sqrt{70}$	1 6 9 5 9 6 1 6 9 &c.
	8 2 1 2 1 2 16 2 1 &c.
$\sqrt{71}$	1 7 5 11 2 11 5 7 1 7 5 &c.
	8 2 2 1 7 1 2 2 16 2 2 &c.
$\sqrt{72}$	1 8 1 8 1 8 &c.
	8 2 16 2 16 2 &c.
$\sqrt{73}$	1 9 8 3 3 8 9 1 9 8 &c.
	8 1 1 5 5 1 1 16 1 1 &c.
$\sqrt{74}$	1 10 7 7 10 1 10 7 &c.
	8 1 1 1 1 16 1 1 &c.
$\sqrt{75}$	1 11 6 11 1 11 6 &c.
	8 1 1 1 16 1 1 &c.
$\sqrt{76}$	1 12 5 8 9 3 4 3 9 8 5 12 1 12 5 &c.
	8 1 2 1 1 5 4 5 1 1 2 1 16 1 2 &c.
$\sqrt{77}$	1 13 4 7 4 13 1 13 4 &c.
	8 1 3 2 3 1 16 1 3 &c.
$\sqrt{78}$	1 14 3 14 1 14 3 &c.
	8 1 4 1 16 1 4 &c.
$\sqrt{79}$	1 15 2 15 1 15 2 &c.
	8 1 7 1 16 1 7 &c.
$\sqrt{80}$	1 16 1 16 1 16 &c.
	8 1 16 1 16 1 &c.
$\sqrt{82}$	1 1 1 1 &c.
	9 18 18 18 &c.
$\sqrt{83}$	1 2 1 2 1 2 &c.
	9 9 18 9 18 9 &c.
$\sqrt{84}$	3 3 1 3 1 3 &c.
	9 6 18 6 18 9 &c.
$\sqrt{85}$	1 4 9 9 4 1 4 9 &c.
	9 4 1 1 4 18 4 1 &c.
$\sqrt{86}$	1 5 10 7 11 2 11 7 10 5 1 5 10 &c.
	9 3 1 1 1 8 1 1 1 3 18 3 1 &c.
$\sqrt{87}$	1 6 1 6 1 6 &c
	9 3 18 3 18 3 &c.
$\sqrt{88}$	1 7 9 8 9 7 1 7 9 &c.
	9 2 1 1 1 2 18 2 1 &c.
$\sqrt{89}$	1 8 5 5 8 1 8 5 &c.
	9 2 3 3 2 18 2 3 &c.
$\sqrt{90}$	1 9 1 9 1 &c.
	9 2 18 2 18 &c.
$\sqrt{91}$	1 10 9 3 14 3 9 10 1 10 9 &c.
	9 1 1 5 1 5 1 1 18 1 1 &c.
$\sqrt{92}$	1 11 8 7 4 7 8 11 1 11 8 &c.
	9 1 1 2 4 2 1 1 18 1 1 &c.

$\sqrt{93}$	1 12 7 11 4 3 4 11 7 12 1 12 7 &c.
	9 1 1 1 4 6 4 1 1 1 18 1 1 &c.
$\sqrt{94}$	1 13 6 5 9 10 3 15 2 15 3 10 9 5 6 13 1 &c.
	9 1 2 3 1 1 5 1 8 1 5 1 1 3 2 1 18 &c.
$\sqrt{95}$	1 14 5 14 1 14 &c.
	9 1 2 1 18 1 &c.
$\sqrt{96}$	1 15 4 15 1 15 &c.
	9 1 3 1 18 1 &c.
$\sqrt{97}$	1 16 3 11 8 9 9 8 11 3 16 1 16 &c.
	9 1 5 1 1 1 1 1 1 5 1 18 1 &c.
$\sqrt{98}$	1 17 2 17 1 17 &c.
	9 1 8 1 18 1 &c.
$\sqrt{99}$	1 18 1 18 1 &c.
	9 1 18 1 18 &c.

Thus, for example, we shall have

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} +, \text{ \&c.}$$

$$\sqrt{3} = 1 + \frac{1}{1} + \frac{1}{2} \text{ \&c.}$$

and so of others.

And, if we form the converging fractions,

$$\frac{p^0}{q^0}, \ \frac{p'}{q'}, \ \frac{p''}{q''}, \ \frac{p'''}{q'''}, \text{ \&c.}$$

according to each of these continued fractions, we shall have

$$(p^0)^2 - 2(q^0)^2 = 1, \ p'^2 - 2q'^2 = -1,$$

$$p''^2 - 2q''^2 = 1, \text{ \&c.}$$

and likewise,

$$(p^0)^2 - 3(q^0)^2 = 1, \ p'^2 - 3q'^2 = -2,$$

$$p''^2 - 3q''^2 = 1, \text{ \&c.}$$

CHAP. III.

Of the Resolution, *in* Integer Numbers, *of* Equations *of the first* Degree, *containing two unknown* Quantities.

[APPENDIX TO CHAP. I.]

42. When we have to resolve an equation of this form,

$$ax - by = c,$$

in which a, b, c, are given integer numbers, positive, or negative, and in which the two unknown quantities, x and y, must also be integers, it is sufficient to know one solution, in order to deduce with ease all the other solutions that are possible.

For, suppose we know that these values, $x = \alpha$, and $y = \beta$, satisfy the conditions of the equation proposed, α and β being any integer numbers, we shall then have $a\alpha - b\beta = c$; and, consequently,

$$ax - by = a\alpha - b\beta, \text{ or } a(x - \alpha) - b(y - \beta) = 0;$$

whence we find $\frac{x-\alpha}{y-\beta} = \frac{b}{a}$. Let us reduce the fraction $\frac{b}{a}$ to its least terms, and supposing, in consequence of this reduction, that it becomes $\frac{b'}{a'}$, where b' and a' will be prime to one another, it is evident that the equation,

$$\frac{x-\alpha}{y-\beta} = \frac{b'}{a'},$$

could not subsist, on the supposition of $x - \alpha$, and $y - \beta$, being integers, unless we have $x - \alpha = mb'$, and $y - \beta = ma'$, m being any integer number; so that we shall have, in general, $x = \alpha + mb'$, and $y = \beta + ma'$; m being an indeterminate integer.

Now, as we may take m either positive, or negative, it is easy to perceive, that we may always determine the number m in such a manner, that the value of x may not be greater than $\frac{b'}{2}$, or that of y not greater than $\frac{a'}{2}$, (abstracting from the signs of these quantities); whence it follows, that if the given equation $ax - by = c$, is resolvible in integer numbers, and we successively substitute for x all the integer numbers, positive as well as negative, contained between these two limits $\frac{b'}{2}$, and $\frac{-b'}{2}$, we shall necessarily find one that will satisfy this equation: and we shall likewise find a satisfactory value of y among the positive, or negative whole numbers, contained between the limits $\frac{a'}{2}$, and $\frac{-a'}{2}$.

By these means we may find the first solution of the equation proposed; after which, we shall have all the others by the preceding formulæ.

43. But, without employing the method of trial, which we have now proposed, and which would sometimes be very laborious, we may make use of the very simple and direct method explained in Chap. I. of the preceding Treatise, or of the following method.

First, if the numbers a and b are not prime to each other, the equation cannot subsist in integer numbers, unless the given number, c, be divisible by the greatest common measure of a and b. Supposing, therefore, the division performed, and expressing the quotients by a', b', c', we shall have to resolve the equation,

$$a'x - b'y = c',$$

where a' and b' are prime to each other.

Secondly, if we can find values of p and q that satisfy the equation, $a'p - b'q = \pm 1$, we may resolve the preceding equation; for it is evident that, by multiplying these values by $\pm c'$, we shall have values that will satisfy the equation,

$$a'x - b'y = c';$$

that is to say, we shall have

$$x = \pm pc', \text{ and } y = \pm qc'.$$

Now, the equation $a'p - b'q = \pm 1$ is always resolvible in integers, as we have demonstrated, Art. 23; and, in order to find the least values of p and q that can satisfy it, we shall only have to convert the fraction $\frac{b'}{a'}$, into a continued fraction by the method of Art. 4, and then deduce from it a series of *principal* fractions, converging to the same fraction, $\frac{b'}{a'}$, by the formulæ of Art. 10; the last of these fractions will be the same fraction $\frac{b'}{a'}$; and if we represent the last but one by $\frac{p}{q}$, we shall have, by the law of these fractions, (Art. 12) $a'p - b'q = \pm 1$; the upper sign being for the case, in which the rank of the fraction is *even*, and the under for that in which it is *odd*.

These values of p and q being thus known, we shall first have $x = \pm pc'$, and $y = \pm qc'$, and then taking these values for α and β, we shall have, in general, (Art. 42),

$$x = \pm pc' + mb', y = \pm qc' + ma',$$

expressions which necessarily include all the solutions of the given equation that are possible in integer numbers.

That we may leave no obstacle to the practice of this method, we shall observe, that although the numbers a and b may be positive, or negative, we may notwithstanding take them always positive, provided we give contrary signs to x, when a is negative, and to y, when b is negative.

44. *Example.* To give an example of the preceding method, we shall take that of Art. 14, Chap. I. of the preceding Treatise, where it is required to resolve the equation, $39p = 56q + 11$. Changing p into x, and q into y, we shall have $39x - 56y = 11$.

So that we shall make $a = 39$, $b = 56$, and $c = 11$; and as 56 and 39 are already prime to each other, we shall have $a' = 39$, $b' = 56$, $c' = 11$. We must therefore reduce the fraction $\frac{b'}{a'} = \frac{56}{39}$, to a continued fraction; and, for this purpose, as we have already done (Art. 20), we shall make the following calculation;

```
39)56(1
   39
   --
   17)39(2
      34
      --
       5)17(3
         15
         --
          2)5(2
            4
            -
            1)2(2
              2
              -
              0.
```

Then, with the quotients 1, 2, 3, &c. we may form the fractions,

$$\begin{array}{ccccc} 1, & 2, & 3, & 2, & 2, \\ \frac{1}{1}, & \frac{3}{2}, & \frac{10}{7}, & \frac{23}{16}, & \frac{56}{39}, \end{array}$$

and the last fraction but one, $\frac{23}{16}$, will be that which we have expressed in general by $\frac{p}{q}$; so that we shall have $p = 23$, $q = 16$; and, as this fraction is the fourth, and consequently, of an even rank, we must take the upper sign; so that we shall have, in general,

$$x = 23 \times 11 + 56m, \text{ and}$$
$$y = 16 \times 11 + 39m;$$

m being any integer whatever, positive, or negative.

45. *Scholium.* We owe the first solution of this problem to M. Bachet de Meziriac, who gave it in the second edition of his Mathematical Recreations, entitled *Problemes plaisans et delectables*, &c. The first edition of this work appeared in 1612; but the solution in question is there only announced, and is only found complete in the edition of 1624. The method of M. Bachet is very direct and ingenious, and cannot be rendered more elegant, or more general.

I seize with pleasure the present opportunity of doing justice to this learned author, having observed that the mathematicians, who have since resolved the same problem, have never taken any notice of his labors.

The method of M. Bachet may be explained in a few words. After having shewn how the solution of equations of the form $ax - by = c$, (a and b being prime to each other), may be reduced to that of $ax - by = \pm 1$, he applies to the resolution of this last equation; and, for this purpose, prescribes the same operation with regard to the numbers a and b, as if we wished to find their greatest common divisor, (and this is what we have just done); then calling c, d, e, f, &c. the remainders arising from the different divisions, and supposing, for example, that f is the last remainder, which will necessarily be equal to unity (because a and b are prime to one another, by hypothesis), he makes, when the number of remainders is even, as in the present case,

$$e \mp 1 = \varepsilon, \frac{\varepsilon d \pm 1}{c} = \delta, \frac{\delta c \mp 1}{d} = \gamma, \frac{\gamma b \pm 1}{c} = \beta,$$
$$\frac{\beta a \mp 1}{b} = \alpha;$$

and these last numbers β, and α, will be the least values of x and y.

If the number of the remainders were odd, g for instance being the last remainder $= 1$, then we must make

$$f \pm 1 = \zeta, \frac{\zeta e \mp 1}{f} = \varepsilon, \frac{\varepsilon d \pm 1}{c} = \delta, \text{\&c.}$$

It is easy to see that this method is fundamentally the same as that of Chap. I.; but it is less convenient, because it requires divisions. Those who are curious in such speculations, will see with pleasure, in the work of M. Bachet, the artifices which he has employed to arrive at the foregoing Rule, and to deduce from it a complete solution of equations of the form, $ax - by = c$.

CHAP. IV.

General method for resolving, in Integer Numbers, Equations *with two unknown* Quantities, *of which one does not exceed the first* Degree.

[APPENDIX TO CHAP. III.

46. Let the general equation,

$$a + bx + cy + dx^2 + exy + gx^2y + fx^3 + hx^4 + kx^3y +, \text{\&c.}$$

$= 0$ be proposed, in which the coefficients a, b, c, &c. are given integer numbers, and x and y two indeterminate numbers, which must also be integers.

Deducing the value of y from this equation, we shall have

$$y = -\frac{a + bx + dx^2 + fx^3 + hx^4 +, \text{\&c.}}{c + ex + gx^2 + kx^3 +, \text{\&c.}}$$

so that the question will be reduced to finding an integer number, which, when taken for x, makes the numerator of this fraction divisible by its denominator.

Let us suppose

$$p = a + bx + dx^2 + fx^3 + hx^4 +, \text{\&c.}$$
$$q = c + ex + gx^2 + kx^3 +, \text{\&c.}$$

and taking x out of both these equations by the ordinary rules of Algebra, we shall have a final equation of this form,

$$A + Bp + Cq + Dp^2 + Epq + Fq^2 + Gp^3 +, \text{\&c.} = 0,$$

where the coefficients A, B, C, &c. will be rational and integer functions of the numbers a, b, c, &c.

Now, since $y = -\frac{p}{q}$, we shall also have $p = -qy$; so that by substituting this value of p, we shall get

$$\text{A} - \text{B}yq + \text{C}q + \text{D}y^2q^2 - \text{E}yq^2 + \text{F}q^2 +, \&c. = 0,$$

where all the terms are multiplied by q, except the first, A; therefore the number A must be divisible by the number q, otherwise it would be impossible for the numbers q and y to be both integers.

We shall therefore seek all the divisors of the known integer number A, and shall successively take each of these divisors for q; from each of which suppositions we shall have a determinate equation in x, the integer and rational roots of which, if it have any, will be found by the known methods; then substituting these roots for x, we shall see whether the values of p and q, which result, are such, that $\frac{p}{q}$ may be an integer number. By these means, we shall certainly find all the integer values of x, which may likewise give integer values of y in the equation proposed.

Hence we see, that the number of integer solutions of such equations must always be limited; but there is one case which must be excepted, and which does not fall under the preceding method.

47. This case is when there are no coefficients c, g, k, &c. So that we have simply,

$$y = -\frac{a + bx + dx^2 + fx^3 + hx^4 +, \&c.}{c}$$

In order to find all the values of x, that will render the quantity $a + bx + dx^2 + fx^3 + hx^4 +$, &c. divisible by the quantity c, we must proceed as follows. Suppose we have already found an integer, n, which satisfies this condition; it is evident that every number of the form $n \pm \mu c$ will likewise satisfy it, μ being any integer number; farther, if n is $> \frac{c}{2}$ (abstracting from the signs of n and c), we may always determine the number μ, and the sign which precedes it so, that the number $n \pm \mu c$, may become $< \frac{c}{2}$; and it is easy to perceive that this could only be done in one way, the values of n and c being given; wherefore, if we express by n' that value of $n \pm \mu c$, which is $< \frac{c}{2}$, and which satisfies the condition in question, we shall have, in general, $n = n' \mp \mu c$, μ being any number whatever.

Whence I conclude, that if we substitute successively, in the formula, $a + bx + dx^2 + fx^3 +$, &c. instead of x, all the integers positive, or negative, that do not exceed $\frac{c}{2}$, and if we denote by n', n'', n''', &c. such of those numbers as will render the quantity, $a + bx + dx^2 +$, &c. divisible by c, all the other numbers that do the same, will necessarily be included in the formulæ $n' \pm \mu' c$, $n'' \pm \mu'' c$, $n''' \pm \mu''' c$, &c. μ', μ'', μ''', &c. being any integer numbers.

Various remarks might here be made to facilitate the finding of the numbers n', n'', n''', &c. but it is the more unnecessary to enlarge upon this subject, as I have already had occasion to treat of it, in a Memoir published among those of the Academy of Berlin for the year 1768, and entitled *Nouvelle Methode pour resoudre les Problemes indeterminés.*

48. I shall, however, say a word on the method of determining two numbers, x and y, so that the fraction

$$\frac{ay^m + by^{m-1}x + dy^{m-2}x^2 + fy^{m-3}x^3 +, \text{ \&c.}}{c}$$

may become an integer number, as this investigation will be very useful to us in the sequel.

Supposing that y and x must be prime to each other, and farther, that y must be prime to c, we may always make $x = ny - cz$; n and z being indeterminate numbers; for, considering x, y, and c, as given numbers, we shall have an equation always resolvible in whole numbers by the method of Chap. III., because y and c have no common measure, by the hypothesis. Now, if we substitute this expression of x in the quantity, $ay^m + by^{m-1}x + dy^{m-2}x^2 +$, &c. it will become,

$$\begin{aligned}&(a + bn + dn^2 + fn^3 +, \text{ \&c.})\, y^m \\ &\quad - (b + 2dn + 3fn^2 +, \text{ \&c.})\, cy^{m-1}z \\ &\quad + (d + 3fn +, \text{ \&c.})\, c^2y^{m-2}z^2 \\ &\quad -, \text{ \&c.}\end{aligned}$$

and it is evident, that this quantity could not be divisible by c, unless the first term, $(a + bn + dn^2 + fn^3 +, \text{ \&c.})\, y^m$ were so, since all the other terms are multiplied by c. Therefore, as c and y are supposed to be prime to each other, the quantity $a + bn + dn^2 + fn^3 +$, &c. must itself be divisible by c; so that we shall only have to seek, by the method of the preceding Article, all the values of n that can satisfy this condition, and then we shall have, in general, $x = ny - az$, z being any integer number whatever.

It is proper to observe, that although we have supposed the numbers x and y to be prime to each other, as well as the numbers y and c, our solution is still no less general; for if x and y had a common measure α, we should only have to substitute $\alpha x'$ and $\alpha y'$, instead of x and y, and should then consider x' and y' as prime to each other; likewise if y' and c were a common measure β, we might put $\beta y''$, instead of y', and consider y'' and c as prime to each other.

CHAP. V.

