4. To divide the number 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

5. To find three such numbers, that the sum of the first and second multiplied into the third, may be equal to 63; and the sum of the second and third, multiplied into the first, may be equal to 28; also, that the sum of the first and third, multiplied into the second, may be equal to 55.

Ans. 2, 5, 9.

6. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132? *Ans.* 14 and 8.

CHAP. VI.

Of the Resolution *of* Mixt Equations *of the* Second Degree.

638. An equation of the second degree is said to be *mixt*, or complete, when three terms are found in it, namely, that which contains the square of the unknown quantity, as ax^2; that, in which the unknown quantity is found only in the first power, as bx; and, lastly, the term which is composed of only known quantities. And since we may unite two or more terms of the same kind into one, and bring all the terms to one side of the sign $=$, the general form of a mixt equation of the second degree will be

$$ax^2 \pm bx \pm c = 0.$$

In this chapter, we shall shew how the value of x may be derived from such equations: and it will be seen, that there are two methods of obtaining it.

639. An equation of the kind that we are now considering may be reduced, by division, to such a form, that the first term may contain only the square, x^2, of the unknown quantity x. We shall leave the second term on the same side with x, and transpose the known term to the other side of the sign $=$. By these means our equation will assume the form of $x^2 \pm px = \pm q$, in which p and q represent any known numbers, positive or negative; and the whole is at present reduced to determining the true value of x. We shall begin by remarking, that if $x^2 + px$ were a real square, the resolution would be attended with no difficulty, because it would only be required to take the square root of both sides.

640. But it is evident that $x^2 + px$ cannot be a square; since we have already seen, (Art. 307.) that if a root consists of two terms, for example, $x + n$, its square always contains three terms, namely, twice the product of the two parts, beside the square of each part; that is to say, the square of $x + n$ is $x^2 + 2nx + n^2$. Now, we have already on one side $x^2 + px$; we may, therefore, consider x^2 as the square of the first part of the root, and in this case px must represent twice the product of x, the first part of the root, by the second part: consequently, this second part must be $\frac{1}{2}p$, and in fact the square of $x + \frac{1}{2}p$, is found to be

$$x^2 + px + \tfrac{1}{4}p^2.$$

641. Now, $x^2 + px + \frac{1}{4}p^2$ being a real square, which has for its root $x + \frac{1}{2}p$, if we resume our equation $x^2 + px = q$, we have only to add $\frac{1}{4}p^2$ to both sides, which gives us $x^2 + px + \frac{1}{4}p^2 = q + \frac{1}{4}p^2$, the first side being actually a square, and the other containing only known quantities. If, therefore, we take the square root of both sides, we find $x + \frac{1}{2}p = \sqrt{(\frac{1}{4}p^2 + q)}$; subtracting $\frac{1}{2}p$, we obtain $x = -\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 + q)}$; and, as every square root may be taken either affirmatively or negatively, we shall have for x two values expressed thus;

$$x = -\tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{4}p^2 + q)}.$$

642. This formula contains the rule by which all quadratic equations may be resolved; and it will be proper to commit it to memory, that it may not be necessary, every time, to repeat the whole operation which we have gone through. We may always arrange the equation in such a manner, that the pure square x^2 may be found on one side, and the above equation have the form $x^2 = -px + q$, where we see immediately that $x = -\frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 + q)}$.

643. The general rule, therefore, which we deduce from that, in order to resolve the equation $x^2 = -px + q$, is founded on this consideration;

That the unknown quantity x is equal to half the coefficient, or multiplier of x on the other side of the equation, *plus* or *minus* the square root of the square of this number, and the known quantity which forms the third term of the equation.

Thus, if we had the equation $x^2 = 6x + 7$, we should immediately say, that $x = 3 \pm \sqrt{(9 + 7)} = 3 \pm 4$, whence we have these two values of x, namely, $x = 7$, and $x = -1$. In the same manner, the equation $x^2 = 10x - 9$, would give $x = 5 \pm \sqrt{(25 - 9)} = 5 \pm 4$, that is to say, the two values of x are 9 and 1.

644. This rule will be still better understood, by distin-

guishing the following cases: 1st, When p is an even number; 2d, When p is an odd number; and 3d, When p is a fractional number.

