CHAP. VII.

Of the Extraction *of the* Roots *of* Polygon Numbers.

656. We have shewn, in a preceding chapter*, how polygonal numbers are to be found; and what we then called *a side*, is also called *a root*. If, therefore, we represent the root by x, we shall find the following expressions for all polygonal numbers:

the IIIgon, or triangle, is $\frac{x^2+x}{2}$,

the IVgon, or square, - x^2,

the Vgon - - - - $\frac{3x^2-x}{2}$,

the VIgon - - - - $2x^2-x$,

the VIIgon - - - - $\frac{5x^2-3x}{2}$,

the VIIIgon - - - $3x^2-2x$,

the IXgon - - - - $\frac{7x^2-5x}{2}$,

the Xgon - - - - - $4x^2-3x$,

the $n$gon - - - - $\frac{(n-2)x^2-(n-4)x}{2}$.

657. We have already shewn, that it is easy, by means of these formulæ, to find, for any given root, any polygon number required: but when it is required reciprocally to find the side, or the root of a polygon, the number of whose sides is known, the operation is more difficult, and always requires the solution of a quadratic equation; on which account the subject deserves, in this place, to be separately considered. In doing this we shall proceed regularly, beginning with the triangular numbers, and passing from them to those of a greater number of angles.

658. Let therefore 91 be the given triangular number, the side or root of which is required.

If we make this root $=x$, we must have

* Chap. 5, Sect. III.

$\frac{x^2+x}{2} = 91$; or $x^2 + x = 182$, and $x^2 = -x + 182$; consequently,

$x = -\frac{1}{2} + \sqrt{(\frac{1}{4} + 182)} = -\frac{1}{2} + \sqrt{(\frac{729}{4})} = -\frac{1}{2} + \frac{27}{2} = 13$; from which we conclude, that the triangular root required is 13; for the triangle of 13 is 91.

659. But, in general, let a be the given triangular number, and let its root be required.

Here if we make it $= x$, we have $\frac{x^2+x}{2} = a$, or $x^2 + x = 2a$; therefore, $x^2 = -x + 2a$, and by the rule $x = -\frac{1}{2} + \sqrt{(\frac{1}{4} + 2a)}$, or $x = \frac{-1+\sqrt{(8a+1)}}{2}$.

This result gives the following rule: To find a triangular root, we must multiply the given triangular number by 8, add 1 to the product, extract the root of the sum, subtract 1 from that root, and lastly, divide the remainder by 2.

660. So that all triangular numbers have this property; that if we multiply them by 8, and add unity to the product, the sum is always a square; of which the following small Table furnishes some examples:

Triangles	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	&c.
8 times + 1 =	9,	25,	49,	81,	121,	169,	225,	289,	361,	441,	&c.

If the given number a does not answer this condition, we conclude, that it is not a real triangular number, or that no rational root of it can be assigned.

661. According to this rule, let the triangular root of 210 be required; we shall have $a = 210$, and $8a + 1 = 1681$, the square root of which is 41; whence we see, that the number 210 is really triangular, and that its root is $\frac{41-1}{2} = 20$. But if 4 were given as the triangular number, and its root were required, we should find it $= \frac{\sqrt{33}}{2} - \frac{1}{2}$, and consequently irrational. However, the triangle of this root, $\frac{\sqrt{33}}{2} - \frac{1}{2}$, may be found in the following manner:

Since $x = \frac{\sqrt{33}-1}{2}$, we have $x^2 = \frac{17-\sqrt{33}}{2}$, and adding

$x = \frac{\sqrt{33}-1}{2}$ to it, the sum is $x^2 + x = \frac{16}{2} = 8$. Consequently, the triangle, or the triangular number, $\frac{x^2+x}{2} = 4$.

662. The quadrangular numbers being the same as squares, they occasion no difficulty. For, supposing the given quadrangular number to be a, and its required root x, we shall have $x^2 = a$, and consequently, $x = \sqrt{a}$; so that the square root and the quadrangular root are the same thing.

663. Let us now proceed to pentagonal numbers.

Let 22 be a number of this kind, and x its root; then, by the third formula, we shall have $\frac{3x^2-x}{2} = 22$, or $3x^2 - x = 44$, or $x^2 = \frac{1}{3}x + \frac{44}{3}$; from which we obtain,

$x = \frac{1}{6} + \sqrt{(\frac{1}{36} + \frac{44}{3})}$, or $x = \frac{1+\sqrt{(529)}}{6} = \frac{1}{6} + \frac{23}{6} = 4$;

and consequently 4 is the pentagonal root of the number 22.

664. Let the following question be now proposed: the pentagon a being given, to find its root.

Let this root be x, and we have the equation

$\frac{3x^2-x}{2} = a$, or $3x^2 - x = 2a$, or $x^2 = \frac{1}{3}x + \frac{2a}{3}$; by means of which we find $x = \frac{1}{6} + \sqrt{(\frac{1}{36} + \frac{2a}{3})}$, that is,

$x = \frac{1+\sqrt{(24a+1)}}{6}$. Therefore, when a is a real pentagon, $24a + 1$ must be a square.