A direct and general method for finding the values of x, *that will render* Rational Quantities *of the form* $\sqrt{(a + bx + cx^2)}$, *and for resolving, in* Rational Numbers, *the indeterminate* Equations *of the second* Degree, *which have two unknown* Quantities, *when they admit of* Solutions *of this kind.*

[APPENDIX TO CHAP. IV.]

49. I suppose first that the known numbers a, b, c, are integers; for if they were fractions, we should only have to reduce them to a common square denominator, and then it is evident, that we might always abstract from their denominator; but with respect to the number x, we shall suppose that it may be integer, or fractional, and shall see, in what follows, how the question is to be resolved, when we admit only integer numbers.

Let then $\sqrt{(a + bx + cx^2)} = y$, and we shall have $2cx + b = \sqrt{(4cy^2 + b^2 - 4ac)}$; so that the difficulty will be reduced to rendering rational the quantity,

$$\sqrt{(4cy^2 + b^2 - 4ac)}.$$

50. Let us suppose, therefore, in general, that we have to make rational the quantity $\sqrt{(\text{A}y^2 + \text{B})}$; that is to say, to make $\text{A}y^2 + \text{B}$ equal to a square, A and B being given integer numbers positive or negative, and y an indeterminate number, which must be rational.

It is evident that if one of the numbers A, or B, were 1, or any other square, the problem would be resolvible by the known methods of Diophantus, which are detailed in

Chap. IV.; we shall therefore abstract from those cases, or rather we shall endeavour to reduce all the rest to them.

Farther, if the numbers A and B were divisible by any square numbers, we might likewise abstract from those divisors; that is to say, suppress them, only by taking for A and B the quotients, which we should have, after dividing the given values by the greatest squares possible; in fact, supposing $A = \alpha^2 A'$, and $\beta = \beta^2 B'$, we shall have to make the number, $A'\alpha^2 y^2 + B'\beta^2$ a square; therefore, dividing by β^2, and making $\frac{\alpha y}{\beta} = y'$; we shall have to determine the unknown quantity y'; so that $A'y'^2 + B'$ may be a square.

Whence it follows that, when we have found a value of y that will make $Ay^2 + B$ become a square (rejecting in the given values of A and B the square factors α^2 and β^2, which they might contain), we shall only have to multiply the value found for y by $\frac{\beta}{\alpha}$, in order to have that which answers to the quantity proposed.

51. Let us, therefore, consider the formula $Ay^2 + B$, in which A and B are given integers, not divisible by any square; and, as we suppose that y may be a fraction, let us make $y = \frac{p}{q}$, p and q being integers prime to each other, in order that the fraction may be reduced to its least terms; we shall therefore have the quantity $\frac{Ap^2}{q^2} + B$, which must be a square; wherefore, $Ap^2 + Bq^2$ must be a square also; so that we shall have to resolve the equation, $Ap^2 + Bq^2 = z^2$, supposing p, q, and z, to be integer numbers.

Now, I say that q must be prime to A, and p prime to B; for if q and A had a common divisor, it is evident that the term Bq^2 would be divisible by the square of that divisor; and the term Ap^2 would only be divisible by the first power of the same divisor, because p and q are prime to each other, and A is supposed not to contain any square factor; wherefore the number $Ap^2 + Bq^2$ would only be once divisible by the common divisor of q and A; consequently, it would be impossible for that number to be a square. In the same manner, it may be proved, that p and B can have no common divisor.

Resolution of the Equation $Ap^2 + Bq^2 = z^2$ *in integer Numbers.*

52. Supposing A greater than B, the equation will be written thus,

$$Ap^2 = z^2 - Bq^2,$$

and as the numbers p, q, and z must be integer, $z^2 - Bq^2$ must be divisible by A.

Now, since A and q are prime to each other (Art. 51), we shall, according to the method of Art. 48, make

$$z = nq - Aq',$$

n and q' being two indeterminate integers; which will change the formula, $z^2 - Bq^2$, into $(n^2 - B)q^2 - 2nAqq' + A^2q'^2$, in which $n^2 - B$ must be divisible by A, taking for n an integer number, not $> \frac{A}{2}$.

We shall try therefore for n all the integer numbers that do not exceed $\frac{A}{2}$, and if we find none that makes $n^2 - B$ divisible by A, we conclude immediately, that the equation $Ap^2 = z^2 - Bq^2$ is not resolvible in whole numbers, and therefore that the quantity $Ay^2 + B$ can never become a square.

But if we find one or more satisfactory values of n, we must substitute them, one after the other, for n, and proceed in the calculation, as shall now be shewn.

I shall only remark farther, that it would be useless to give n values greater than $\frac{A}{2}$, for calling n', n'', n''', &c. the values of n less than $\frac{A}{2}$, which will render $n^2 - B$ divisible by A, all the other values of n that will have the same effect will be contained in these formulæ, $n' \pm \mu'A$, $n'' \pm \mu''A$, $n''' \pm \mu'''A$, &c. (Chap. IV. 47). Now, substituting these values for n, in the formula, $(n^2 - B)q^2 - 2nAqq' + A^2q'^2$, that is to say, $(nq - Aq')^2 - Bq^2$, it is evident that we shall have the same results, as if we only put n', n'', n''', &c. instead of n, and added to q' the quantities $\mp\mu'q$, $\mp\mu''q$, $\mp\mu'''q$, &c. so that, as q' is an indeterminate number, these substitutions would not give formulæ different from what we should have, by the simple substitution of the values n', n'', n''', &c.

53. Since, therefore, $n^2 - B$ must be divisible by A, let A'

be the quotient of this division, so that $AA' = n^2 - B$, and the equation,

$$Ap^2 = z^2 - Bq^2 = (n^2 - B)q^2 - 2nAqq' + A^2q'^2,$$

being divided by A, will become

$$p^2 = A'q^2 - 2nqq' + Aq'^2,$$

where A' will necessarily be less than A, because

$$A' = \frac{n^2 - B}{A}, \text{ and } B \angle A, \text{ and } n \text{ not } 7\ \frac{A}{2}.$$

First, if A' be a square number, it is evident this equation will be resolvible by the known methods; and the simplest solution will be obtained, by making $q' = 0$, $q = 1$, and $p = \sqrt{A'}$.

Secondly, if A' be not a square, we must ascertain whether it be less than B, or at least whether it be divisible by any square number, so that the quotient may be less than B, abstracting from the signs; then we must multiply the whole equation by A', and, because $AA' - n^2 = - B$, we shall have $A'p^2 = (A'q - nq')^2 - Bq'^2$; so that $Bq'^2 + A'p^2$ must be a square; hence, dividing by p^2, and making $\frac{q'}{p} = y'$, and $A' = C$, we shall have to make a square of the formula $By'^2 + C$, which evidently resembles that of Art. 52. Thus, if C contains a square factor γ^2, we may suppress it, by multiplying the value which we shall find for y' by γ, in order to have its true value; and we shall have a formula similar to that of Art. 51, but with this difference, that the coefficients, B and C, of our last will be less than the coefficients, A and B, of the other.

54. But if A' be not less than B, nor becomes so when divided by the greatest square, which measures it, then we must make $q = \nu q' + q''$; and, substituting this value in the equation, it will become

$$p^2 = A'q''^2 - 2n'q''q' + A''q'^2,$$

where $n' = n - \nu A'$,

$$\text{and } A'' = A''\nu^2 - 2n\nu + A = \frac{n'^2 - B}{A'}.$$

We must determine the whole number ν, which is always possible, so, that n' may not be $7\ \frac{A'}{2}$, abstracting from the signs, and then it is evident, that A'' will become $\angle A'$, because $A'' = \frac{n'^2 - B}{A'}$, and $B =$, or $\angle A'$, and $n =$, or $\angle \frac{A'}{2}$.

We shall therefore apply the same reasoning here that we did in the preceding Article; and if A'' is a square, we shall have the resolution of the equation: but if A'' is not a square, and $\angle B$, or becomes so, when divided by a square, we must multiply the equation by A', and shall thus have, by making $\frac{p}{q''} = y'$, and $A'' = C$, the formula $By'^2 + C$, which must be a square, and in which the coefficients, B and C, (after having suppressed in C the square divisors, if there are any), will be less than those of the formula $Ay^2 + B$ of Art. 51. But if these cases do not take place, we shall, as before, make $q' = \nu'q'' + q'''$, and the equation will be changed into this,

$$p^2 = A'''q''^2 - 2n''q''q''' + A''q'''^2,$$

where $n'' = n' - n'A''$,

$$\text{and } A''' = A''n'^2 - 2n'\nu' + A' = \frac{n''^2 - B}{A''}.$$

We shall therefore take for ν' such an integer number, that n'' may not be $7 \frac{A''}{2}$, abstracting from the signs; and, as B is not $7 A''$ (*hyp.*), it follows, from the equation, $A''' = \frac{n''^2 - B}{A''}$, that A''' will be $\angle A''$; so that we may go over the same reasoning as before, and shall draw from it similar conclusions.

Now, as the numbers A, A', A'', A''', &c. form a decreasing series of integer numbers, it is evident, that, by continuing this series, we shall necessarily arrive at a term less than the given number B; and then calling this term C, we shall have, as we have already seen, the formula $By'^2 + C$ to make equal to a square. So that by the operations we have now explained, we may always be certain of reducing the formula, $Ay^2 + B$, to one more simple, such as $By'^2 + C$; at least, if the problem is resolvible.

55. Now, in the same manner as we have reduced the

formula, $\text{A}y^2 + \text{B}$, to $\text{B}y'^2 + \text{C}$, we might reduce this last to $\text{C}y''^2 + \text{D}$, where D will be less than C, and so on; and as the numbers A, B, C, D, &c. form a decreasing series of integers, it is evident that this series cannot go on to infinity, and therefore the operation must always terminate. If the question admits of no solution in rational numbers, we shall arrive at an impossible condition; but, if the question is resolvible, we shall always be brought to an equation like that of Art. 53, in which one of the coefficients, as A′, will be a square; so that the known methods will be applicable to it: this equation being resolved, we may, by inverting the operation, successively resolve all the preceding equations, up to the first $\text{A}p^2 + \text{B}q^2 = z^2$.

We will illustrate this method by some examples.

56. *Example* 1. Let it be proposed to find a rational value of x, such, that the formula, $7 + 15x + 13x^2$, may become a square *.

Here, we shall have $a = 7$, $b = 15$, $c = 13$; and therefore $4c = 4 \times 13$, and $b^2 - 4ac = -139$; so that calling the root of the square in question y, we shall have the formula $4 \times 13y^2 - 139$, which must be a square. We shall also have $\text{A} = 4 \times 13$, and $\text{B} = -139$, where it will at once be observed, that A is divisible by the square 4; so that we must reject this square divisor, and simply suppose $\text{A} = 13$; but we must then divide the value found for y by 2, as is shewn, Art. 50.

Making, therefore, $y = \frac{p}{q}$, we shall have the equation, $13p^2 - 139q^2 = z^2$; or, because 139 is > 13, let us make $y = \frac{q}{p}$, in order to have $-139p^2 + 13q^2 = z^2$, an equation which we may write thus, $-139p^2 = z^2 - 13q^2$.

We shall now make (Art. 52) $z = nq - 139q'$, and must take for n an integer number not $> \frac{139}{2}$, that is to say, < 70 such, that $n^2 - 13$ may be divisible by 139. Assuming now $n = 41$, we have $n^2 - 13 = 1668 = 139 \times 12$; so that by making the substitution, and then dividing by -139, we shall have the equation,

$$p^2 = -12q^2 + 2 \times 41qq' - 139q'^2.$$

Now, as -12 is not a square, this equation has not the

* See Chap. IV. Art. 57, of the preceding Treatise.

requisite conditions; since 12 is already less than 13, we shall multiply the whole equation by -12, and it will become $-12p^2 = (-12q + 41q')^2 - 13q'^2$; so that $13q'^2 - 12p^2$ must be a square; or, making $\frac{q'}{p} = y'$, $13y'^2 - 12$ must be so too. Where, it is evident, we should only have to make $y' = 1$; but as we have got this value merely by chance, let us proceed in the calculation according to our method, until we arrive at a formula, to which the ordinary methods may be applied. As 12 is divisible by 4, we may reject this square divisor, remembering, however, that we must multiply the value of y' by 2; we have therefore to make a square of the formula $13y'^2 - 3$; or making $y' = \frac{r}{s}$, (supposing r and s to be integers prime to each other; so that the fraction $\frac{r}{s}$ is already reduced to its least terms, as well as the fraction $\frac{q}{p}$), the formula $13r^2 - 3s^2$ must be a square.

Let the root be z', which gives $13r^2 = z'^2 + 3s^2$; and, making $z' = ms - 13s'$, m being an integer not $> \frac{13}{2}$, that is, < 7, and such, that $m^2 + 3$ may be divisible by 13. Assuming $m = 6$, which gives $m^2 + 3 = 39 = 13 \times 3$, we have, by substituting the value of z', and dividing the whole equation by 13, $r^2 = 3s^2 - 2 \times 6ss' + 13s'^2$. As the coefficient 3 of s^2 is neither a square, nor less than that of s'^2, in the preceding equation, let us make (Art. 54), $s = \mu s' + s''$, and substituting, we shall have the transformed equation,

$$r^2 = 3s''^2 - 2(6 - 3\mu)s''s' + (3\mu^2 - 2 \times 6\mu + 13)s'^2;$$

and here we must determine μ so, that $6 - 3\mu$ may not be $> \frac{3}{2}$, and it is clear that we must make $\mu = 2$, which gives $6 - 3\mu = 0$; and the equation will become $r^2 = 3s''^2 + s'^2$, which is evidently reduced to the form required, as the coefficient of the square of one of the two indeterminate quantities of the second side is also a square. In order to have the most simple solution, we shall make $s'' = 0$, $s' = 1$, and $r = 1$; therefore, $s = \mu = 2$, hence $y' = \frac{r}{s} = \frac{1}{2}$; but we know that we must multiply the value of y' by 2; so that we shall have $y' = 1$; wherefore, tracing back the steps, we obtain $\frac{q'}{p} = 1$; whence $q' = p$; and the equation

$$-12p^2 = (-12q + 41q')^2 - 13q'^2 \text{ will give}$$
$$(-12q + 41p)^2 = p^2;$$

that is, $-12q + 41p = p$; so that $12q = 40p$; therefore, $y = \frac{q}{p} = \frac{40}{12} = \frac{10}{3}$; but as we must divide the value of y by 2, we shall have $y = \frac{5}{3}$; which will be the root of the given formula, $7 + 15x + 13x^2$; so that making $7 + 15x + 13x^2 = \frac{25}{9}$, we shall find, by resolving the equation, that $26x + 15 = \pm\frac{7}{3}$; whence, $x = -\frac{19}{39}$, or $= -\frac{2}{3}$.

We might have also taken $-12q + 41p = -p$, and should have had $y = \frac{q}{p} = \frac{21}{6}$; and, dividing by 2, $y = \frac{21}{12}$; then making $7 + 15x + 13x^2 = (\frac{21}{12})^2$, we shall find $26x + 15 = \pm\frac{9}{2}$; whence, $x = -\frac{21}{52}$, or $= -\frac{3}{4}$.

If we wished to have other values of x, we should only have to seek other solutions of the equation $r^2 = 3s''^2 + s'^2$, which is resolvible in general by the methods that are known; but when we know a single value of x, we may immediately deduce from it all the other satisfactory values, by the method explained in Chap. IV. of the preceding Treatise.

57. *Scholium.* Suppose, in general, that the quantity $a + bx + cx^2$ becomes equal to a square g^2, when $x = f$, so that we have $a + bf + cf^2 = g^2$; then $a = g^2 - bf - cf^2$; substituting this value in the given formula, it will become $g^2 + b(x - f) + c(x^2 - f^2)$. Now, let us take $g + m(x - f)$ for the root of this quantity, (m being an indeterminate number), and we shall have the equation,

$$g^2 + b(x - f) + c(x^2 - f^2) =$$
$$g^2 + 2mg(x - f) + m^2(x - f)^2$$

that is, expunging g^2 on both sides, and then dividing by $x - f$, we have

$$b + c(x + f) = 2mg + m^2(x - f);$$

whence we find $x = \frac{fm^2 - 2gm + b + cf}{m^2 - c}$. And it is evident, on account of the indeterminate number m, that this expression of x must comprehend all the values that can be given to x, in order to make the formula proposed a square; for whatever be the square number, to which this formula may be equal, it is evident, that the root of this number may always be represented by $g + m(x - f)$, giving to m a suitable value. So that when we have found, by the method above explained, a single satisfactory value of x, we have only to take it for f, and the root of the square which results for g; and, by the preceding formula, we shall have all the other possible values of x.

In the preceding example, we found $y = \frac{5}{3}$, and $x = -\frac{2}{3}$; so that, making $g = \frac{5}{3}$, and $f = -\frac{2}{3}$, we shall have

$$x = \frac{19 - 10m - 2m^2}{3(m^2 - 13)},$$

which is a general expression for the rational values of x, by which the quantity $7 + 15x + 13x^2$ may be made a square.

58. *Example* 2. Let it also be proposed to find a rational value of y, so that $23y^2 - 5$ may be a square.

As 23 and 5 are not divisible by any square number, we shall have no reduction to make. So that making

$y = \dfrac{p}{q}$, the formula $23p^2 - 5q^2$ must become a square, z^2; so that we shall have the equation $23p^2 = z^2 + 5q^2$.

We shall therefore make $z = nq - 23q'$, and we must take for n an integer number, not $> \frac{23}{2}$, such, that $n^2 + 5$ may be divisible by 23. I find $n = 8$, which gives $n^2 + 5 = 23 \times 3$, and this value of n is the only one that has the requisite conditions. Substituting, therefore, $8q - 23q'$, in the room of z, and dividing the whole equation by 23, we

shall have $p^2 = 3q^2 - 2 \times 8qq' + 23q'^2$, in which we see that the coefficient 3 is already less than the value of B, which is 5, abstracting from the sign. Art. 52.

Thus, we shall multiply the whole equation by 3, and

shall have $3p^2 = (3q - 8q')^2 + 5q'^2$; so that making $\dfrac{q'}{p} = y'$, the formula $-5y'^2 + 3$ must be a square, the coefficients 5 and 3 admitting of no reduction.

Therefore, let $y' = \dfrac{r}{s}$ (r and s being supposed prime to each other, whereas q' and p cannot be), and we shall have to make a square of the quantity $-5r^2 + 3s^2$; so that calling the root z', we shall have $-5r^2 + 3s^2 = z'^2$, and thence $-5r^2 = z'^2 - 3s^2$.