1st, Let p be an even number, and the equation such, that $x^2 = 2px + q$; we shall, in this case, have

$$x = p \pm \sqrt{(p^2 + q)}.$$

2d, Let p be an odd number, and the equation $x^2 = px + q$; we shall here have $x = \frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 + q)}$; and since $\frac{1}{4}p^2 + q = \frac{p^2+4q}{4}$, we may extract the square root of the denominator, and write

$$x = \frac{1}{2}p \pm \frac{\sqrt{(p^2+4q)}}{2} = \frac{p \pm \sqrt{(p^2+4q)}}{2}.$$

3d, Lastly, if p be a fraction, the equation may be resolved in the following manner. Let the equation be $ax^2 = bx + c$, or $x^2 = \frac{bx}{a} + \frac{c}{a}$, and we shall have, by the rule, $x = \frac{b}{2a} \pm \sqrt{(\frac{b^2}{4a^2} + \frac{c}{a})}$. Now, $\frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2+4ac}{4a^2}$, the denominator of which is a square; so that

$$x = \frac{b \pm \sqrt{(b^2+4ac)}}{2a}.$$

645. The other method of resolving mixt quadratic equations is, to transform them into pure equations; which is done by substitution: for example, in the equation $x^2 = px + q$, instead of the unknown quantity x, we may write another unknown quantity, y, such, that $x = y + \frac{1}{2}p$; by which means, when we have determined y, we may immediately find the value of x.

If we make this substitution of $y + \frac{1}{2}p$ instead of x, we have $x^2 = y^2 + py + \frac{1}{4}p^2$, and $px = py + \frac{1}{2}p^2$; consequently, our equation will become

$$y^2 + py + \frac{1}{4}p^2 = py + \frac{1}{2}p^2 + q;$$

which is first reduced, by subtracting py, to

$$y^2 + \frac{1}{4}p^2 = \frac{1}{2}p^2 + q;$$

and then, by subtracting $\frac{1}{4}p^2$, to $y^2 = \frac{1}{4}p^2 + q$. This is a pure quadratic equation, which immediately gives

$$y = \pm \sqrt{(\frac{1}{4}p^2 + q)}.$$

Now, since $x = y + \frac{1}{2}p$, we have

$$x = \frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 + q)},$$

as we found it before. It only remains, therefore, to illustrate this rule by some examples.

646. *Question* 1. There are two numbers; the one exceeds the other by 6, and their product is 91: what are those numbers?

If the less be x, the other will be $x + 6$, and their product $x^2 + 6x = 91$. Subtracting $6x$, there remains $x^2 = 91 - 6x$, and the rule gives

$x = -3 \pm \sqrt{(9 + 91)} = -3 \pm 10$; so that $x = 7$, or $x = -13$.

The question therefore admits of two solutions;

By the one, the less number $x = 7$, and the greater $x + 6 = 13$.

By the other, the less number $x = -13$, and the greater $x + 6 = -7$.

647. *Question* 2. To find a number such, that if 9 be taken from its square, the remainder may be a number, as much greater than 100, as the number itself is less than 23.

Let the number sought be x. We know that $x^2 - 9$ exceeds 100 by $x^2 - 109$: and since x is less than 23 by $23 - x$, we have this equation

$$x^2 - 109 = 23 - x.$$

Therefore $x^2 = -x + 132$, and, by the rule, $x = -\frac{1}{2} \pm \sqrt{(\frac{1}{4} + 132)} = -\frac{1}{2} \pm \sqrt{(\frac{529}{4})} = -\frac{1}{2} \pm \frac{23}{2}$. So that $x = 11$, or $x = -12$.

Hence, when only a positive number is required, that number will be 11, the square of which *minus* 9 is 112, and consequently greater than 100 by 12, in the same manner as 11 is less than 23 by 12.

648. *Question* 3. To find a number such, that if we multiply its half by its third, and to the product add half the number required, the result will be 30.

Supposing the number to be x, its half, multiplied by its third, will give $\frac{1}{6}x^2$; so that $\frac{1}{6}x^2 + \frac{1}{2}x = 30$; and multiplying by 6, we have $x^2 + 3x = 180$, or $x^2 = -3x + 180$, which gives $x = -\frac{3}{2} \pm \sqrt{(\frac{9}{4} + 180)} = -\frac{3}{2} \pm \frac{27}{2}$.

Consequently, either $x = 12$, or $x = -15$.

649. *Question* 4. To find two numbers, the one being double the other, and such, that if we add their sum to their product, we may obtain 90.