Let 330, for example, be the given pentagon, the root will be $x = \frac{1+\sqrt{(7921)}}{6} = \frac{1+89}{6} = 15$.

665. Again, let a be a given hexagonal number, the root of which is required.

If we suppose it $= x$, we shall have $2x^2 - x = a$, or $x^2 = \frac{1}{2}x + \frac{1}{2}a$; and this gives

$$x = \frac{1}{4} + \sqrt{(\tfrac{1}{16} + \tfrac{1}{2}a)} = \frac{1+\sqrt{(8a+1)}}{4}.$$

So that, in order that a may be really a hexagon, $8a + 1$ must become a square; whence we see, that all hexagonal numbers are contained in triangular numbers; but it is not the same with the roots.

For example, let the hexagonal number be 1225, its root will be $x = \dfrac{1+\sqrt{9801}}{4} = \dfrac{1+99}{4} = 25.$

666. Suppose a an heptagonal number, of which the root is required.

Let this root be x, then we shall have $\dfrac{5x^2-3x}{2} = a$, or $x^2 = \frac{3}{5}x + \frac{2}{5}a$, which gives

$$x = \tfrac{3}{10} + \sqrt{(\tfrac{9}{100} + \tfrac{2}{5}a)} = \frac{3+\sqrt{(40a+9)}}{10};$$

therefore the heptagonal numbers have this property, that if they be multiplied by 40, and 9 be added to the product, the sum will always be a square.

Let the heptagon, for example, be 2059; its root will be found $= x = \dfrac{3+\sqrt{(82369)}}{10} = \dfrac{3+287}{10} = 29.$

667. Let us suppose a an octagonal number, of which the root x is required.

We shall here have $3x^2 - 2x = a$, or $x^2 = \frac{2}{2}x + \frac{1}{3}a$, whence results $x = \frac{1}{3} + \sqrt{(\frac{1}{9} + \frac{1}{3}a)} = \dfrac{1+\sqrt{(3a+1)}}{3}.$

Consequently, all octagonal numbers are such, that if multiplied by 3, and unity be added to the product, the sum is constantly a square.

For example, let 3816 be an octagon; its root will be $x = \dfrac{1+\sqrt{11449}}{3} = \dfrac{1+107}{3} = 36.$

668. Lastly, let a be a given n-gonal number, the root of which it is required to assign; we shall then, by the last formula, have this equation:

$\dfrac{(n-2)x^2-(n-4)x}{2} = a$, or $(n-2)x^2 - (n-4)x = 2a$;

consequently, $x^2 = \dfrac{(n-4)x}{n-2} + \dfrac{2a}{n-2}$; whence,

$$x = \frac{n-4}{2(n-2)} + \sqrt{\left(\frac{(n-4)^2}{4(n-2)^2} + \frac{2a}{n-2}\right)}, \text{ or}$$

$$x = \frac{n-4}{2(n-2)} + \sqrt{\left(\frac{(n-4)^2}{4(n-2)^2} + \frac{8(n-2)a}{4(n-2)^2}\right)}, \text{ or}$$

$$x = \frac{n-4+\sqrt{(8(n-2)a+(n-4)^2)}}{2(n-2)}.$$

This formula contains a general rule for finding all the possible polygonal roots of given numbers.

For example, let there be given the XXIV-gonal number, 3009: since a is here $= 3009$ and $n = 24$, we have $n - 2 = 22$ and $n - 4 = 20$; wherefore the root, or

$$x, = \frac{20+\sqrt{(529584+400)}}{44} = \frac{20+728}{44} = 17.$$

CHAP. VIII.

Of the Extraction *of the* Square Roots *of* Binomials.

669. By a *binomial** we mean a quantity composed of two parts, which are either both affected by the sign of the square root, or of which one, at least, contains that sign.

For this reason $3 + \sqrt{5}$ is a binomial, and likewise $\sqrt{8} + \sqrt{3}$; and it is indifferent whether the two terms be joined by the sign $+$ or by the sign $-$. So that $3 - \sqrt{5}$ and $3 + \sqrt{5}$ are both binomials.

670. The reason that these binomials deserve particular attention is, that in the resolution of quadratic equations we are always brought to quantities of this form, when the resolution cannot be performed. For example, the equation $x^2 = 6x - 4$ gives $x = 3 + \sqrt{5}$.

It is evident, therefore, that such quantities must often occur in algebraic calculations; for which reason, we have already carefully shewn how they are to be treated in the ordinary operations of addition, subtraction, multiplication, and division. but we have not been able till now to shew how their square roots are to be extracted; that is, so far as that extraction is possible; for when it is not, we must be satisfied with affixing to the quantity another radical sign. Thus, the square root of $3 + \sqrt{2}$ is written $\sqrt{\overline{3 + \sqrt{2}}}$; or $\sqrt{(3 + \sqrt{2})}$.

671. It must here be observed, in the first place, that the

* In algebra we generally give the name *binomial* to any quantity composed of two terms; but Euler has thought proper to confine this appellation to those expressions, which the French analysts call *quantities partly commensurable, and partly incommensurable.* F. T.