We shall, therefore, take $z' = ms + 5s'$, and m must be an integer number not $7\frac{5}{2}$, and such, that $m^2 - 3$ may be divisible by 5. Now, this is impossible; for we can only take $m = 1$, or $m = 2$, which gives $m^2 - 3 = -2$, or $= 1$. From this, therefore, we may conclude that the problem is not resolvible; that is to say, it is impossible for the formula $23y^2 - 5$ ever to become a square, whatever number we substitute for y *.

59. *Corollary.* If we had a quadratic equation, with two unknown quantities, such as
$a + bx + cy + dx^2 + exy + fy^2 = 0$, and it were proposed to find rational values of x and y that would satisfy the conditions of this equation, we might do this, when it is possible, by the method already explained.

Taking the value of y in x, we have
$2fy + ex + c = \surd((c - ex)^2 - 4f(a + bx + dx^2))$;
or, making $\alpha = c^2 - 4af$, $\beta = 2ce - 4bf$, $\gamma = e^2 - 4df$, $2fy + ex + c = \surd(\alpha + \beta x + \gamma x^2)$; the question will be reduced to finding the values of x, that will render rational the radical quantity $\surd(\alpha + \beta x + \gamma x^2)$.

60. *Scholium.* I have already considered this subject, rather differently, in the Memoirs of the Academy of Sciences at Berlin, for the year 1767, and, I believe, first gave a direct method, without the necessity of trial, for solving indeterminate problems of the second degree. The reader, who wishes to investigate this subject fully, may consult those Memoirs; where he will, in particular, find new and important remarks on the investigation of such integer numbers as, when taken for n, will render $n^2 - $ B divisible by A, A and B being given numbers.

* The impossibility of the formula $23y^2 - 5 = z^2$ is readily demonstrated: for y^2 must be of one of the forms $4n$, or $4n + 1$. In the first case, $23y^2 - 5$ is of the form $23 \times 4n - 5$, which is the same as $4n - 1$, and this is an impossible form for square numbers. In the second case, $23y^2 - 5$ is of the form $23 \times (4n + 1) - 5$, which is the same as $4n - 18$, or $4n - 2$, and this again is an impossible form for square numbers. Therefore, the formula $23y^2 - 5 = z^2$ is always impossible. B.

In the Memoirs for 1770, and the following years, investigations will be found on the form of divisors of the numbers represented by $z^2 - Bq^2$; so that by the mere form of the number A, we shall often be able to judge of the impossibility of the equation $Ap^2 = z^2 - Bq^2$, where $Ay^2 + B = \square$, (Art. 52).

CHAP. VI.

Of Double *and* Triple Equalities.

61. We shall here say a few words on the subject of double and triple equalities, which are much used in the analysis of Diophantus, and for the solution of which, that great mathematician, and his commentators, have thought it necessary to give particular rules.

When we have a formula, containing one or more unknown quantities, to make equal to a perfect power, such as a square, or a cube, &c. this is called, in the Diophantine analysis, a simple equality; and when we have two formulæ, containing the same unknown quantity, or quantities, to make equal each to a perfect power, this is called a double equality, and so on.

Hitherto, we have seen how to resolve simple equalities, in which the unknown quantity does not exceed the second degree, and the power proposed is the square.

Let us now see how double and triple equalities of the same kind are to be managed.

62. Let us first propose this double equality,

$$a + bx = \square;$$
$$c + dx = \square;$$

where the unknown quantity is found only in the first degree.

Making $a + bx = t^2$, and $c + dx = u^2$, and expunging x from both equations, we have $ad - bc = dt^2 - bu^2$; therefore,

$$dt^2 = bu^2 + ad - bc, \text{ and } (dt)^2 = dbu^2 + (ad - bc)d;$$

so that the difficulty will be reduced to finding a rational value of u, such, that $dbu^2 + ad^2 - bcd$ may become a square. This simple equality will be resolved by the method already explained, and knowing u, we shall likewise have $x = \frac{u^2 - c}{d}$.

If the double equality were

$$ax^2 + bx = \square,$$
$$cx^2 + dx = \square,$$

we should only have to make $x = \frac{1}{x'}$, and then multiplying both formulæ by the square x'^2, we should get these two equalities, $a + bx' = \square$, and $c + dx' = \square$, which are similar to the preceding,

Thus, we may resolve, in general, all the double equalities, in which the unknown quantity does not exceed the first degree, and those in which the unknown quantity is found in all the terms, provided it does not exceed the second degree; but it is not the same when we have equalities of this form,

$$a + bx + cx^2 = \square,$$
$$\alpha + \beta x + \gamma x^2 = \square.$$

If we resolve the first of these equalities by our method, and call f the value of x, which makes $a + bx + cx^2 = g^2$, we shall have, in general (Art. 57.),

$$x = \frac{fm^2 - 2gm + b + cf}{m^2 - c};$$

wherefore, substituting this expression of x in the other formula; $\alpha + \beta x + \gamma x^2$, and then multiplying it by $(m^2 - c)^2$, we shall have to resolve the equality,

$$\alpha(m^2 - c)^2 + \beta(m^2 - c) \times (fm^2 - 2gm + b + cf) + \gamma(fm^2 - 2gm + b + cf)^2 = \square;$$

in which, the unknown quantity, m, rises to the fourth degree.

Now, we have not yet any general rule for resolving such equalities; and all we can do is to find successively different solutions, when we already know one. (See Chap. IX.)

63. If we had the triple equality,

$$\left.\begin{array}{l} ax + by \\ cx + dy \\ hx + ky \end{array}\right\} = \square,$$

we must make $ax + by = t^2$, $cx + dy = u^2$, and $hx + ky = s^2$, and, expunging x and y from these three equations, we should have

$$(ak - bh)u^2 - (ck - dh)t^2 = (ad - cb)s^2;$$

so that, making $\frac{u}{t} = z$, the difficulty would be reduced to resolving the simple equality,

$$\frac{ak-bh}{ad-cb}z^2 - \frac{ck-dh}{ad-cb} = \square,$$

which is evidently a case of our general method.

Having found the value of z, we shall have $u = tz$, and the two first equations will give

$$x = \frac{d-bz^2}{ad-cb}t^2, \; y = \frac{az^2-c}{ad-cb}t^2.$$

But if the given triple equality contained only one variable quantity, we should then again have an equality with the unknown quantity rising to the fourth degree.

In fact, it is evident that this case may be deduced from the preceding, by making $y = 1$; so that we must have $\frac{az^2-c}{ad-cb}t^2 = 1$; and, consequently, $\frac{az^2-c}{ad-cb} = \square$.

Now, calling f one of the values of z, which can satisfy the above equality, and, in order to abridge, making $\frac{ak-bh}{ad-cb} = e$, we shall have, in general, (Art. 57.)

$$z = \frac{fm^2 - 2gm + ef}{m^2 - e}.$$

Then, substituting this value of z in the last equality, and multiplying the whole of it by the square of m^2-e, we shall have, $\frac{a(fm^2-2gm+ef)^2-c(m^2-e)^2}{ad-cb} = \square$, where the unknown quantity, m, evidently rises to the fourth power.

CHAP. VII.

A direct and general method for finding all the values of y *expressed in* Integer Numbers, *by which we may render* Quantities *of the form* $\sqrt{(Ay^2 + B)}$, *rational;* A *and* B *being given* Integer Numbers; *and also for finding all the possible* Solutions, *in* Integer Numbers, *of indeterminate* Quadratic Equations *of two unknown* Quantities.

[APPENDIX TO CHAP. VI.]

64. Though by the method of Art. 5, general formulæ may be found, containing all the rational values of y, by which $Ay^2 + B$ may be made eqnal to a square; yet those formulæ are of no use, when the values of y are required to be expressed in integer numbers: for which reason, we must here give a particular method for resolving the question in the case of integer numbers.

Let then $Ay^2 + B = x^2$; and as A and B are supposed to be integer numbers, and y must also be integer, it is evident that x ought likewise to be integer; so that we shall have to resolve, in integers, the equation $x^2 - Ay^2 = B$. Now, I begin by remarking, that if B is not divisible by a square number, y must necessarily be prime to B; for suppose, if possible, that y and B have a common divisor α, so that $y = \alpha y'$, and $B = \alpha B'$; we shall then have $x^2 = A\alpha^2 y'^2 = \alpha B'$, whence it follows that x^2 must be divisible by α; and as α is neither a square, nor divisible by any square (*hyp.*), because α is a factor of B, x must be divisible by α. Making then $x = \alpha x'$, we shall have $\alpha^2 x'^2 = \alpha^2 A y'^2 + \alpha B'$; or, dividing by α, $\alpha x'^2 = \alpha A y'^2 + B'$; whence it is evident, that B' must still be divisible by α, which is contrary to the hypothesis.

It is only, therefore, when B contains square factors, that y can have a common measure with B; and it is easy to see, from the preceding demonstration, that this common measure of y and B can only be the root of one of the square factors of B, and that the number x must have the same common measure; so that the whole equation will be divisible by the square of this common divisor of x, y, and B.

Hence I conclude, 1st. That if B is not divisible by any square, y and B will be prime to each other.

2dly. That if B is divisible by a single square α^2, y may be either prime to B, or divisible by α, which makes two cases to be separately examined. In the first case, we shall resolve the equation $x^2 - Ay^2 = B$, supposing y and B prime to one another; in the second, we shall have to resolve the equation, $x^2 - Ay^2 = B'$, B$'$ being $= \frac{B}{\alpha^2}$, supposing also y and B$'$ prime to each other; but it will then be necessary to multiply by α the values found for y and x, in order to have values corresponding to the equation proposed.

3dly. If B is divisible by two different squares, α^2 and β^2, we shall have three cases to consider. In the first, we shall resolve the equation $x^2 - Ay^2 = B$, considering y and B as prime to each other. In the second, we shall likewise resolve the equation, $x^2 - Ay^2 = B'$, B$'$ being $= \frac{B}{\alpha^2}$, on the supposition of y and B being prime to each other, and we shall then multiply the values of x and y by α. In the third, we shall resolve the equation $x^2 - Ay^2 = B''$, B$''$ being $= \frac{B}{\beta^2}$, on the supposition of y and B$''$ being prime to each other, and we shall then multiply the values of x and y by β.

4thly, &c. Thus, we shall have as many different equations to resolve, as there may be different square divisors of B; but those equations will be all of the same form, $x^2 - Ay^2 = B$, and y also will always be prime to B.

65. Let us therefore consider, generally, the equation $x^2 - Ay^2 = B$; where y is prime to B; and, as x and y must be integers, $x^2 - Ay^2$ must be divisible by B.

By the method, therefore, of Chap. IV. 48, we shall make $x = ny - Bz$, and shall have the equation,
$(n^2 - A)y^2 - 2nByz + B^2z^2 = B$, from which we perceive, that the term, $(n^2 - A)y^2$, must be divisible by B, since all the others are so of themselves; wherefore, as y is prime to B, (*hyp.*) $n^2 - A$ must be divisible by B; so that making $\frac{n^2 - A}{B} = C$, and dividing by B, we shall have,
$Cy^2 - 2nyz + Bz^2 = 1$. Now, this equation is simpler than the one proposed, because the second side is equal to unity.

We shall seek, therefore, the values of n, which may render $n^2 - A$ divisible by B; for this it will be sufficient, (Art. 47), to try for n all the integer numbers, positive or negative, not $> \frac{B}{2}$; and if among these we find no one satisfactory, we shall at once conclude that it is impossible for $n^2 - A$ to be divisible by B, and therefore that the given equation is not resolvible in integer numbers.

But if, in this manner, we find one, or more satisfactory numbers, we must take them, one after another, for n, which will give as many different equations, to be separately considered, each of which will furnish one, or more solutions, of the given question.

With regard to such values of n as would exceed that of $\frac{B}{2}$, we may neglect them, because they would give no equations different from those, which will result from the values of n that are not $> \frac{B}{2}$, as we have already shewn (Art. 52.)

Lastly, as the condition from which we must determine n is, that $n^2 - A$ may be divisible by B, it is evident, that each value of n may be negative, as well as positive; so that it will be sufficient to try, successively, for n, all the natural numbers, that are not greater than $\frac{B}{2}$, and then to take the satisfactory values of n, both in *plus* and in *minus*.

We have elsewhere given rules for facilitating the investigation of the values of n, that may have the property required, and even for finding those values *à priori* in a great number of cases. See the Memoirs of Berlin for the year 1767, pages 194, and 274.

Resolution *of the* Equation $Cy^2 - 2nyz + Bz^2 = 1$, *in* Integer Numbers.

This equation may be resolved by two different methods.

First Method.

66. As the quantities C, n, B are supposed to be integer numbers, as well as the indeterminate quantities y and z, it is evident, that the quantity $Cy^2 - 2nyz + Bz^2$ must always be equal to integer numbers; consequently, unity will be its least possible value, unless it may become 0, which can only happen, when this quantity may be resolved into two rational

factors. As this case is attended with no difficulty, we shall at once neglect it, and the question will be reduced to finding such values of y and z, as will make the quantity in question the least possible. If the *minimum* is equal to unity, we shall have the resolution of the proposed equation; otherwise, we shall be assured, that it admits of no solution in integer numbers. So that the present problem falls under the third problem of Chap. II., and admits of a similar solution. Now, as we have here $(2n)^2 - 4\text{BC} = 4\text{A}$ (Art. 65), we must make two distinct cases, according as A shall be positive, or negative.

First case, when $n^2 - \text{BC} = \text{A} \angle 0$.

67. According to the method of Art. 32, we must reduce the fraction $\frac{n}{\text{C}}$, taken positively, to a continued fraction; this may be done by the rule of Art. 4; then, by the formulæ of Art. 10, we shall form the series of fractions converging towards $\frac{n}{\text{C}}$, and shall have only to try, successively, the numerators of those fractions for the number y, and the corresponding denominators for the number z: if the given formula is resolvible in integers, we shall in this way find the satisfactory values of y and z; and, conversely, we may be certain, that it admits not of any solution in integer numbers, if no satisfactory values are found among the numbers that we have tried.

Second case, when $n^2 - \text{BC} = \text{A} > 0$.

68. We shall here employ the method of Art. 33 *et seq.* so that, because $\text{E} = 4\text{A}$, we shall at once consider the quantity (Art. 39), $a = \frac{n \pm \sqrt{\text{A}}}{\text{C}}$, in which we must determine the signs both of the value of n, which we have seen may be either positive or negative, and of $\sqrt{\text{A}}$, so that it may become positive; we shall then make the following calculation:

$$\text{Q}^0 = -n, \qquad \text{P}^0 = \text{C}, \qquad \mu \angle \frac{-\text{Q}^0 \pm \sqrt{\text{A}}}{\text{P}^0}.$$

$$\text{Q}' = -\mu\text{P}^0 + \text{Q}^0, \quad \text{P}' = \frac{\text{Q}'^2 - \text{A}}{\text{P}^0}, \qquad \mu' \angle \frac{-\text{Q}' \mp \sqrt{\text{A}}}{\text{P}'}.$$

$$Q'' = \mu' P' + Q', \qquad P'' = \frac{Q''^2 - A}{P'}, \qquad \mu'' < \frac{-Q'' \pm \sqrt{A}}{P'''}.$$

$$Q''' = {}'' P'' + Q'', \qquad P''' = \frac{Q'''^2 - A}{P''}, \qquad \mu''' < \frac{-Q''' \mp \sqrt{A}}{P'''}.$$

&c. &c. &c.

and we shall only continue these series until two corresponding terms of the first and the second series appear again together; then, if among the terms of the second series, P^0, P', P'', &c. there be found one positive, and equal to unity, this term will give a solution of the proposed equation; and the values of y and z will be the corresponding terms of the two series p^0, p', p'', &c. and q^0, q', q'', calculated according to the formulæ of Art. 25; otherwise, we may immediately conclude, that the given equation is not resolvible in integer numbers. *See the example of* Art. 40.

Third case, when A *is a* square.

69. In this case, the quantity $\sqrt{A}$ will become rational, and the quantity $Cy^2 - 2nyz + Bz^2$ will be resolvible into two rational factors. Indeed, this quantity is no other than $\frac{(Cy - nz)^2 - Az^2}{C}$, which, supposing $A = a^2$, may be thrown into this form, $\frac{(Cy \pm (n + a)z)(Cy \pm (n - a)z)}{C}$.

Now, as $n^2 - a^2 = AC = (n + a) \times (n - a)$, the product of $n + a$ by $n - a$ must be divisible by C; and, consequently, one of these two numbers $n + a$, and $n - a$, must be divisible by one of the factors of C, and the other by the other factor. Let us, therefore, suppose $C = bc$, $n + a = fb$, and $n - a = gc$, f and b being whole numbers, and the preceding quantity will become the product of these two linear factors, $cy \pm fz$, and $by \pm gz$; therefore, since these two factors are both integers, it is evident that their product could not be $= 1$, as the given equation requires, unless each of them were separately $= \pm 1$; we shall therefore make $cy \pm fz = \pm 1$, and $by \pm gz = \pm 1$, and by these means we shall determine the numbers y and z. If we find these numbers integer, we shall have the solution of the equation proposed; otherwise, it will be irresolvible, at least in whole numbers.

Second Method.

70. Let the formula $cy^2 - 2nyz + Bz^2$ undergo such transformations as those we have already made (Art. 54), and we shall invariably be brought by the transformations, to an equation, such as $L\xi^2 - 2M\xi\psi + N\psi^2$, the numbers L, M, N, being whole numbers, depending upon the given numbers C, B, n, so that we have $M^2 - LN = n^2 - CB = A$; and farther, that 2M may not be greater (abstracting from the signs) than the number L, nor the number N, the numbers ξ and ψ will likewise be integer, but depending on the indeterminate numbers y and z.

For example, let C be less than B, and let us put the formula in question into this form,

$$B'y^2 - 2nyy' + By'^2,$$

making $C = B'$, and $z = y'$; if $2n$ be not greater than B', it is evident that this formula will already of itself have the requisite conditions; but if $2n$ be greater than B', then we must suppose $y = my' + y''$; and, by substitution, we shall have the transformed formula,

$$B'y''^2 - 2n'y''y' + B''y'^2,$$

where

$$n' = n - mB', \text{ and } B'' = m^2B' - 2mn + B = \frac{n'^2 - A}{B'}.$$

Now, as the number m is indeterminate, we may, by supposing it an integer, take it such, that the number $n - mB'$ may not be greater than $\frac{1}{2}B'$, abstracting from the sign; then $2n'$ will not surpass B'. So that, if $2n'$ does not even exceed B'', the preceding transformed formula will already be in the case which we have seen; but if $2n'$ is greater than B'', we shall then continue to suppose $y' = m'y'' + y'''$, which will give this new transformation,

$$B'''y''^2 - 2n''y''y''' + B''y'''^2,$$

where

$$n'' = n' - m'B'', \text{ and } B''' = m'^2B'' - 2m'n' + B' = \frac{n''^2 - A}{B''}.$$

We shall now determine the whole number m', so that $n' - m'B''$ may not be greater than $\frac{B''}{2}$, by which means $2n''$ will not exceed B''; so that we shall have the required transformation, if $2n''$ does not even exceed B''; but if $2n''$ exceed B''', we shall again suppose $y'' = m''y''' + y^{iv}$, &c. &c.