Let one of the numbers be x, then the other will be $2x$; their product also will be $2x^2$, and if we add to this $3x$, or their sum, the new sum ought to make 90. So that $2x^2 + 3x = 90$; or $2x^2 = 90 - 3x$; whence $x^2 = -\frac{3}{2}x + 45$, and thus we obtain

$$x = -\tfrac{3}{4} \pm \sqrt{(\tfrac{9}{16} + 45)} = -\tfrac{3}{4} \pm \tfrac{27}{4}.$$

Consequently $x = 6$, or $x = -7\tfrac{1}{2}$.

650. *Question* 5. A horse-dealer bought a horse for a certain number of crowns, and sold it again for 119 crowns, by which means his profit was as much per cent as the horse cost him; what was his first purchase?

Suppose the horse cost x crowns; then, as the dealer gains x per cent, we have this proportion:

$$\text{As } 100 : x :: x : \frac{x^2}{100};$$

since therefore he has gained $\frac{x^2}{100}$, and the horse originally cost him x crowns, he must have sold it for $x + \frac{x^2}{100}$; therefore $x + \frac{x^2}{100} = 119$; and subtracting x, we have $\frac{x^2}{100} = -x + 119$; then multiplying by 100, we obtain $x^2 = -100x + 11900$. Whence, by the rule, we find $x = -50 \pm \sqrt{(2500 + 11900)} = -50 \pm \sqrt{14400} = -50 \pm 120 = 70$.

The horse therefore cost 70 crowns, and since the horse-dealer gained 70 per cent when he sold it again, the profit must have been 49 crowns. So that the horse must have been sold again for $70 + 49$, that is to say, for 119 crowns.

651. *Question* 6. A person buys a certain number of pieces of cloth: he pays for the first 2 crowns, for the second 4 crowns, for the third 6 crowns, and in the same manner always 2 crowns more for each following piece. Now, all the pieces together cost him 110: how many pieces had he?

Let the number sought be x; then, by the question, the purchaser paid for the different pieces of cloth in the following manner:

for the 1, 2, 3, 4, 5 .... x pieces
he pays 2, 4, 6, 8, 10 .... $2x$ crowns.

It is therefore required to find the sum of the arithmetical progression $2 + 4 + 6 + 8 + \ldots\ldots 2x$, which consists of x terms, that we may deduce from it the price of all the pieces of cloth taken together. The rule which we have already given for this operation requires us to add the last term to the first; and the sum is $2x + 2$; which must be multiplied by the number of terms x, and the product will be $2x^2 + 2x$; lastly, if we divide by the difference 2, the quotient will be $x^2 + x$, which is the sum of the progression; so that we have $x^2 + x = 110$; therefore $x^2 = -x + 110$, and $x = -\frac{1}{2} + \sqrt{(\frac{1}{4} + 110)} = -\frac{1}{2} + \frac{21}{2} = 10$.

And hence the number of pieces of cloth is 10.

652. *Question* 7. A person bought several pieces of cloth for 180 crowns; and if he had received for the same sum 3 pieces more, he would have paid 3 crowns less for each piece. How many pieces did he buy?

Let us represent the number sought by x; then each piece will have cost him $\frac{180}{x}$ crowns. Now, if the purchaser had had $x + 3$ pieces for 180 crowns, each piece would have cost $\frac{180}{x+3}$ crowns; and, since this price is less than the real price by three crowns, we have this equation,

$$\frac{180}{x+3} = \frac{180}{x} - 3.$$

Multiplying by x, we obtain $\frac{180x}{x+3} = 180 - 3x$; dividing by 3, we have $\frac{60x}{x+3} = 60 - x$; and again, multiplying by $x + 3$, gives $60x = 180 + 57x - x^2$; therefore adding x^2, we shall have $x^2 + 60x = 180 + 57x$; and subtracting $60x$, we shall have $x^2 = -3x + 180$.

The rule consequently gives,

$$x = -\frac{3}{2} + \sqrt{(\frac{9}{4} + 180)}, \text{ or } x = -\frac{3}{2} + \frac{27}{2} = 12.$$

He therefore bought for 180 crowns 12 pieces of cloth at 15 crowns the piece; and if he had got 3 pieces more, namely, 15 pieces for 180 crowns, each piece would have cost only 12 crowns, that is to say, 3 crowns less.

653. *Question* 8. Two merchants enter into partnership with a stock of 100 pounds; one leaves his money in the partnership for three months, the other leaves his for two months, and each takes out 99 pounds of capital and profit. What proportion of the stock did they separately furnish?