Now, it is evident, that these operations cannot go on to infinity; for since $2n$ is greater than B', and $2n'$ is not, n' will evidently be less than n; in the same manner, $2n'$ is greater than B'', and $2n''$ is not, wherefore n'' will be less than n', and so on; so that the numbers n, n', n'', &c. will form a decreasing series of integers, which of course cannot go on to infinity. We shall therefore arrive at a formula, in which the coefficient of the middle term will not be greater than those of the two extreme terms, and which will likewise have the other properties already mentioned; as is evident from the nature of the transformations employed.

In order to facilitate the transformation of the formula,

$$Cy^2 - 2nyz + Bz^2$$

into this,

$$L\xi^2 - 2M\xi\psi + N\psi^2,$$

let us denote by D the greater of the two extreme coefficients C and B, and the other coefficient by D'; and, *vice versâ*, let us denote by θ the variable quantity, whose square shall be found multiplied by D', and the other variable quantity by θ'; so that the given formula may take this form,

$$D'\theta^2 - 2n\theta\theta' + D\theta'^2,$$

where D' is less than D; then we have only to make the following calculation.

$$m = \frac{n}{D'},\ n' = n - mD',\ D'' = \frac{n'^2 - A}{D'},\ \theta = m\theta' + \theta'',$$

$$m' = \frac{n'}{D''},\ n'' = n' - m'D'',\ D''' = \frac{n''^2 - A}{D''},\ \theta' = m'\theta'' + \theta''',$$

$$m'' = \frac{n''}{D'''},\ n''' = n'' - m''D''',\ D^{iv} = \frac{n'''^2 - A}{D'''},\ \theta'' = m''\theta''' + \theta^{iv},$$

&c. &c. &c.

where it must be observed, that the sign $=$, which is put after the letters m, m', m'', &c. does not express a perfect equality, but only an equality as approximate as possible, so long as we understand only integer numbers by m, m', m'', &c. The sign $=$ being only employed for want of a better.

These operations must be continued, until in the series n, n', n'', &c. we find a term, as n_ϱ, which (abstracting from the sign) does not exceed the half of the corresponding term,

D^{ρ} of the series D', D'', D''', &c. any more than the half of the following term $D^{\rho+1}$. Then we may make $D^{\rho} = L$, $n^{\rho} = N$, $D^{\rho+1} = M$, and $\theta^{\rho} = \psi$, $\theta^{\rho+1} = \xi$, or $D^{\rho} = M$, $D^{\rho+1} = L$, and $\theta^{\rho} = \xi$, $\theta^{\rho+1} = \psi$. We must always suppose, as we proceed, that we have taken, for M, the less of the two numbers D^{ρ}, $D^{\rho+1}$.

71. The equation, $Cy^2 - 2nyz + Dz^2 = 1$, will therefore be reduced to this,

$$L\xi^2 - 2N\xi\psi + M\psi^2 = 1,$$

where $N^2 - LM = A$, and where 2N is neither $> L$, nor $> M$, (abstracting from the signs). Now, M being the less of the two coefficients L and M, let us multiply the whole of the equation by the coefficient M; and making

$$v = M\psi - N\xi,$$

it is evident, that it will be changed into

$$v^2 - A\xi^2 = M,$$

in which we must make a distinction between the two cases of A positive, and A negative.

1st. Let A be negative, and $= -a$ (a being a positive number), the equation will then be

$$v^2 + a\xi^2 = M.$$

Now, as $N^2 - LM = A$, we shall have $a = LM - N^2$; whence we immediately perceive, that the numbers L and M must have the same signs; otherwise, 2N can neither be $> L$, nor $> M$; wherefore N^2 will not be $> \frac{LM}{4}$; therefore, $a =$, or $> \frac{3}{4}LM$; and since M is supposed to be less than L, or at least not greater than L, we shall have, *à fortiori*, $a =$, or $> \frac{3}{4}M^2$; whence $M =$, or $< \sqrt{\frac{4a}{3}}$; and $M < \frac{4}{3}\sqrt{a}$.

Hence, we see that the equation, $v^2 + a\xi^2 = M$, could not exist on the supposition of v and ξ being whole numbers, unless we made $\xi = 0$, and $v^2 = M$, which requires M to be a square number.

Let us, therefore, suppose $M = \mu^2$, and we shall have $\xi = 0$, $v = \pm\mu$, wherefore, from the equation, $v = M\psi - N\xi$, we shall have $\mu^2\psi = \pm\mu$, and, consequently, $\psi = \pm\frac{1}{\mu}$; so that ψ cannot be a whole number, as it ought, by the hypothesis, unless μ be equal to unity, or $= \pm 1$, and, consequently, $M = 1$.

Hence, therefore, we may infer, that the given equation is not resolvible in integers, unless M be found equal to unity, and positive. If this condition takes place, then we make $\xi = 0$, $\psi = \pm 1$, and go back from these values to those of y and z.

This method is founded on the same principles as that of Art. 67; but it has the advantage of not requiring any trial.

2dly. Let A be now a positive number, and we shall have $A = N^2 - LM$. And as N^2 cannot be greater than $\frac{LM}{4}$, it is evident that the equation cannot subsist, unless $-LM$ be a positive number; that is to say, unless L and M have contrary signs. Thus, A will necessarily be $\angle - LM$, or at farthest $= - LM$, if $N = 0$; so that we shall have $- LM =$, or $\angle A$; and, consequently, $M^2 =$, or $\angle A$, or $M =$, or $\angle \sqrt{A}$.

The case of $M = \sqrt{A}$ cannot take place, except when A is a square; consequently, this case may be easily resolved by the method already given, (Art. 69).

There remains, now, only the case in which A is not a square, and in which we shall necessarily have $M \angle \sqrt{A}$ (abstracting from the sign of M); then the equation, $\upsilon^2 - A\xi^2 = M$, will come under the case of the theorem, Art. 38, and may therefore be resolved by the method there explained.

Hence, we have only to make the following calculation:

$$Q^0 = 0, \qquad P^0 = 1, \qquad \mu \angle \sqrt{A}$$

$$Q' = \mu, \qquad P' = Q'^2 - A, \qquad \mu' \angle \frac{-Q' - \sqrt{A}}{P'}$$

$$Q'' = \mu' P' + Q', \qquad P'' = \frac{Q''^2 - A}{P'}, \qquad \mu'' \angle \frac{-Q'' + \sqrt{A}}{P''}$$

$$Q''' = \mu'' P'' + Q'', \qquad P''' = \frac{Q'''^2 - A}{P''}, \qquad \mu''' \angle \frac{-Q''' - \sqrt{A}}{P'''},$$

&c. &c. &c.

continuing it until two corresponding terms of the first and second series appear again together; or until in the series P', P'', P''', &c. there be found a term equal to unity, and positive; that is to say, $= P^0$: for then all the succeeding terms will return in the same order in each of the three series (Art. 37). If in the series P', P'', P''', &c. there be found a term equal to M, we shall have the resolution of the given equation; for we shall only have to take, for υ and ξ, the corresponding terms of the series p', p'', p''', &c. q', q'', q''', &c. calculated according to the formulæ of Art. 25; and we may even find an infinite number of satisfactory values for υ and ξ, by continuing the same series to infinity.

Now, as soon as we know two values of υ and ξ, we shall have, from the equation, $\upsilon = M\psi - N\xi$, that of ψ, which will also be a whole number; then we may go back from these values of ξ and ψ, that is to say, of $\theta_{\varrho+1}$, and θ_{ϱ}, to those of θ and θ', or of y and z (Art. 70).

But if in the series P', P'', P''', &c. there is no term $= M$, we are sure that the equation proposed admits of no solution in whole numbers.

It is proper to observe, that, as the series P^0, P', P'', &c. as well as the two others, Q^0, Q', Q'', &c. and μ, μ', μ'', &c. depend only on the number A; the calculation, once made for a given value of A, will serve for all the equations in which A, or $n^2 - CB$, shall have the same value; and hence the foregoing method is preferable to that of Art. 68, which requires a new calculation for each equation.

Lastly, so long as A does not exceed 100, we may make use of the Table given, Art. 41, which contains for each radical $\sqrt{A}$, the values of the terms of the two series P^0, $-P'$, P'', $-P'''$, &c. and μ, μ', μ'', &c. continued, until one of the terms P', P'', P''', &c. becomes $= 1$; after which, all the succeeding terms of both series return in the same order. So that, by means of this Table, we may judge, immediately, whether the equation, $\upsilon^2 - A\xi^2 = M$, be resolvible, or not.

Of the manner of finding all the possible solutions of the equation, $Cy^2 - 2nyz + Bz^2 = 1$, *when we know only one of them.*

72. Though, by the methods just given, we may successively find all the solutions of this equation, when it is resolvible in integer numbers; yet this may be done, in a manner still more simple, as follows:

Call p and q the values found for y and z; so that we have

$$Cp^2 - 2npq + Bq^2 = 1,$$

and take two other whole numbers, r and s, such, that $ps - qr = 1$; which is always possible, because p and q are necessarily prime to each other; then suppose

$$y = pt + ru, \text{ and } z = qt + su,$$

t and u being two new indeterminate numbers; substituting these expressions in the equation,

$$Cy^2 - 2nyz + Bz^2 = 1,$$

and, in order to abridge, making

$$\text{P} = \text{C}p^2 - 2npq + \text{B}q^2,$$
$$\text{Q} = \text{C}pr - n(ps + qr) + \text{B}qs,$$
$$\text{R} = \text{C}r^2 - 2nrs + \text{B}s^2,$$

we shall have the equation transformed into this,

$$\text{P}t^2 + 2\text{Q}tu + \text{R}u^2 = 1.$$

Now we have, by hypothesis, $\text{P} = 1$; farther, if we call ϱ and σ, two values of r and s that satisfy the equation, $ps - qr = 1$, we shall have, in general, (Art. 42),

$$r = \varrho + mp,\ s = \sigma + mq,$$

m being any whole number; therefore, putting these values into the expression of Q, it will become

$$\text{Q} = \text{C}p\varrho - n(p\sigma + q\varrho) + \text{B}q\sigma + m\text{P};$$

so that, as $\text{P} = 1$, we may make $\text{Q} = 0$, by taking

$$m = -\text{C}p\varrho + n(p\sigma + q\varrho) - \text{B}q\sigma.$$

We now observe, that the value of $\text{Q}^2 - \text{PR}$ is reduced (after the above substitutions and reductions), to this; $(n^2 - \text{CB}) \times (ps - qr)^2$; so that as $ps - qr = 1$, we shall have $\text{Q}^2 - \text{PR} = n^2 - \text{CB} = \text{A}$; therefore, making $\text{P} = 1$, and $\text{Q} = 0$, we shall have $-\text{R} = \text{A}$, that is, $\text{R} = -\text{A}$; so that the equation before transformed will become $t^2 - \text{A}u^2 = 1$. Now, as y, z, p, q, r, and s are whole numbers, by the hypothesis, it is easy to perceive, that t and u will also be whole numbers; for, deducing their values from the equations, $y = pt + ru$, and $z = qt + su$, we have

$$t = \frac{sy - rz}{ps - qr}, \text{ and } u = \frac{qy - pz}{qr - ps};$$

that is to say, (because $ps - qr = 1$), $t = sy - rs$, and $u = pz - qy$.

We shall therefore only have to resolve, in whole numbers, the equation $t^2 - \text{A}u^2 = 1$, and each value of t and u will give new values of y and z.

For, substituting the value of the number m, already found, in the general values of r and s, we shall have

$$r = \varrho(1 - \text{C}p^2) - \text{B}pq\sigma + np(p\sigma + q\varrho),$$
$$s = \sigma(1 - \text{B}q^2) - \text{C}pq\varrho + nq(p\sigma + q\varrho);$$

or, because $\text{C}p^2 - 2npq + \text{B}q^2 = 1$,

$$r = (\text{B}q - np) \times (q\varrho - p\sigma) = -\text{B}q + np,$$
$$s = (\text{C}p - nq) \times (p\sigma - q\varrho) = \text{C}p - nq.$$

Therefore, putting these values of r and s in the foregoing expressions of y and z, we shall have, in general,

$$y = pt - (Bq - np)u,$$
$$z = qt + (Cp - nq)u.$$

73. The whole therefore is reduced to resolving the equation $t^2 - Au^2 = 1$.

Now, 1st, if A be a negative number, it is evident, that this equation cannot subsist, in whole numbers, except by making $u = 0$, and $t = 1$, which would give $y = p$, and $z = q$. Whence we may conclude that, in the case of A being a negative number, the proposed equation,

$$Cy^2 - 2nyz + Bz^2 = 1,$$

can never admit but of one solution in whole numbers.

The case would be the same, if A were a positive square number; for making $A = a^2$, we should have $(t + au) \times (t - au) = 1$; wherefore, $t + au = \pm 1$, and $t - au = \pm 1$; wherefore, $2au = 0$, $u = 0$, and consequently $t = \pm 1$.

2dly. But if A be a positive number, not square, then the equation, $t^2 - Au^2 = 1$, is always capable of an infinite number of solutions, in whole numbers, (Art. 37), which may be found by the formulæ already given (Art. 71); but it will be sufficient to find the least values of t and u; and, for this purpose, as soon as we have arrived, in the series P', P'', P''', &c. at a term equal to unity, we shall have only to calculate, by the formulæ of Art. 25, the corresponding terms of the two series p', p'', p''', &c. and q', q'', q''', &c. for these will be the values required of t and u. Whence it is evident, that the same calculation made for resolving the equation $v^2 - A\xi^2 = M$, will serve also for the equation

$$t^2 - Au^2 = 1.$$

Provided that A does not exceed 100, we have the least values of t and u calculated in the Table, at the end of Chap. VII. of the preceding Treatise, and in which the numbers a, m, n, are the same as those that are here called A, t and u.

74. Let us denote by t', u', the least values of t, u, in the equation $t^2 - Au^2 = 1$; and in the same manner as these values may serve to find new values of y and z, in the equation, $Cy^2 - 2nyz + Bz^2 = 1$, so they will likewise serve for finding new values of t and u in the equation $t^2 - Au^2 = 1$, which is only a particular case of the former. For this purpose, we shall only have to suppose $C = 1$, and $n = 0$, which gives $-B = A$, and then take t, u, instead of y, z, and t', u', instead of p, q. Making these substitutions, therefore, in the general expressions of y and z (Art. 72), and farther, putting T, V, instead of t, u, we shall have, generally,

$$t = \text{T}t' + \text{AV}u',$$
$$u = \text{T}u' + \text{V}t',$$

and, for the determination of T and V, we shall have the equation $\text{T}^2 - \text{AV}^2 = 1$, which is similar to the one proposed.

Thus, we may suppose $\text{T} = t'$, and $\text{V} = u'$, which will give

$$t = t'^2 + \text{A}u'^2,\ u = t'u' + t'u'.$$

Calling t'', u'' the second values of t and u, we shall have

$$t'' = t'^2 + \text{A}u'^2,\ u'' = 2t'u'.$$

Now, it is evident, that we may take these new values t'', u'', instead of the first t', u'; so that we shall have

$$t = \text{T}t'' + \text{AV}u'',$$
$$u = \text{T}u'' + \text{V}t'',$$

where we may again suppose $\text{T} = t'$, $\text{V} = u'$, which will give

$$t = t't'' + \text{A}u'u'',\ u = t'u'' + u't''.$$

Thus, we shall have new values of t and u, which will be

$$t''' = t't'' + \text{A}u'u'' = t'(t'^2 + 3\text{A}u'^2),$$

$$u''' = t'u'' + u't'' = u'(3t'^2 + \text{A}u'^2),$$

and so on.

75. The foregoing method only enables us to find the values t'', t''', &c. u'', u''', &c. successively; let us now consider how this investigation may be generalised. We have first,

$$t = \text{T}t' + \text{AV}u',\ u = \text{T}u' + \text{V}t';$$

whence this combination,

$$t \pm u\sqrt{\text{A}} = (t' \pm u'\sqrt{\text{A}}) \times (\text{T} \pm \text{V}\sqrt{\text{A}});$$

then supposing $\text{T} = t'$, and $\text{V} = u'$, we shall have

$$t'' \pm u''\sqrt{\text{A}} = (t' \pm u'\sqrt{\text{A}})^2.$$

Let us now substitute these values of t'' and u'', instead of those of t' and u', and we shall have

$$t \pm u\sqrt{\text{A}} = (t' \pm u'\sqrt{\text{A}})^2 \times (\text{T} \pm \text{V}\sqrt{\text{A}}),$$

where, again making $\text{T} = t'$, and $\text{V} = u'$, and calling t''', u''', the resulting values of t and u, there will arise

$$t''' \pm u'''\sqrt{\text{A}} = (t' \pm u'\sqrt{\text{A}})^3.$$

In the same manner, we shall find

$$t^{\text{iv}} \pm u^{\text{iv}}\sqrt{\text{A}} = (t' \pm u'\sqrt{\text{A}})^4,$$

and so on.

Hence, in order to simplify, if we now call T and V the first and the least values of t, u, which we before called t', u', we shall have, in general,

$$t \pm u\sqrt{A} = (T \pm V\sqrt{A})^m,$$

m being any positive whole number; whence, on account of the ambiguity of the signs, we derive

$$t = \frac{(T+V\sqrt{A})^m + (T-V\sqrt{A})^m}{2}$$

$$u = \frac{(T+V\sqrt{A})^m - (T-V\sqrt{A})^m}{2\sqrt{A}}.$$

Though these expressions appear under an irrational form, it is easy to see that they will become rational, if we involve the powers of $T \pm V\sqrt{A}$; for it is well known that

$$(T \pm V\sqrt{A})^m = T^m \pm mT^{m-1}V\sqrt{A} + \frac{m(m-1)}{2}T^{m-2}V^2A$$
$$+ \frac{m(m-1)\times(m-2)}{2\times 3}T^{m-3}V^3A\sqrt{A} +, \text{\&c.}$$

Wherefore,

$$t = T^m + \frac{m(m-1)}{2}AT^{m-2}V^2$$
$$+ \frac{m(m-1)\times(m-2)\times(m-3)}{2\times 3\times 4}A^2T^{m-4}V^4 +, \text{\&c.}$$

$$u = mT^{m-1}V + \frac{m(m-1)\times(m-2)}{2\times 3}AT^{m-3}V^3$$
$$+ \frac{m(m-1)\times(m-2)\times(m-3)\times(m-4)}{2\times 3\times 4\times 5}A^2T^{m-5}V^5 +, \text{\&c.}$$

Where we may take for m any positive whole numbers whatever.

It is evident that, by successively making $m = 1, 2, 3, 4$, &c. we shall have values of t and u, that will go on increasing.

I shall now shew that, in this manner, we may obtain all the possible values of t and u, provided T and V are the least of them. For this purpose, it is sufficient to prove, that between the values of t and u, which answer to m, any number whatever, and those which would answer to the number, $m + 1$, it is impossible to find any intermediate values, that will satisfy the equation $t^2 - Au^2 = 1$.

For example, let us make the values t''', u''', which result from the supposition of $m = 3$, and the values t^{iv}, u^{iv}, which result from the supposition of $m = 4$, and let us suppose it possible that there are other intermediate values, θ and υ, which would likewise satisfy the equation $t^2 - Au^2 = 1$.

Since we have $t'''^2 - Au'''^2 = 1$, $t^{iv\,2} - Au^{iv\,2} = 1$, and $\theta^2 - Av^2 = 1$,

we shall have $\theta^2 - t'''^2 = A(v^2 - u'''^2)$, and $t^{iv\,2} - \theta^2 = A(u^{iv\,2} - v^2)$; whence we see that, if $\theta > t'''$ and $< t^{iv}$, we shall also have $v > u'''$, and $< u^{iv}$. Farther, we shall also have these other values of t and u; namely, $t = \theta t^{iv} - Avu^{iv}$, $u = \theta u^{iv} - vt^{iv}$, which will satisfy the same equation, $t^2 - Au^2 = 1$; for, by substitution, we shall have

$$(\theta t^{iv} - Avu^{iv})^2 - A(vt^{iv} - \theta u^{iv})^2 = (\theta^2 - Av^2) \times (t^{iv\,2} - Au^{iv\,2}) = 1,$$

an identical equation, because $\theta^2 - Av^2 = 1$, and $t^{iv\,2} - Au^{iv\,2} = 1$ (*hyp.*). Now, these two last equations give

$$\theta - v\sqrt{A} = \frac{1}{\theta + v\sqrt{A}}, \text{ and } t^{iv} - u^{iv}\sqrt{A} = \frac{1}{t^{iv} + u^{iv}\sqrt{A}};$$

hence, substituting instead of θ, in the expression,

$$u = \theta u^{iv} - vt^{iv},$$

the quantity $v\sqrt{A} + \dfrac{1}{\theta + v\sqrt{A}}$; and, instead of t^{iv}, the quantity $u^{iv}\sqrt{A} + \dfrac{1}{t^{iv} + u^{iv}\sqrt{A}}$, we shall have

$$u = \frac{v^{iv}}{\theta + v\sqrt{A}} - \frac{v}{t^{iv} + u^{iv}\sqrt{A}}.$$

In the same manner, if we consider the quantity $t'''u^{iv} - u'''t^{iv}$, it may likewise, on account of $t'''^2 - Au'''^2 = 1$, be put into the form, $\dfrac{u^{iv}}{t''' + u'''\sqrt{A}} + \dfrac{u'''}{t^{iv} + u^{iv}\sqrt{A}}$.

Now, it is easy to perceive, that the preceding quantity must be less than this, because $\theta > t'''$, and $v > u'''$; therefore, we shall have a value of u, which will be less than the quantity $t'''u^{iv} - u'''t^{iv}$; but this quantity is equal to V; for

$$t''' = \frac{(T + V\sqrt{A})^3 + (T - V\sqrt{A})^3}{2},$$

$$t^{iv} = \frac{(T + V\sqrt{A})^4 + (T - V\sqrt{A})^4}{2},$$

$$u''' = \frac{(T + V\sqrt{A})^3 - (T - V\sqrt{A})^3}{2\sqrt{A}},$$

$$u^{iv} = \frac{(T + V\sqrt{A})^4 - (T - V\sqrt{A})^4}{2\sqrt{A}}, \text{ whence,}$$

$$t'''u^{iv} - t^{iv}u''' = \frac{(T - V\sqrt{A})^3 \times (T + V\sqrt{A})^4 - (T - V\sqrt{A})^4 \times (T + V\sqrt{A})^3}{2\sqrt{A}}.$$

Farther, $(T - V\sqrt{A})^3 \times (T + V\sqrt{A})^3 = (T^2 - AV^2)^3 = 1$, since $T^2 - AV^2 = 1$, by hypothesis; whence

$$(T - V\sqrt{A})^3 \times (T + V\sqrt{A})^4 = T + V\sqrt{A}, \text{ and}$$
$$(T - V\sqrt{A})^4 \times (T + V\sqrt{A})^3 = T - V\sqrt{A};$$

so that the value of $t'''u^{iv} - u'''t^{iv}$ will be reduced to

$$\frac{2V\sqrt{A}}{2\sqrt{A}} = V.$$

It would follow from this, that we should have a value of $u \angle V$, which is contrary to the hypothesis; since V is supposed to be the least possible value of u. There cannot, therefore, be any intermediate values of t and u between these, t''', t^{iv}, and u''', u^{iv}. And, as this reasoning may be applied, in general, to all the values of t and u, which would result from the above formulæ, by making m equal to any whole number, we may infer, that those formulæ actually contain all the possible values of t and u.

It is unnecessary to observe, that the values of t and u may be taken either positive, or negative; for this is evident from the equation itself, $t^2 - Au^2 = 1$.

Of the manner of finding all the possible Solutions, *in whole numbers, of indeterminate* Quadratic Equations *of two unknown quantities.*

76. The methods, which we have just explained, are sufficient for the complete solution of equations of the form $Ay^2 + B = x^2$; but we may have to resolve equations of a more complicated form: for which reason, it is proper to shew how such solutions are to be obtained.

Let there be proposed the equation

$$ar^2 + brs + cs^2 + dr + es + f = o,$$

where a, b, c, d, e, f, are given whole numbers, and r and s are two unknown numbers, that must likewise be integer.

I shall first have, by the common solution,

$$2ar + bs + d = \sqrt{((bs + d)^2 - 4a(cs^2 + es + d))},$$

whence we see, that the difficulty is reduced to making

$$(bs + d)^2 - 4a(cs^2 + es + d) \text{ a square.}$$

In order to simplify, let us suppose

$$b^2 - 4ac = \text{A},$$
$$bd - 2ae = g,$$
$$d^2 - 4af = h,$$

and $\text{A}s^2 + 2gs + h$ must be a square; representing this square by y^2, in order that we may have the equation,

$$\text{A}s^2 + 2gs + h = y^2,$$

and taking the value of y, we shall have

$$\text{A}s + g = \sqrt{(\text{A}y^2 + g^2 - \text{A}h)};$$

so that we shall only have to make a square of the formula, $\text{A}y^2 + g^2 - \text{A}h$.

If, therefore, we also make $g^2 - \text{A}h = \text{B}$, we shall have to render rational the radical quantity, $\sqrt{(\text{A}y^2 + \text{B})}$; which we may do by the known methods.

Let $\sqrt{(\text{A}y^2 + \text{B})} = x$, so that the equation to be resolved may be $\text{A}y^2 + \text{B} = x^2$; we shall then have $\text{A}s + g = \pm x$. Now, we already have $2ar + bs + d = \pm y$; so that, when we have found the values of x and y, we shall have those of r and s, by the two equations,

$$s = \frac{\pm x - g}{\text{A}},\ r = \frac{\pm y - d - bs}{2a}.$$

Now, as r and s must be whole numbers, it is evident, 1st, that x and y must be whole numbers likewise; 2dly, that $\pm x - g$ must be divisible by A, and $\pm y - d - bs$ by $2a$. Thus, after having found all the possible values of x and y, in whole numbers, it will still remain to find those among them that will render r and s whole numbers. If A is a negative number, or a positive square number, we have seen that the number of possible solutions in whole numbers is always limited; so that in these cases, we shall only have to try, successively, for x and y, the values found; and if we meet with none that give whole numbers for r and s, we conclude that the proposed equation admits of no solution of this kind.

There is no difficulty, therefore, but in the case of A being a positive number, not a square; in which we have seen, that the number of possible solutions in whole numbers may be infinite. In this case, as we should have an infinite number of values to try, we could never judge of the solvibility of the proposed equation, without having a rule, by which the trial may be reduced within certain limits. This we shall now investigate.

77. Since we have (Art. 65), $x = ny - \text{B}z$, and (Art. 72), $y = pt - (\text{B}q - np)u$, and $z = qt + (\text{C}p - nq)u$, it is easy to perceive, that the general expressions of r and s will take this form,

$$r = \frac{\alpha t + \beta u + \gamma}{\delta}, \quad s = \frac{\alpha' t + \beta' u + \gamma'}{\delta'},$$

$\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$, being known whole numbers, and t, u, being given by the formulæ of Art. 75, in which the exponent m may be any positive whole number; thus, the question is reduced to finding what value we must give to m, in order that the values of r and s may be whole numbers.

78. I observe, first, that it is always possible to find a value of u divisible by any given number, Δ; for, supposing $u = \Delta\omega$, the equation, $t^2 - Au^2 = 1$, will become $t^2 - A\Delta^2\omega^2 = 1$, which is always resolvible in whole numbers; and we shall find the least values of t and ω, by making the same calculation as before, only taking $A\Delta^2$, instead of A. Now, as these values also satisfy the equation $t^2 - Au^2 = 1$, they will necessarily be contained in the formulæ of Art. 75. Thus, we shall necessarily have a value of m, which will make the expression of u divisible by Δ.

Let us denote this value of m by μ, and I say that, if we make $m = 2\mu$, in the general expressions of t and u of the Article just quoted, the value of u will be divisible by Δ; and that of t being divided by Δ will give 1 for a remainder.

For, if we express by T' and V' the values of t and u, in which $m = \mu$, and by T'' and V'' those in which $m = 2\mu$, we shall have (Art. 75),

$$T' \pm V'\sqrt{A} = (T \pm V\sqrt{A})^{\mu}, \text{ and}$$
$$T'' \pm V''\sqrt{A} = (T \pm V\sqrt{A})^{2\mu}; \text{ therefore,}$$
$$(T' \pm V'\sqrt{A})^2 = (T'' \pm V''\sqrt{A}),$$

that is to say, comparing the rational part of the first side with the rational part of the second, and the irrational with the irrational,

$$T'' = T'^2 + AV'^2, \text{ and } V'' = 2T'V';$$

hence, since V' is divisible by Δ, V'' will be so likewise; and T'' will leave the same remainder that T'^2 would leave; but we have $T'^2 - AV'^2 = 1$ (*hyp.*), therefore $T'^2 - 1$ must be divisible by Δ, and even by Δ^2, since V'^2 is so already; wherefore, T'^2, and, consequently, T'' likewise, being divided by Δ, will leave the remainder 1.

Now, I say that the values of t and u, which answer to any exponent whatever, m, being divided by Δ, will leave the same remainders as the values of t and u, which would answer to the exponent $m + 2\mu$. For, denoting these last by θ and υ, we shall have,

$$t \pm u\sqrt{A} = (T \pm V\sqrt{A})^m, \text{ and}$$
$$\theta \pm \upsilon\sqrt{A} = (T \pm V\sqrt{A})^{m+2\mu}; \text{ wherefore,}$$
$$\theta \pm \upsilon\sqrt{A} = (t \pm u\sqrt{A}) \times (T \pm V\sqrt{A})^{2\mu},$$

but we have just before found

$$T'' \pm V''\sqrt{A} = (T \pm V\sqrt{A})^{2\mu};$$

whence we shall have

$$\theta \pm \upsilon\sqrt{A} = (t \pm u\sqrt{A}) \times (T'' \pm V''\sqrt{A});$$

then, by multiplying and comparing the rational parts, and the irrational parts, respectively, we derive

$$\theta = tT'' + AuV'', \ \upsilon = tV'' + uT''.$$

Now, V'' is divisible by Δ, and T'' leaves the remainder 1; therefore θ will leave the same remainder as t, and υ the same remainder as u.

In general, therefore, the remainders of the values of t and u, corresponding to the exponents $m + 2\mu$, $m + 4\mu$, $m + 6\mu$, &c. will be the same as those of the values, which correspond to any exponent whatever, m.

Hence, therefore, we may conclude, that, if we wish to have the remainders arising from the division of the terms t', t'', t''', &c. and u', u'', u''', &c. which correspond to $m = 1$, 2, 3, &c. by the number Δ, it will be sufficient to find these remainders as far as the terms $t^{2\mu}$ and $u^{2\mu}$ inclusive; for, after these terms, the same remainders will return in the same order; and so on to infinity.

With regard to the terms $t^{2\mu}$ and $u^{2\mu}$, at which we may stop, one of them $u^{2\mu}$ will be exactly divisible by Δ, and the other $t^{2\mu}$ will leave unity for a remainder; so that we shall only have to continue the divisions until we arrive at the remainders 1 and 0; we may then be sure that the succeeding terms will always give a repetition of the same remainders as those we have already found.

We might also find the exponent, 2μ, *a priori;* for we should only have to perform the calculation pointed out, Art. 71, in the first place, for the number A, and then for the number $A\Delta^2$; and if π be the rank of the term of the series P', P'', P''', &c. which, in the first case, will be $= 1$, and ϱ the rank of the term that will be $= 1$, in the second case, we shall only have to seek the smallest multiple of π

and ϱ, which being divided by π, will give the required value of μ.

Thus, for example, if we have $A = 6$, and $\Delta = 3$, we shall find for the radical $\sqrt{6}$, in the Table of Art. 41, $P^0 = 1$, $P' = -2$, $P'' = 1$; therefore, $\pi = 2$. Then we shall find, in the same Table, for the radical $\sqrt{(6 \times 9)} = \sqrt{54}$, $P^0 = 1$, $P' = -5$, $P'' = 9$, $P''' = -2$, $P^{iv} = 9$, $P^{v} = -5$, $P^{vi} = 1$; and hence $\varrho = 6$. Now, the least multiple of 2 and 6 is 6, which being divided by 2 gives the remainder 3; so that we shall here have $\mu = 3$, and $2\mu = 6$.

Therefore, in order to have, in this case, all the remainders of the division of the terms t', t'', t''', &c. and u', u'', u''', &c. by 3, it will be sufficient to find those of the six leading terms of each series; for the succeeding terms will always give a repetition of the same remainders: that is to say, the seventh terms will give the same remainders as the first, the eighth terms, the same as the second; and so on to infinity.

Lastly, the terms t^{μ} and u^{μ} may sometimes happen to have the same properties as the terms $t^{2\mu}$ and $u^{2\mu}$; that is to say, u^{μ} may be divisible by Δ, and t^{μ} may leave unity for a remainder. In such cases, we may stop at these very terms; for the remainders of the succeeding terms, $t^{\mu+1}$, $t^{\mu+2}$, &c. $u^{\mu+1}$, $u^{\mu+2}$, &c. will be the same as those of the terms t', t'', &c. u', u'', &c. and so of the others.

In general, we shall denote by M the least value of the exponent m, that will render $t - 1$, and u, divisible by Δ.

79. Let us now suppose that we have any expression whatever, composed of t and u, and given whole numbers, so that it may always represent whole numbers; and that it is required to find the values, which must be given to the exponent m, in order that this expression may become divisible by any given number whatever, Δ: we shall only have to make, successively, $m = 1, 2, 3$, &c. as far as M; and if none of these suppositions render the given expression divisible by Δ, we may conclude, with certainty, that it can never become so, whatever values we give to m.

But if in this manner we find one, or more values of m, which render the given expression divisible by Δ, then calling each of these values N, all the values of m that can possibly do the same, will be N, $N + M$, $N + 2M$, $N + 3M$, &c. and, in general, $N + \lambda M$; λ being any whole number whatever.

In the same manner, if we had another expression composed likewise of t, u, and given whole numbers, and, at the same time, divisible by any other given number whatever,

Δ', we should in like manner seek the corresponding values of M and N, which we shall here express by M′ and N′, and all the values of the exponent m, that will satisfy the condition proposed, will be contained in the formula $N' + \lambda' M'$; λ' being any whole number whatever. So that we shall only have to seek the values, which we must give to the whole numbers λ and λ', in order that we may have

$$N + \lambda M = N' + \lambda' M, \text{ or } M\lambda - M\lambda' = N' - N,$$

an equation resolvible by the method of Art. 42.

It is easy to apply what we have just now said to the case of Art. 77, where the given expressions have the form, $\alpha t + \beta u + \gamma$, $\alpha' t + \beta' u + \gamma'$, and the divisors are δ and δ'.

We must only recollect to take the numbers t and u, successively, positive and negative, in order to have all the cases that are possible.

80. *Scholium* If the equation proposed for resolution, in whole numbers, were of the form

$$ar^2 + 2brs + cs^2 = f,$$

we might immediately apply to it the method of Art. 65; for, 1st, it is evident that r and s could have no common divisor, unless the number f were at the same time divisible by the square of that divisor; so that we may always reduce the question to the case, in which r and s shall be prime to each other. 2dly, It is evident, also, that s and f could have no common divisor, unless that divisor were one also of the number a, supposing r prime to s; so that we may also reduce the question to the case, in which s and f shall be prime to each other. (See Art. 64).

Now, s being supposed prime to f, and to r, we may make $r = ns - fz$; and, in order that the equation may be resolvible in whole numbers, there must be a value of n, positive or negative, not greater than $\frac{f}{2}$, which may render the quantity $an^2 + 2bn + c$ divisible by f. This value being substituted for n, the whole equation will become divisible by f, and will be found reduced to the case of Art. 66, *et seq.*

It is easy to perceive, that the same method may serve for reducing every equation of the form,

$$ar^m + br^m s + cr^{m-1}s^2 +, \text{ \&c. } + ks^m = f,$$

a, b, c, &c. being given whole numbers, and r and s being two indeterminate numbers, which must likewise be integers, in another similar equation, but in which the whole known term is unity, and then we may apply to it the general method of Art. 2. See the *Scholium* of Art. 30.

81. *Example* 1. Let it be proposed to render rational the quantity, $\sqrt{(30 + 62s - 7s^2)}$, by taking only whole numbers for s.

We shall here have to resolve this equation,

$$30 + 62s - 7s^2 = y^2,$$

which being multiplied by 7, may be put into this form,

$$7 \times 30 + (31)^2 - (7s - 31)^2 = 7y^2,$$

or, making $7s - 31 = x$, and transposing,

$$x^2 = 1171 - 7y^2, \text{ or } x^2 + 7y^2 = 1171.$$

This equation now comes under the case of Art. 64; so that we shall have $A = -7$, and $B = 1171$, from which we instantly perceive, that y and B must be prime to each other, since this last number contains no square factor.