Suppose the first partner contributed x pounds, the other will have contributed $100 - x$. Now, the former receiving 99*l.*, his profit is $99 - x$, which he has gained in three months with the principal x; and since the second receives also 99*l.*, his profit is $x - 1$, which he has gained in two months with the principal $100 - x$; it is evident also, that the profit of this second partner would have been

$\frac{3x-3}{2}$, if he had remained three months in the partnership: and as the profits gained in the same time are in proportion to the principals, we have the following proportion,

$$x : 99 - x :: 100 - x : \frac{3x-3}{2}.$$

And the equality of the product of the extremes to that of the means, gives the equation,

$$\frac{3x^2-3x}{2} = 9900 - 199x + x^2;$$

then multiplying this by 2, we have $3x^2 - 3x = 19800 - 398x + 2x^2$; and subtracting $2x^2$, we obtain $x^2 - 3x = 19800 - 398x$. Adding $3x$, gives $x^2 = 19800 - 395x$; then by the rule,
$x = -\frac{395}{2} + \sqrt{(\frac{156025}{4} + \frac{79200}{4})} = -\frac{395}{2} + \frac{485}{2} = \frac{90}{2} = 45$.

The first partner therefore contributed 45*l.* and the other 55*l.* The first having gained 54*l.* in three months, would have gained in one month 18*l.*; and the second having gained 44*l.* in two months, would have gained 22*l.* in one month: now these profits agree; for if, with 45*l.*, 18*l.* are gained in one month, 22*l.* will be gained in the same time with 55*l.*

654. *Question* 9. Two girls carry 100 eggs to market; the one had more than the other, and yet the sum which they both received for them was the same. The first says to the second, If I had had your eggs, I should have received 15 pence. The other answers, If I had had yours, I should have received $6\frac{2}{3}$ pence. How many eggs did each carry to market?

Suppose the first had x eggs; then the second must have had $100 - x$.

Since, therefore, the former would have sold $100 - x$ eggs for 15 pence, we have the following proportion:

$$(100 - x) : 15 :: x : \frac{15x}{100-x}.$$

Also, since the second would have sold x eggs for $6\frac{2}{3}$ pence, we readily find how much she got for $100 - x$ eggs, thus:

$$\text{As } x : (100 - x) :: \tfrac{20}{3} : \frac{2000-20x}{3x}.$$

Now, both the girls received the same money; we have consequently the equation, $\frac{15x}{100-x} = \frac{2000-20x}{3x}$, which becomes $25x^2 = 200000 - 4000x$; and, lastly,

$$x^2 = -160x + 8000;$$

whence we obtain

$$x = -80 + \surd(6400 + 8000) = -80 + 120 = 40.$$

So that the first girl had 40 eggs, the second had 60, and each received 10 pence.

655. *Question* 10. Two merchants sell each a certain quantity of silk; the second sells 3 ells more than the first, and they received together 35 crowns. Now, the first says to the second, I should have got 24 crowns for your silk; the other answers, And I should have got for yours 12 crowns and a half. How many ells had each?

Suppose the first had x ells; then the second must have had $x+3$ ells; also, since the first would have sold $x+3$ ells for 24 crowns, he must have received $\frac{24x}{x+3}$ crowns for his x ells. And, with regard to the second, since he would have sold x ells for $12\frac{1}{2}$ crowns, he must have sold his $x+3$ ells for $\frac{25x+75}{2x}$; so that the whole sum they received was

$$\frac{24x}{x+3} + \frac{25x+75}{2x} = 35 \text{ crowns}.$$

This equation becomes $x^2 = 20x - 75$; whence we have $x = 10 \pm \surd(100 - 75) = 10 \pm 5$.

So that the question admits of two solutions: according to the first, the first merchant had 15 ells, and the second had 18; and since the former would have sold 18 ells for 24 crowns, he must have sold his 15 ells for 20 crowns. The second, who would have sold 15 ells for 12 crowns and a half, must have sold his 18 ells for 15 crowns; so that they actually received 35 crowns for their commodity.

According to the second solution, the first merchant had 5 ells, and the other 8 ells; and since the first would have sold 8 ells for 24 crowns, he must have received 15 crowns for his 5 ells; also, since the second would have sold 5 ells for 12 crowns and a half, his 8 ells must have produced him 20 crowns; the sum being, as before, 35 crowns.