According to the method of Art. 65, we shall make $x = ny - 1171z$; and, in order that the equation may be resolvible, we must find for n a positive, or negative integer, not $> \frac{B}{2}$; that is, not > 580, such that $n^2 - A$, or $n^2 + 7$, may be divisible by B, or by 1171.

I find $n = \pm 321$, which gives $n^2 + 7 = 1171 \times 88$; so that I substitute, in the preceding equation, $\pm 321y - 1171z$, instead of x; by which means, the whole is now divisible by 1171, and when the division is performed, it becomes

$$88y^2 \mp 642yz + 1171z^2 = 1.$$

In order to resolve this equation, I shall employ the second method, explained in Art. 70, because it is in fact simpler and more convenient than the first. Now, as the coefficient of y^2 is less than that of z^2, we shall here have $D = 1171$, $D' = 88$, and $n = \pm 321$; wherefore retaining, for the sake of simplifying, the letter y, instead of θ, and putting y', instead of z, I shall make the following calculation, first supposing $n = 321$;

$$m = \tfrac{321}{88} = 4, \qquad n' = 321 - 4 \times 88 = -31,$$

$$D'' = \frac{31^2+7}{88} = 11, \; y = 4y' + y'',$$

$$m' = \frac{-31}{11} = -3, \; n'' = -31 + 3 \times 11 = 2,$$

$$D''' = \frac{4+7}{11} = 1, \quad y' = -3y'' + y''',$$

$$m'' = \tfrac{2}{1} = 2, \qquad n''' = 2 - 2 \times 1 = 0,$$

$$D^{iv} = \tfrac{7}{1} = 7, \qquad y'' = 2y''' + y^{iv}.$$

Since $n''' = 0$, and consequently $\angle \frac{D'''}{2}$, and $\angle \frac{D^{iv}}{2}$, we shall here stop, and make $D''' = M = 1$, $D^{iv} = L = 7$, $n''' = 0 = N$, and $y''' = \xi$, $y^{iv} = \psi$, because D''' is $\angle D^{iv}$.

I now observe, that A being $= -7$, and consequently negative, in order that the equation may be resolvible, we must have $M = 1$, as we have just now found; so that we may conclude, that the resolution is possible. We shall therefore suppose $\xi = y''' = 0$, $\psi = y^{iv} = \pm 1$; and we shall have, from the foregoing formulæ,

$$y'' = \pm 1, y' = \mp 3 = z, y = \mp 12 \pm 1 = \mp 11,$$

the doubtful signs being arbitrary. Therefore,

$$x = 321y - 1171z = \mp 18; \text{ and, consequently,}$$

$$s = \frac{x+31}{7} = \frac{31 \mp 18}{7} = \tfrac{13}{7}, \text{ or } = \tfrac{49}{7} = 7.$$

Now, as the value of s is required to be a whole number, we can only take $s = 7$.

It is remarkable, that the other value of s, namely $\frac{13}{7}$, although fractional, gives nevertheless a whole number for the value of the radical, $\sqrt{(30 + 62s - 7s^2)}$, and the same number, 11, which the value $s = 7$ gives; so that these two values of s will be the roots of the equation,

$$30 + 62s - 7s^2 = 121.$$

We have supposed $n = 321$. Now, we may likewise make $n = -321$; but it is easy to foresee, that the whole change that would result from it, in the preceding formulæ, would be a change of the sign of the values of m, m', m'', and of n', n'', by which means the values of y', and of y, will also have different signs; we should not therefore have any new result, since these values already have the doubtful sign $\pm$.

It will be the same in all other cases; so that we need not take the value of n, successively, positive and negative.

The value $s = 7$, which we have just found, results from the value of $n = \pm 321$: and we may find other values of s, if we have found other values of n having the requisite condition; but, as the divisor $B = 1171$, is a prime number, there can be no other values of n, with the same property, as we have elsewhere demonstrated *, whence we must conclude, that the number 7 is the only one that satisfies the question.

* Memoirs of Berlin, for the year 1767, page 194.

The preceding problem may be resolved more easily by mere trial; for when we have arrived at the equation, $x^2 = 1171 - 7y^2$, we shall only have to try, for y, all the whole numbers, whose squares multiplied by 7 do not exceed 1171; that is to say, all the numbers $\angle \sqrt{\frac{1171}{7}} \angle 13$.

It is the same with all the equations, in which A is a negative number; for when we are brought to the equation, $x^2 = B + Ay^2$, where making $A = -a$, and $x^2 = B - ay^2$, it is evident, that the satisfactory values of y, if there are any, can only be found among the numbers, $\angle \sqrt{\frac{B}{a}}$. So that I have not given particular methods for the case of A negative, only because these methods are intimately connected with those concerning the case of A positive, and because all these methods, being so nearly alike, reciprocally illustrate and confirm each other.

82. *Example* 2. Let us now give some examples for the case of A positive, and let it be proposed to find all the whole numbers, which we may take for y, in order that the radical quantity, $\sqrt{(13y^2 + 101)}$, may become rational.

Here, we shall have (Art. 64), $A = 13$, $B = 101$; and the equation to be resolved in integers will be, $x^2 - 13y^2 = 101$, in which, because 101 is not divisible by any square, y must be prime to 101.

We shall therefore make (Art. 65), $x = ny - 101z$, and $n^2 - 13$ must be divisible by 101, taking $n \angle \frac{101}{2} \angle 51$.

I find $n = 35$, which gives $n^2 = 1225$, and

$$n^2 - 13 = 1212 = 101 \times 12;$$

so that we may take $n = \pm 35$, and substituting $\pm 35y - 101z$, instead of x, we shall have an equation wholly divisible by 101, which, after the division, will be $12y^2 \mp 70yz + 101z^2 = 1$.

In order to resolve this equation, let us also employ the method of Article 70; let us make $D' = 12$, $D = 101$, $n = \pm 35$; but, instead of the letter θ, we shall preserve the letter y, and shall only change z into y', as in the preceding example.

1st. If $n = 35$, we shall make the following calculation:

$$m = \tfrac{35}{12} = 3, \qquad n' = 35 - 3 \times 12 = -1,$$

$$D'' = \frac{1 - 13}{12} = -1, \qquad y = 3y' + y'',$$

$$m' = \frac{-1}{-1} = 1, \qquad n'' = -1 + 1 = 0,$$

$$D''' = \frac{-13}{-1} = 13, \quad y = y'' + y'''.$$

As $n'' = 0$, and, consequently, $< \frac{D''}{2}$, and $< \frac{D'''}{2}$, we shall stop here, and shall have the transformed equation,

$$\overset{\prime\prime\prime\,\prime\prime}{D}y^2 - 2n''y''y''' + \overset{\prime\prime\,\prime\prime\prime}{D}y^2 = 1, \text{ or } 13\overset{\prime\prime}{y}{}^2 - \overset{\prime\prime\prime}{y}{}^2 = 1;$$

which being reduced to the form, $\overset{\prime\prime\prime}{y}{}^2 - 13y'' = 1$, will admit of the method of Art. 71; and, as $A = 13$ is < 100, we may make use of the Table, Art. 41.

Thus, we shall only have to see, whether, in the upper series of numbers belonging to $\sqrt{13}$, there be found the number 1 in an even place; for, in order that the preceding equation may be resolvible, we must find in the series P^0, P', P'', &c. a term $= -1$; but we have $P^0 = 1$, $-P' = 4$, $P'' = 3$, &c. wherefore, &c. Now, in the series 1, 4, 3, 3, 4, 1, &c. we find 1 in the sixth place; so that $P^v = -1$; and hence we shall have a solution of the given equation, by taking $y''' = p^v$, and $y'' = q^v$, the numbers p^v, q^v, being calculated according to the formulæ of Article 25, giving to μ, μ', μ'', &c. the values 3, 1, 1, 1, 1, 6, &c. which form the lower series of numbers belonging to $\sqrt{13}$ in the same Table.

We shall therefore have

$$\begin{array}{lll} p^0 = 1 & p^{iv} = p''' + p'' = 11 & q'' = 1 \\ p' = 3 & p^{v} = p^{iv} + p''' = 18 & q''' = q'' + q' = 2 \\ p'' = p' + p^0 = 4 & q^0 = 0 & q^{iv} = q''' + q'' = 3 \\ p''' = p'' + p' = 7 & q' = 1 & q^{v} = q^{iv} + q''' = 5. \end{array}$$

So that $y''' = 18$, and $y'' = 5$; therefore,

$$y' = y'' + y''' = 23, \text{ and } y = 3y' + y'' = 74.$$

We have supposed $n = 35$; but we may also take $n = -35$.

Let therefore $n = -35$, we shall make

$$m = \frac{-35}{12} = -3, \qquad n' = -35 + 3 \times 12 = 1,$$

$$D'' = \frac{1 - 13}{12} = -1, \qquad y = -3y' + y'',$$

$$m' = \frac{1}{-1} = -1, \qquad n'' = 1 - 1 = 0,$$

$$D''' = \frac{-13}{-1} = 13, \qquad y' = -y'' + y'''.$$

Thus, we have the same values of D'', D''', and n'', as before; so that the transformed equation in y'', and y''', will likewise be the same.

We shall, therefore, have also $y''' = 18$, and $y'' = 5$; wherefore, $y' = -y'' + y''' = 13$, and $y = -3y' + y'' = -34$.

So that we have found two values of y, with the corresponding values of y', or z; and these values result from the supposition of $n = \mp 35$. Now, as we cannot find any other value of n, with the requisite conditions, it follows that the preceding values will be the only *primitive* values that we can have; but we may then find from them an infinite number of *derivative* values by the method of Art. 72.

Taking, therefore, these values of y and z for p and q, we shall have, in general, by the same Article,

$$y = 74t - (101 \times 23 - 35 \times 74)u = 74t + 267u$$
$$z = 23t + (12 \times 74 - 35 \times 23)u = 23t + 83u; \text{ or}$$

$$y = -34t - (101 \times 13 - 35 \times 34)u = -34t - 123u$$
$$z = 13t + (-12 \times 34 + 35 \times 13)u = 13t + 47u;$$

and we shall only have farther to deduce the values of t and u from the equation, $t^2 - 13u^2 = 1$. Now, all these values may be found already calculated in the Table at the end of Chap. VII. of the preceding Treatise: we shall therefore immediately have $t = 649$, and $u = 180$; so that taking these values for T and V, in the formulæ of Art. 75, we shall have, in general,

$$t = \frac{(649 + 180\sqrt{13})^m + (649 - 180\sqrt{13})^m}{2},$$

$$u = \frac{(649 + 180\sqrt{13})^m - (649 - 180\sqrt{13})^m}{2\sqrt{13}};$$

where we may give to m whatever value we choose, provided we take only positive whole numbers.

Now, as the values of t and u may be taken both positive and negative, the values of y, which satisfy the question, will all be contained in these two formulæ,

$$y = \pm 74t \pm 267u,$$
$$\text{and } y = \pm 34t \pm 123u,$$

the doubtful signa being arbitrary.

If we make $m = 0$, we shall have $t = 1$, and $u = 0$; wherefore, $y = \pm 74$, or $= \pm 34$; and this last value is the least that will resolve the problem.

I have already resolved this problem in the Memoirs of Berlin, for the year 1768, page 243; but as I have there employed a method somewhat different from the foregoing, and fundamentally the same as the *first* method of Art. 66, it was thought proper to repeat it here, in order that the comparison of the results, which are the same by both methods, might serve, if necessary, as a confirmation of them.

83. *Example* 3. Let it be proposed to find whole numbers, which being taken for y, may render rational the quantity, $\sqrt{(79y^2 + 101)}$.

Here we shall have to resolve, in integers, the equation,

$$x^2 - 79y^2 = 101,$$

in which y will be prime to 101, since this number does not contain any square factor.

If we therefore suppose $x = ny - 101z$, $n^2 - 79$ must be divisible by 101, taking $n \angle \frac{101}{2} \angle 51$; we find $n = 33$, which gives $n^2 - 13 = 1010 = 101 \times 10$; thus, we may take $n = \pm 33$, and these will be the only values that have the condition required.

Substituting, therefore, $\pm 33y - 101z$ instead of x, and then dividing the whole equation by 101, we shall have it transformed into $10y^2 \mp 66yz + 101z^2 = 1$. Let us, therefore, make $D' = 10$, $D = 101$, $n = \pm 33$, and first taking n positive, we shall work as in the preceding example; thus, we shall have $m = \frac{33}{10} = 3$, $n' = 33 - 3 \times 10 = 3$,

$$D'' = \frac{9 - 79}{10} = -7, y = 3y' + y''.$$

Now, as $n' = 3$ is already $\angle \frac{D'}{2}$, and $\angle \frac{D''}{2}$, it is not necessary to proceed any farther: so that the equation will be transformed to this,

$$-7y'^2 - 6y'y'' + 10y''^2 = 1,$$

which being multiplied by -7, may be put into this form,

$$(7y' + 3y'')^2 - 79y''^2 = -7.$$

Since, therefore, 7 is $\angle \sqrt{79}$, if this equation be resolvible, the number 7 must be found among the terms of the upper series of numbers answering to $\sqrt{79}$ in the Table (Art. 41), and also hold an even place there, since it has the sign $-$.

But the series in question contains only the numbers 1, 15, 2, always repeated; therefore, we may immediately conclude, that the last equation is not resolvible; and, consequently, the equation proposed is not, at least when we take $n = 33$.

It only remains, therefore, to try the other value of $n = -33$, which will give

$$m = \frac{-33}{10} = -3,\ n' = -33 + 3 \times 10 = -3,$$

$$D'' = \frac{9 - 97}{10} = -7,\ y = -3y' + y'';$$

so that we shall have the equation transformed into

$$-7y'^2 + 6y'y'' + 10y''^2 = 1,$$

which may be reduced to the form,

$$(7y' - 3y'')^2 - 79y''^2 = -7,$$

which is similar to the preceding. Whence I conclude, that the given equation absolutely admits of no solution in whole numbers.

84. *Scholium.* M. Euler, in an excellent Memoir printed in Vol. IX. of the New Commentaries of Petersburg, finds by induction this rule for determining the resolvibility of every equation of the form $x^2 - Ay^2 = B$, when B is a prime number: it is, that the equation must be possible, whenever B shall have the form $4An + r^2$, or $4An + r^2 - A$; but the foregoing example shews this rule to be defective; for 101 is a prime number, of the form $4An + r^2 - A$, making $A = 79$, $n = -4$, and $r = 38$; yet the equation, $x^2 - 79y^2 = 101$, admits of no solution in whole numbers.

If the foregoing rule were true, it would follow, that, if the equation $x^2 - Ay^2 = B$ were possible, when B has any value whatever, b, it would be so likewise, when we have taken $B = 4An + b$, provided B were a prime number. We might limit this last rule, by requiring b to be also a prime number; but even with this limitation the preceding example would shew it to be false; for we have $101 = 4An + b$, by taking $A = 79$, $n = -2$, and $b = 733$; now, 733 is a prime number, of the form $x^2 - 79y^2$, making $x = 38$, and $y = 3$; yet 101 is not of the same form, $x^2 - 79y^2$.

CHAP. VIII.

Remarks *on* Equations *of the form* $p^2 = Aq^2 + 1$, *and on the common method of resolving them in* Whole Numbers.

85. The method of Chap. VII. of the preceding Treatise, for resolving equations of this kind, is the same that Wallis gives in his Algebra (Chap. 98), and ascribes to Lord Brouncker. We find it, also, in the Algebra of Ozanam, who gives the honor of it to M. de Fermat. Whoever was the inventor of this method, it is at least certain, that M. de Fermat was the author of the problem which is the subject of it. He had proposed it as a challenge to all the English mathematicians, as we learn from the *Commercium Epistolicum* of Wallis; which led Lord Brouncker to the invention of the method in question. But it does not appear that this author was fully apprised of the importance of the problem which he resolved. We find nothing on the subject, even in the writings of Fermat, which we possess, nor in any of the works of the last century, which treat of the Indeterminate Analysis. It is natural to suppose that Fermat, who was particularly engaged in the theory of integer numbers, concerning which he has left us some very excellent theorems, had been led to the problem in question by his researches on the general resolution of equations of the form,

$$x^2 = Ay^2 + B,$$

to which all quadratic equations of two unknown quantities are reduced. However, we are indebted to Euler alone for the remark, that this problem is necessary for finding all the possible solutions of such equations *.

The method which I have pursued for demonstrating this proposition, is somewhat different from that of M. Euler; but it is, if I am not mistaken, more direct and more general. For, on the one hand, the method of M. Euler naturally leads to fractional expressions, where it is required to avoid them; and, on the other, it does not appear very evidently, that the suppositions, which are made in order to remove the fractions, are the only ones that could have taken place. Indeed, we have elsewhere shewn, that the finding of one solution of the equation $x^2 = Ay^2 + B$, is not always sufficient to enable us to

* See Chap. VI. of the preceding Treatise, Vol. VI. of the Ancient Commentaries of Petersburg, and Vol. IX. of the New.

deduce others from it, by means of the equation $p^2 = Aq^2 + 1$; and that, frequently, at least when B is not a prime number, there may be values of x and y, which cannot be contained in the general expressions of M. Euler *.

With regard to the manner of resolving equations of the form $p^2 = Aq^2 + 1$, I think that of Chap. VII., however ingenious it may be, is still far from being perfect. For, in the first place, it does not shew that every equation of this kind is always resolvible in whole numbers, when a is a positive number not a square. Secondly, it is not demonstrated, that it must always lead to the solution sought for. Wallis, indeed, has professed to prove the former of these propositions; but his demonstration, if I may presume to say so, is a mere *petitio principii*. (See Chap. 99). Mine, I believe, is the first rigid demonstration that has appeared; it is in the *Melanges de Turin*, Vol. IV.; but it is very long, and very indirect: that of Art. 37, is founded on the true principles of the subject, and leaves, I think, nothing to wish for. It enables us, also, to appreciate that of Chap. VII., and to perceive the inconveniences into which it might lead, if followed without precaution. This is what we shall now discuss.

86. From what we have demonstrated, Chap. II., it follows, that the values of p and q, which satisfy the equation $p^2 - Aq^2 = 1$, can only be the terms of some one of the *principal* fractions derived from the continued fraction, which would express the value of $\sqrt{A}$; so that supposing this continued fraction to be represented thus,

$$\mu + \frac{1}{\mu'} + \frac{1}{\mu''} + \frac{1}{\mu'''} + \text{, \&c.}$$

we must have,

$$\frac{p}{q} = \mu + \frac{1}{\mu'} + \frac{1}{\mu''} + \text{, \&c.}$$

$$+ \frac{1}{\mu^{\varrho}};$$

μ^{ϱ} being any term whatever of the infinite series μ', μ'', &c. the rank of which, ϱ, can only be determined *a posteriori*.

We must observe that, in this continued fraction, the numbers μ, μ', μ'', &c. must all be positive, although we have

* See Art. 45 of my Memoir on Indeterminate Problems, in the Memoirs of Berlin. 1767.

seen (Art. 3) that, in general, in continued fractions, we may render the denominators positive or negative, according as we take the approximate values less, or greater, than the real ones; but the method of Problem I. (Art. 23, *et seq.*), absolutely requires the approximate values μ, μ', μ'', &c. to be all taken less than the real ones.

87. Now, since the fraction $\frac{p}{q}$ is equal to a continued fraction, whose terms are μ, μ', μ'', &c. it is evident, from Art. 4, that μ will be the quotient of p divided by q, that μ' will be that of q divided by the remainder, μ'', that of this remainder divided by the second remainder, and so on; so that calling r, s, t, &c. the remainders in question, we shall have, from the nature of division, $p = \mu q + r$, $q = \mu' r + s$, $r = \mu'' s + t$, &c. where the last remainder must be $= 0$, and the one before the last $= 1$, because p and q are numbers prime to each other. Thus, μ will be the approximate integer value of $\frac{p}{q}$, μ' that of $\frac{q}{r}$, μ'' that of $\frac{r}{s}$, &c. these values being all taken less than the real ones, except the last μ^{ϱ}, which will be exactly equal to the corresponding fraction; because the following remainder is supposed to be nothing.

Now, as the numbers μ, μ', μ'', &c. μ^{ϱ}, are the same for the continued fraction, which expresses the value of $\frac{p}{q}$, and for that which expresses the value of $\sqrt{A}$, we may take, as far as the term m^{ϱ}, $\frac{p}{q} = \sqrt{A}$, that is to say, $p^2 - Aq^2 = 0$. Thus, we shall first seek the approximate, deficient value of $\frac{p}{q}$; that is to say, of $\sqrt{A}$, and that will be the value of μ; then we shall substitute in $p^2 - Aq^2 = 0$, instead of p, its value $\mu q + r$, which will give

$$(\mu^2 - A)q^2 + 2\mu qr + r^2 = 0,$$

and we shall again seek the approximate, deficient value of $\frac{q}{r}$; that is, of the positive root of the equation,

$$(\mu^2 - A) \times (\frac{q}{r})^2 + 2\mu\frac{q}{r} + 1 = 0,$$

and we shall have the value of μ'.

Still continuing to substitute $\mu' r + s$, instead of q, in the transformed equation $(\mu^2 - A)q^2 + 2\mu qr + r^2 = 0$; we shall have an equation, whose root will be $\frac{r}{s}$; then taking the approximate, deficient value of this root, we shall have the value of μ''. Here again we shall substitute $\mu''r + s$, instead of r, &c.

Let us now suppose, for example, that t is the last remainder, which must be nothing, then s will be the last but one, which must be $= 1$; wherefore, if the formula $p^2 - Aq^2$, when transformed into terms of s and t, is $Ps^2 + Qst + Rt^2$, by making $t = 0$, and $s = 1$, it must become $= 1$, in order that the given equation, $p^2 - Aq^2 = 1$, may take place; and therefore P must be $= 1$. Thus, we shall only have to continue the above operations and transformations, until we arrive at a transformed formula, in which the coefficient of the first term is equal to unity; then, in that formula, we shall make the first of the two indeterminates, as r, equal to 1, and the second, as s, equal to 0; and, by going back, we shall have the corresponding values of p and q.

We might likewise work with the equation $p^2 - Aq^2 = 1$ itself, only taking care to abstract from the term 1, which is known, and consequently from the other known terms, likewise, that may result from this, in the determination of the approximate values μ, μ', μ'', &c. of $\frac{p}{q}$, $\frac{q}{r}$, $\frac{r}{s}$, &c. In this case, we shall try at each new transformation, whether the indeterminate equation can subsist, by making one of the two indeterminates $= 1$, and the other $= 0$; when we have arrived at such a transformation, the operation will be finished; and we shall have only to go back through the several steps, in order to have the required values of p and q.

Here, therefore, we are brought to the method of Chap. VII. To examine this method in itself, and independently of the principles from which we have just deduced it, it must appear indifferent whether we take the approximate values of μ, μ', μ'', &c. less, or greater than the real values; since, in whatever way we take these values, those of r, s, t, &c. must go on decreasing to 0. (Art. 6.)

Wallis also expressly says, that we may employ the limits for μ, μ', μ'', &c. either in *plus*, or in *minus*, at pleasure; and he even proposes this, as the proper means often of abridging the calculation. This is likewise remarked by Euler, Art. 102, *et seq.* of the chapter just now quoted. However, the ollowing example will shew, that by setting about it in this way, we may run the risk of never arriving at the solution of the equation proposed.

Let us take the example of Art. 101 of that chapter, in which it is required to resolve an equation of this form, $p^2 = 6q^2 + 1$, or $p^2 - 6q^2 = 1$. We have $p = \sqrt{(6q^2 + 1)}$; and, neglecting the constant term 1, $p = q\sqrt{6}$; wherefore $\frac{p}{q} = \sqrt{6} > 2, < 3$. Let us take the limit in *minus*, and make $\mu = 2$, and then $p = 2q + r$; substituting this value, therefore, we shall have $-2q^2 + 4qr + r^2 = 1$; whence, $q = \frac{2r + \sqrt{(6r^2 - 2)}}{2}$; or, rejecting the constant term -2, $q = \frac{2r + r\sqrt{6}}{2}$; whence, $\frac{q}{r} = \frac{2 + \sqrt{6}}{2} > 2, < 3$. Let us again take the limit in *minus*, and make $q = 2r + s$; the last equation will then become $r^2 - 4rs - 2s^2 = 1$; where we at once perceive, that we may suppose $s = 0$, and $r = 1$; so that we shall have $q = 2$, and $p = 5$.

Let us now resume the former transformation,

$$-2q^2 + 4qr + r^2 = 1,$$

where we found $\frac{q}{r} > 2$, and < 3; and, instead of taking the limit in *minus*, let us take it in *plus*, that is to say, let us suppose $q = 3r + s$; or, since s must then be a negative quantity, $q = 3r - s$, we shall then have the following transformation, $-5r^2 + 8rs - 2s^2 = 1$, which will give $r = \frac{4s + \sqrt{(6s^2 - 5)}}{5}$; wherefore, neglecting the constant term 5, $r = \frac{4s + s\sqrt{6}}{5}$, and $\frac{r}{s} = \frac{4 + \sqrt{6}}{5} > 1$, and < 2.

Let us again take the limit in *plus*, and make $r = 2s - t$, we shall now have $-6s^2 + 12st - 5t^2 = 1$; therefore $s = \frac{6t + \sqrt{(6t^2 - 6)}}{6}$; so that, rejecting the term -6,

$$s = \frac{6t + t\sqrt{6}}{6}, \text{ and } \frac{s}{t} = 1 + \frac{\sqrt{6}}{6} > 1, < 2.$$

Let us continue taking the limits in *plus*, and make $s = 2t - u$, we shall next have $-5t^2 + 12tu - 6u^2 = 1$; wherefore,

$$t = \frac{6u + \sqrt{(6u^2 - 5)}}{5}; \text{ and } \frac{t}{u} = \frac{6 + \sqrt{6}}{5} > 1, < 2.$$

Let us, therefore, in the same manner, make $t = 2u - x$, and we shall have $-2u^2 + 8ux - 5x^2 = 1$; wherefore, &c.

Continuing thus to take the limits always in *plus*, we shall never come to a transformed equation, in which the coefficient of the first term is equal to unity, which is necessary to our finding a solution of the equation proposed.

The same thing must happen, whenever we take the first limit in *minus*, and all the succeeding in *plus*; the reason of this might be given *a priori*; but as the reader can easily deduce it from the principles of our theory, I shall not dwell on it. It is sufficient for the present to have shewn the necessity of investigating these problems more fully, and more rigorously, than has hitherto been done.

CHAP. IX.

Of the manner of finding Algebraic Functions *of all* Degrees, *which, when multiplied together, may always produce* Similar Functions.

[APPENDIX TO CHAP. XI. AND XII.]

88. I believe I had, at the same time with M. Euler, the idea of employing the irrational, and even imaginary factors of formulæ of the second degree, in finding the conditions, which render those formulæ equal to squares, or to any powers. On this subject, I read a Memoir to the academy in 1768, which has not been printed; but of which I have given a summary at the end of my researches *on Indeterminate Problems*, which are to be found in the volume for the year 1767, printed in 1769, before even the German translation of M. Euler's Algebra.

In the place now quoted, I have shewn how the same method may be extended to formulæ of higher dimensions than the second; and I have by these means given the solution of some equations, which it would perhaps have been extremely difficult to resolve in any other way. It is here intended to generalise this method still more, as it seems to deserve the attention of mathematicians, from its novelty and singularity.

89. Let α and β be the two roots of the quadratic equation,

$$s^2 - as + b = 0,$$

and let us consider the product of these two factors,

$$(x + \alpha y) \times (x + \beta y),$$

which must be a real product; being equal to

$$x^2 + (\alpha + \beta)xy + \alpha\beta y^2.$$

Now, we have $\alpha + \beta = a$, and $\alpha\beta = b$, from the nature of the equation, $s^2 - as + b = 0$; therefore we shall have this formula of the second degree,

$$x^2 + axy + by^2,$$

which is composed of the two factors,

$$x + \alpha y, \text{ and } x + \beta y.$$

Now, it is evident, that if we have a similar formula,

$$x'^2 + ax'y' + by'^2,$$

and wish to multiply them, the one by the other, we have only to multiply together the two factors $x + \alpha y$, $x' + \alpha y'$, and also the other two factors $x + \beta y$, $x' + \beta y'$, and then the two products, the one by the other. Now, the product of $x + \alpha y$ by $x' + \alpha y'$ is, $x'^2 + \alpha(xy' + yx') + \alpha^2 yy'$; but since α is one of the roots of the equation, $s^2 - as + b = 0$, we shall have $\alpha^2 - a\alpha + b = 0$; whence, $\alpha^2 = a\alpha - b$; and, substituting this value of α^2, in the preceding formula, it will become, $xx' - byy' + \alpha(xy' + yx' + ayy')$; so that, in order to simplify, making

$$X = xx' - byy'$$
$$Y = xy' + yx' + ayy',$$

the product of the two factors $x + \alpha y$, $x' + \alpha y'$, will be $X + \alpha Y$; and, consequently, of the same form as each of them. In the same manner, we shall find, that the product of the two other factors, $x + \beta y$, and $x' + \beta y'$, will be $X + \beta Y$; so that the whole product will be $(X + \alpha Y) \times (X + \beta Y)$; that is, $X^2 + aXY + bY^2$, which is the product of the two similar formulæ,

$$x^2 + axy + by^2, \text{ and } x'^2 + ax'y' + by'^2.$$

If we wished to have the product of these three similar formulæ,

$$x^2 + axy + by^2,\ x'^2 + ax'y' + by'^2,\ x''^2 + ax''y'' + by''^2,$$

we should only have to find that of the formula, $X^2 + aXY + bY^2$, by the last, $x''^2 + ax''y'' + by''^2$; and it is evident, from the foregoing formulæ, that by making

$$X' = XY'' - bYy'',$$
$$Y' = Xy'' + Yx'' + aYy'',$$

the product sought would be

$$X'^2 + aX'Y' + bY'^2.$$

In the same manner, we might find the product of four, or of a still greater number of formulæ similar to this,

$$x^2 + axy + by^2,$$

and these products likewise will always have the same form.

90. If we make $x' = x$, and $y' = y$, we shall have

$$X = x^2 - by^2, \ Y = 2xy + ay^2;$$

and, consequently,

$$(x^2 + axy + by^2)^2 = X^2 + aXY + bY^2.$$

Therefore, if we wish to find rational values of X and Y, such, that the formula $X^2 + aXY + bY^2$ may become a square, we shall only have to give the preceding values to X and Y, and we shall have, for the root of the square, the formula,

$$x^2 + axy + by^2;$$

x and y being two indeterminate numbers.

If we farther make $x'' = x' = x$, and $y'' = y' = y$, we shall have $X' = Xx - bYy$, $Y' = Xy + Yx + aYy$; that is, by substituting the preceding values of X and Y,

$$X' = x^3 - 3bxy^2 + aby^3,$$
$$Y' = 3x^2y + 3axy^2 + (a^2 - b)y^3;$$

wherefore,

$$(x^2 + axy + by^2)^3 = X'^2 + aX'Y' + bY'^2.$$

Thus, if we proposed to find rational values of X' and Y', such, that the formula $X'^2 + aX'Y' + bY'^2$ might become a cube, we should only have to give to X' and Y' the foregoing values, by which means we should have a cube, whose root would be $x^2 + axy + by^2$; x and y being both indeterminate.

In a similar manner, we may resolve questions, in which it is required to produce fourth, fifth powers, &c. but we may, once for all, find general formulæ for any power whatever, m, without passing through the lower powers.

Let it be proposed, therefore, to find rational values of X and Y, such, that the formula, $X^2 + aXY + bY^2$, may become a power, m; that is, let it be required to solve the equation,

$$X^2 + aXY + bY^2 = Z^m.$$

As the quantity $X^2 + aXY + bY^2$ is formed from the product of the two factors, $X + \alpha Y$, and $X + \beta Y$, in order that this quantity may become a power of the dimension m, each of its factors must likewise become a similar power.

Let us, therefore, first make

$$X + \alpha Y = (x + \alpha y)^m,$$

and, expressing this power by Newton's theorem, we shall have

$$x^m + mx^{m-1}y\alpha + \frac{m(m-1)}{2}x^{m-2}y^2\alpha^2$$
$$+ \frac{m(m-1)\times(m-2)}{2\times3}x^{m-3}y^3\alpha^3 +, \text{ \&c.}$$

Now, since α is one of the roots of the equation, $s^2 - as + b = 0$, we shall also have $\alpha^2 - a\alpha + b = 0$; wherefore, $\alpha^2 = a\alpha - b$, $\alpha^3 = a\alpha^2 - b\alpha = (a^2 - b)\alpha - ab$, $\alpha^4 = (a^2 - b)\alpha^2 - ab\alpha = (a^3 - 2ab)\alpha - a^2b + b^2$; and so on. Thus, we shall only have to substitute these values in the preceding formula, and then we shall find it to be compounded of two parts, the one wholly rational, which we shall compare to X, and the other wholly multiplied by the root α, which we shall compare to αY.

If, in order to simplify, we make

$$\begin{array}{ll} A' = 1 & B' = 0 \\ A'' = a & B'' = b \\ A''' = aA'' - bA' & B''' = aB'' - bB' \\ A^{iv} = aA''' - bA'' & B^{iv} = aB''' - bB'' \\ A^{v} = aA^{iv} - bA''', & B^{v} = aB^{iv} - bB''', \end{array}$$

&c. &c. &c. we shall have,

$$\begin{aligned} \alpha &= A'\alpha - B' \\ \alpha^2 &= A''\alpha - B'' \\ \alpha^3 &= A'''\alpha - B''' \\ \alpha^4 &= A^{iv}\alpha - B^{iv}, \text{ \&c.} \end{aligned}$$

Wherefore, substituting these values, and comparing, we shall have

$$X = x^m - mx^{m-1}yB' - \frac{m(m-1)}{2}x^{m-2}y^2B''$$
$$- \frac{m(m-1)\times(m-2)}{2\times3}x^{m-3}y^3B''' -, \text{ \&c.}$$

$$Y = mx^{m-1}yA' + \frac{m(m-1)}{2}x^{m-2}y''A''$$
$$+ \frac{m(m-1)\times(m-2)}{2\times3}x^{m-3}y^3A''' +, \text{ \&c.}$$

Now, as the root α does not enter into the expressions of

X and Y, it is evident, that, having $X + \alpha Y = (x + \alpha y)^m$, we shall likewise have, $X + \beta Y = (x + \beta y)^m$; wherefore, multiplying these two equations together, we shall have,

$$X^2 + aXY + bY^2 = (x^2 + axy + by^2)^m;$$

and, consequently, $Z = x^2 + axy + by^2$. The problem, therefore, is solved.

If a were $= 0$, the foregoing formulæ would become simpler; for we should have $A' = 1$, $A'' = 0$, $A''' = -b$, $A^{iv} = 0$, $A^{v} = b^2$, $A^{vi} = 0$, $A^{vii} = -b^3$, &c. and, likewise, $B' = 0$, $B'' = b$, $B''' = 0$, $B^{iv} = -b^2$, $B^{v} = 0$, $B^{vi} = b^3$, &c.

Therefore, $X = x^m - \frac{m(m-1)}{2}x^{m-2}y^2b + \frac{m(m-1)\times(m-2)\times(m-3)}{2\times3\times4}x^{m-4}y^4b^2 -$, &c.

$Y = mx^{m-1}y + \frac{m(m-1)\times(m-2)}{2\times3}x^{m-3}y^3b + \frac{m(m-1)\times(m-2)\times(m-3)\times(m-4)}{2\times3\times4\times5}x^{m-5}y^3b^2 +$, &c.

And these values will satisfy the equation,

$$X^2 + bY^2 = (x^2 + by^2)^m.$$

91. Let us now proceed to the formulæ of three dimensions; in order to which, we shall denote by α, β, γ, the three roots of the cubic equation, $s^3 - as^2 + bs - c = 0$, and we shall then consider the product of these three factors,

$$(x + \alpha y + \alpha^2z) \times (x + \beta y + \beta^2z) \times (x + \gamma y + \gamma^2z),$$

which must be rational, as we shall perceive. The multiplication being performed, we shall have the following product,

$$x^3 + (\alpha+\beta+\gamma)x^2y + (\alpha^2+\beta^2+\gamma^2)x^2z + (\alpha\beta+\alpha\gamma+\beta\gamma)xy^2 + (\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta)xyz + (\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)xz^2 + \alpha\beta\gamma y^3 + (\alpha^2\beta\gamma+\beta^2\alpha\gamma+\gamma^2\alpha\beta)y^2z + (\alpha^2\beta^2\gamma+\alpha^2\gamma^2\beta+\beta^2\gamma^2\alpha)yz^2 + \alpha^2\beta^2\gamma^2z^3.$$

Now, from the nature of equations, we have

$$\alpha + \beta + \gamma = a,\ \alpha\beta + \alpha\gamma + \beta\gamma = b,\ \alpha\beta\gamma = c.$$

Farther, we shall find

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) = a^2 - 2b$, $\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta = (\alpha+\beta+\gamma)\times(\alpha\beta+\alpha\gamma+\beta\gamma) - 3\alpha\beta\gamma = ab - 3c$; and $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha+\beta+\gamma)\alpha\beta\gamma = b^2 - 2ac$; also, $\alpha^2\beta\gamma+\beta^2\alpha\gamma+\gamma^2\alpha\beta = (\alpha+\beta+\gamma)\alpha\beta\gamma = ac$, and $\alpha^2\beta^2\gamma+\alpha^2\gamma^2\beta+\beta^2\gamma^2\alpha = (\alpha\beta+\alpha\gamma+\beta\gamma)\alpha\beta\gamma = bc$.

Therefore, making these substitutions, the product in question will be

$$x^3 + ax^2y + (a^2-2b)x^2z + bxy^2 + (ab-3c)xyz + (b^2-2ac)xz^2$$
$$+ cy^3 + acy^2z + bcyz^2 + c^2z^3.$$

And this formula will have the property, that if we multiply together as many similar formulæ as we choose, the product will always be a similar formula.

Let us suppose that the product of the foregoing formula by the following was required, namely,

$$x'^3 + ax'^2y' + (a^2 - 2b)x'^2z' + bx'y'^2 + (ab - 3c)x'y'z'$$
$$+ (b^2 - 2ac)x'z'^2 + cy'^3 + acy'^2z' + bcy'z'^2 + c^2z'^3;$$

it is evident, that we have only to seek the product of these six factors,

$$x + \alpha y + \alpha^2 z, \quad x + \beta y + \beta^2 z, \quad x + \gamma y + \gamma^2 z,$$
$$x' + \alpha y' + \alpha^2 z', \quad x' + \beta y' + \beta^2 z', \quad x' + \gamma y' + \gamma^2 z';$$

if we first multiply $x + \alpha y + \alpha^2 z$, by $x' + \alpha y' + \alpha^2 z'$, we shall have this partial product,

$$xx' + \alpha(xy' + yx') + \alpha^2(xz' + zx' + yy') + \alpha^3(yz' + zy') + \alpha^4 zz';$$

now, α being one of the roots of the equation,

$$s^3 - as^2 + bs - c = 0,$$

we shall have $\alpha^3 - a\alpha^2 + b\alpha - c = 0$; consequently,

$$\alpha^3 = a\alpha^2 - b\alpha + c; \text{ whence,}$$

$$\alpha^4 = a\alpha^3 - b\alpha^2 + c\alpha = (a^2 - b)\alpha^2 - (ab - c)\alpha + ac;$$

so that substituting these values, and, in order to abridge, making

$$\text{X} = xx' - c(yz' + zy') + aczz',$$
$$\text{Y} = xy' + yx' - b(yz' + zy') - (ab - c)zz',$$
$$\text{Z} = xz' + zx' + yy' + a(yz' + zy') + (a^2 - b)zz',$$

the product in question will become of this form, $\text{X} + \alpha\text{Y} + \alpha^2\text{Z}$; that is to say, of the same form as each of those from which it has been produced. Now, as the root α does not enter into the values of X, Y, Z, it is evident, that these quantities will be the same, if we change α into β, or γ; wherefore, since we already have

$$(x + \alpha y + \alpha^2 z) \times (x' + \alpha y' + \alpha^2 z') = \text{X} + \alpha\text{Y} + \alpha^2\text{Z},$$

we shall likewise have, by changing α into β,

$$(x + \beta y + \beta^2 z) \times (x' + \beta y' + \beta^2 z') = \text{X} + \beta\text{Y} + \beta^2\text{Z};$$

and, by changing α into γ,

$$(x + \gamma y + \gamma^2 z) \times (x' + \gamma y' + \gamma^2 z') = \text{X} + \gamma\text{Y} + \gamma^2\text{Z};$$

therefore, by multiplying these three equations together, we shall have, on the one side, the product of the two given formulæ, and on the other, the formula,

$$X^3 + aX^2Y + (a^2 - 2b)X^2Z + bXY^2 + (ab - 3c)XYZ + (b^2 - 2ac)XZ^2 + cY^3 + acY^2Z + bcYZ^2 + c^2Z^3,$$

which will therefore be equal to the product required; and is evidently of the same form as each of the two formulæ of which it is composed.

If we had a third formula, such as

$$x''^3 + ax''^2y'' + (a - 2b)x''^2z'' + bx''y''^2 + (ab - 3c)x''y''z'' + (b^2 - 2ac)x''z''^2 + cy''^3 + acy''^2z'' + bcy''z''^2 + c^2z''^3,$$

and if we wished to have the product of this formula and the two preceding, it is evident, that we should only have to make

$$X' = Xx'' - c(Yz'' + Zy'') + acZz'',$$
$$Y' = Xy'' + Yx'' - b(Yz'' + Zy'') - (ab - c)Zz'',$$
$$Z' = Xz'' + Zx'' + Yy'' + a(Yz'' + Zy'') + (a^2 - b)Zz'',$$

and we should have, for the product required,

$$X'^3 + aX'^2Y' + (a^2 - 2b)X'^2Z' + bX'Y'^2 + (ab - 3c)X'Y'Z' + (b^2 - 2ac)X'Z'^2 + cY'^3 + acY'^2Z' + bcY'Z'^2 + c^2Z'^3.$$

92. Let us now make $x' = x$, $y' = y$, $z' = z$, and we shall have,

$$X = x^2 - 2cyz + acz^2,$$
$$Y = 2xy - 2byz - (ab - c)z^2,$$
$$Z = 2xz + y^2 + 2ayz + (a^2 - b)z^2;$$

and these values will satisfy the equation,

$$X^3 + aX^2Y + bXY^2 + cY^3 + (a^2 - 2b)X^2Z + (ab - 3c)XYZ + acY^2Z + (b^2 - 2ac)XZ^2 + bcYZ^2 + c^2Z^3 = V^2, \text{ by taking}$$

$$V = x^3 + ax^2y + bxy^2 + cy^3 + (a^2 - 2b)x^2z + (ab - 3c)xyz + acy^2z + (b^2 - 2ac)xz^2 + bcyz^2 + c^2z^3;$$

wherefore, if we had, for example, to resolve an equation of this form, $X^3 + aX^2Y + bXY^2 + cY^3 = V^2$, a, b, c being any given quantities, we should only have to destroy Z, by making $2xz + y^2 + 2ayz + (a^2 - b^2)z^2 = 0$, whence we derive $x = -\dfrac{y^2 + 2ayz + (a^2 - b^2)z^2}{2z}$; and, substituting this value of x in the foregoing expressions of X, Y, and V, we shall have very general values of these quantities, which will satisfy the equation proposed.

This solution deserves particular attention, on account of its generality, and the manner in which we have arrived at it; which is, perhaps, the only way in which it can be easily resolved.

We should likewise obtain the solution of the equation,

$$\overset{\prime}{\mathrm{X}}{}^3 + a\overset{\prime}{\mathrm{X}}{}^2\mathrm{Y}' + (a^2 - 2b)\overset{\prime}{\mathrm{X}}{}^2\mathrm{Z}' + b\overset{\prime}{\mathrm{X}}\overset{\prime}{\mathrm{Y}}{}^2 + (ab - 3c)\mathrm{X}'\mathrm{Y}'\mathrm{Z}'$$

$$+ (b^2 - 2ac)\overset{\prime}{\mathrm{X}}\overset{\prime}{\mathrm{Z}}{}^2 + c\overset{\prime}{\mathrm{Y}}{}^3 + ac\overset{\prime}{\mathrm{Y}}{}^2\mathrm{Z}' + bc\overset{\prime}{\mathrm{Y}}\overset{\prime}{\mathrm{Z}}{}^2 + c^2\overset{\prime}{\mathrm{Z}}{}^3 = \mathrm{V}^3,$$

by making, in the foregoing formulæ,

$$x'' = x' = x, \quad y'' = y' = y, \quad z'' = z' = z,$$

and taking

$$\mathrm{V} = x^3 + ax^2y + (a^2 - 2b)x^2z + bxy^2 + (ab - 3c)xyz$$
$$+ (b^2 - 2ac)xz^2 + cy^3 + acy^2z + bcyz^2 + c^2z^3.$$

And we might resolve, successively, the cases in which, instead of the third power V^3, we should have V^4, V^5, &c. But we are going to consider these questions in a general manner, as we have done Art. 90.

93. Let it be proposed, therefore, to resolve an equation of this form,

$$\mathrm{X}^3 + a\mathrm{X}^2\mathrm{Y} + (a^2 - 2b)\mathrm{X}^2\mathrm{Z} + b\mathrm{X}\mathrm{Y}^2 + (ab - 3c)\,\mathrm{XYZ} +$$
$$(b^2 - 2ac)\mathrm{XZ}^2 + c\mathrm{Y}^3 + ac\mathrm{Y}^2\mathrm{Z} + bc\mathrm{YZ}^2 + c^2\mathrm{Z}^3 = \mathrm{V}^m.$$

Since the quantity, which forms the first side of this equation, is nothing more than the product of these three factors,

$$(\mathrm{X} + \alpha\mathrm{Y} + \alpha^2\mathrm{Z}) \times (\mathrm{X} + \beta\mathrm{Y} + \beta^2\mathrm{Z}) \times (\mathrm{X} + \gamma\mathrm{Y} + \gamma^2\mathrm{Z}),$$

it is evident that, in order to render this quantity equal to a power of the dimension m, we have only to make each of its factors separately equal to such a power.

Let then $\quad \mathrm{X} + \alpha\mathrm{Y} + \alpha^2\mathrm{Z} = (x + \alpha y + \alpha^2 z)^m.$

We shall begin by expressing the mth power of $x + \alpha y + \alpha^2 z$ according to Newton's theorem, which will give

$$x^m + mx^{m-1}(y + \alpha z)\alpha + \frac{m(m-1)}{2}x^{m-2}(y + \alpha z)^2\alpha^2$$

$$+ \frac{m(m-1) \times (m-2)}{2 \times 3}x^{m-3}(y + \alpha z)^3\alpha^3 +, \text{ \&c.}$$

Or rather, forming the different powers of $y + \alpha z$, and then arranging them, according to the dimensions of α,

$$x^m + mx^{m-1}y\alpha + (mx^{m-1}z + \frac{m(m-1)}{2}x^{m-2}y^2)\alpha^2$$

$$+ (m(m-1)x^{m-2}yz + \frac{m(m-1)\times(m-2)}{2\times 3}x^{m-3}y^3)\alpha^3 +, \text{\&c.}$$

But as in this formula we do not easily perceive the law of the terms, we shall suppose, in general,

$$(x + \alpha y + \alpha^2 z)^m = \text{P} + \text{P}'\alpha + \text{P}''\alpha^2 + \text{P}'''\alpha^3 + \text{P}^{iv}\alpha^4 +, \text{\&c.}$$

and we shall find,

$$\text{P} = x^m,$$

$$\text{P}' = \frac{my\text{P}}{x},$$

$$\text{P}'' = \frac{(m-1)y\text{P}' + 2mz\text{P}}{2x},$$

$$\text{P}''' = \frac{(m-2)y\text{P}'' + (2m-1)z\text{P}'}{3x},$$

$$\text{P}^{iv} = \frac{(m-3)y\text{P}''' + (2m-2)z\text{P}''}{4x}, \text{\&c.}$$

which may easily be demonstrated by the differential calculus.

Now, since α is one of the roots of the equation,

$s^3 - as^2 + bs - c = 0$, we shall have

$\alpha^3 - a\alpha^2 + b\alpha - c = 0$; whence,

$\alpha^3 = a\alpha^2 - b\alpha + c$; wherefore,

$\alpha^4 = a\alpha^3 - b\alpha^2 + c\alpha = (a^2 - b)\alpha^2 - (ab - c)\alpha + ac,$

$\alpha^5 = (a^2 - b)\alpha^3 - (ab - c)\alpha^2 + ac\alpha = (a^3 - 2ab + c)\alpha^2$

$-(a^2b - b^2 - ac)\alpha + (a^2 - b)c$; and so on.

So that if, in order to simplify, we make

$\text{A}' = 0 \qquad \text{A}^{iv} = a\text{A}''' - b\text{A}'' + c\text{A}'$

$\text{A}'' = 1 \qquad \text{A}^{v} = a\text{A}^{iv} - b\text{A}''' + c\text{A}''$

$\text{A}''' = a \qquad \text{A}^{vi} = a\text{A}^{v} - b\text{A}^{iv} + c\text{A}'''$, &c.

$\text{B}' = 1 \qquad \text{C}' = 0$

$\text{B}'' = 0 \qquad \text{C}'' = 0$

$\text{B}''' = b \qquad \text{C}''' = c$

$\text{B}^{iv} = a\text{B}''' - b\text{B}'' + c\text{B}' \qquad \text{C}^{iv} = a\text{C}''' - b\text{C}'' + c\text{C}'$

$\text{B}^{v} = a\text{B}^{iv} - b\text{B}''' + c\text{B}'' \qquad \text{C}^{v} = a\text{C}^{iv} - b\text{C}''' + c\text{C}''$

$\text{B}^{vi} = a\text{B}^{v} - b\text{B}^{iv} + c\text{B}'''$, &c. $\qquad \text{C}^{vi} = a\text{C}^{v} - b\text{C}^{iv} + c\text{C}'''$, &c.

we shall have,

$\alpha = \text{A}'\alpha^2 - \text{B}'\alpha + \text{C}' \qquad \alpha^3 = \text{A}'''\alpha^2 - \text{B}'''\alpha + \text{C}'''$

$\alpha^2 = \text{A}''\alpha^2 - \text{B}''\alpha + \text{C}'' \qquad \alpha^4 = \text{A}^{iv}\alpha^2 - \text{B}^{iv}\alpha + \text{C}^{iv}$, &c.

Substituting these values, therefore, in the expression

$(x + \alpha y + \alpha^2 z)^m$, it will be found composed of three parts, one all rational, another all multiplied by α, and the third all multiplied by α^2; so that we shall only have to compare the first to X, the second to αY, and the third to α^2Z, and, by these means, we shall have

$$\mathrm{X} = \mathrm{P} + \mathrm{P}'\mathrm{C}' + \mathrm{P}''\mathrm{C}'' + \mathrm{P}'''\mathrm{C}''' + \mathrm{P}^{iv}\mathrm{C}^{iv}, \text{ \&c.}$$
$$\mathrm{Y} = -\mathrm{P}'\mathrm{B}' - \mathrm{P}''\mathrm{B}'' - \mathrm{P}'''\mathrm{B}''' - \mathrm{P}^{iv}\mathrm{B}^{iv}, \text{ \&c.}$$
$$\mathrm{Z} = \mathrm{P}'\mathrm{A}' + \mathrm{P}''\mathrm{A}'' + \mathrm{P}'''\mathrm{A}''' + \mathrm{P}^{iv}\mathrm{A}^{iv}, \text{ \&c.}$$

These values, therefore, will satisfy the equation,

$$\mathrm{X} + \alpha\mathrm{Y} + \alpha^2\mathrm{Z} = (\mathrm{X} + \alpha y + \alpha^2 z)^m;$$

and as the root α does not enter into the expressions of X, Y, and Z, it is evident, that we may change α into β, or into γ; so that we shall have both

$$\mathrm{X} + \beta\mathrm{Y} + \beta^2\mathrm{Z} = (x + \beta y + \beta^2 z)^m, \text{ and}$$
$$\mathrm{X} + \gamma\mathrm{Y} + \gamma^2\mathrm{Z} = (x + \gamma y + \gamma^2 z)^m.$$

If we now multiply these three equations together, it is evident, that the first member will be the same as that of the given equation, and that the second will be equal to a power m, the root of which being called V, we shall have

$$\mathrm{V} = x^3 + ax^2y + (a^2 - 2b)x^2z + bxy^2 + (ab - 3c)xyz$$
$$+ (b^2 - 2ac)xz^2 + cy^3 + acy^2z + bcyz^2 + c^2z^3.$$

Thus, we shall have the values required of X, Y, Z, and V, which will contain three indeterminates, x, y, z.

94. If we wished to find formulæ of four dimensions, having the same properties as those we have now examined, it would be necessary to consider the product of four factors of this form,

$$x + \alpha y + \alpha^2 z + \alpha^3 t$$
$$x + \beta y + \beta^2 z + \beta^3 t$$
$$x + \gamma y + \gamma^2 z + \gamma^3 t$$
$$x + \delta y + \delta^2 z + \delta^3 t,$$

supposing $\alpha, \beta, \gamma, \delta$ to be the roots of a biquadratic equation, such as $s^4 - as^3 + bs^2 - cs + d = 0$; we shall thus have

$$\alpha + \beta + \gamma + \delta = a,$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = b,$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = c,$$
$$\alpha\beta\gamma\delta = d,$$

by which means we may determine all the coefficients of the different terms of the product in question, without knowing the roots $\alpha, \beta, \delta, \gamma$. But as this requires different reductions, which are not easily performed, we may set about it, if it be judged more convenient, in the following manner.

Let us suppose, in general,

$$x + sy + s^2z + s^3t = \varrho;$$

and, as s is determined by the equation,

$$s^4 - as^3 + bs^2 - cs + d = 0,$$

let us take away s from these two equations by the common rules, and the equation, which results, after expunging s, being arranged according to the unknown quantity ϱ, will rise to the fourth degree; so that it may be put into this form, $\varrho^4 - \mathrm{N}\varrho^3 + \mathrm{P}\varrho^2 - \mathrm{Q}\varrho + \mathrm{R} = 0$.

Now, the cause of this equation in ϱ rising to the fourth degree is, that s may have the four values α, β, γ, δ; and also that ϱ may likewise have these four corresponding values,

$$x + \alpha y + \alpha^2 z + \alpha^3 t$$
$$x + \beta y + \beta^2 z + \beta^3 t$$
$$x + \gamma y + \gamma^2 z + \gamma^3 t$$
$$x + \delta y + \delta^2 z + \delta^3 t,$$

which are nothing but those factors, the product of which is required. Wherefore, since the last term R must be the product of all the four roots, or values of ϱ, it follows, that this quantity, R, will be the product required.

But we have now said enough on this subject, which we might resume, perhaps, on some other occasion.

I shall here close these Additions, which the limits I prescribed to myself will not permit me to carry any farther; perhaps they have already been found too long: but the subjects I have considered being rather new and little known, I thought it incumbent on me to enter into several details, necessary for the full illustration of the methods which I have explained, and of their different uses.

THE END.