so that $s = 1537$; which is the whole number of soldiers. By this means we find,

$$x = 1802 - 1537 = 265;$$
$$2y = 2703 - 1537 = 1166, \text{ or } y = 583;$$
$$3z = 3604 - 1537 = 2067, \text{ or } z = 689.$$

The company of Swiss therefore has 265 men; that of Swabians 583; and that of Saxons 689.

CHAP. V.

Of the Resolution *of* Pure Quadratic Equations.

623. An equation is said to be of the second degree, when it contains the square, or the second power, of the unknown quantity, without any of its higher powers; and an equation, containing likewise the third power of the unknown quantity, belongs to cubic equations, and its resolution requires particular rules.

624. There are, therefore, only three kinds of terms in an equation of the second degree:

1. The terms in which the unknown quantity is not found at all, or which is composed only of known numbers.

2. The terms in which we find only the first power of the unknown quantity.

3. The terms which contain the square, or the second power, of the unknown quantity.

So that x representing an unknown quantity, and the letters a, b, c, d, &c. the known quantities, the terms of the first kind will have the form a, the terms of the second kind will have the form bx, and the terms of the third kind will have the form cx^2.

625. We have already seen, how two or more terms of the same kind may be united together, and considered as a single term.

For example, we may consider the formula
$ax^2 - bx^2 + cx^2$ as a single term, representing it thus,
$(a - b + c)x^2$; since, in fact, $(a - b + c)$ is a known quantity.

And also, when such terms are found on both sides of the sign $=$, we have seen how they may be brought to one side, and then reduced to a single term. Let us take, for example, the equation,

$$2x^2 - 3x + 4 = 5x^2 - 8x + 11;$$

we first subtract $2x^2$, and there remains

$$-3x + 4 = 3x^2 - 8x + 11;$$

then adding $8x$, we obtain,

$$5x + 4 = 3x^2 + 11;$$

lastly, subtracting 11, there remains $3x^2 = 5x - 7$.

626. We may also bring all the terms to one side of the sign $=$, so as to leave *zero*, or 0, on the other; but it must be remembered, that when terms are transposed from one side to the other, their signs must be changed.

Thus, the above equation will assume this form, $3x^2 - 5x + 7 = 0$; and, for this reason also, the following general formula represents all equations of the second degree;

$$ax^2 \pm bx \pm c = 0;$$

in which the sign $\pm$ is read *plus* or *minus*, and indicates, that such terms as it stands before may be sometimes positive, and sometimes negative.

627. Whatever therefore be the original form of a quadratic equation, it may always be reduced to this formula of three terms. If we have, for example, the equation

$$\frac{ax+b}{cx+d} = \frac{ex+f}{gx+h}$$

we may, first, destroy the fractions; multiplying, for this purpose, by $cx + d$, which gives

$ax + b = \dfrac{cex^2 + cfx + edx + fd}{gx+h}$, then by $gx + h$, we have

$$agx^2 + bgx + ahx + bh = cex^2 + cfx + edx + fd,$$

which is an equation of the second degree, reducible to the three following terms, which we shall transpose by arranging them in the usual manner:

$$\left.\begin{matrix} ag \\ -ce \end{matrix}\right\} x^2 + \left\{\begin{matrix} +bg \\ +ah \\ -cf \\ -ed \end{matrix}\right\} x + \left\{\begin{matrix} +bh \\ -fd \end{matrix}\right\} = 0.$$

We may exhibit this equation also in the following form, which is still more clear:

$$(ag - ce)x^2 + (bg + ah - cf - ed)x + bh - fd = 0.$$

628. Equations of the second degree, in which all the three kinds of terms are found, are called *complete*, and the resolution of them is attended with greater difficulties; for

which reason we shall first consider those, in which one of the terms is wanting.

Now, if the term x^2 were not found in the equation, it would not be a quadratic, but would belong to those of which we have already treated; and if the term, which contains only known numbers, were wanting, the equation would have this form, $ax^2 \pm bx = 0$, which being divisible by x, may be reduced to $ax \pm b = 0$, which is likewise a simple equation, and belongs not to the present class.

629. But when the middle term, which contains the first power of x, is wanting, the equation assumes this form, $ax^2 \pm c = 0$, or $ax^2 = \mp c$; as the sign of c may be either positive, or negative.

We shall call such an equation a *pure* equation of the second degree, and the resolution of it is attended with no difficulty; for we have only to divide by a, which gives $x^2 = \frac{c}{a}$; and taking the square root of both sides, we find $x = \sqrt{\frac{c}{a}}$; by which means the equation is resolved.

630. But there are three cases to be considered here. In the first, when $\frac{c}{a}$ is a square number (of which we can therefore really assign the root) we obtain for the value of x a rational number, which may be either integral, or fractional. For example, the equation $x^2 = 144$, gives $x = 12$. And $x^2 = \frac{9}{16}$, gives $x = \frac{3}{4}$.

The second case is, when $\frac{c}{a}$ is not a square, in which case we must therefore be contented with the sign $\sqrt{\ }$. If, for example, $x^2 = 12$, we have $x = \sqrt{12}$, the value of which may be determined by approximation, as we have already shewn.

The third case is that, in which $\frac{c}{a}$ becomes a negative number: the value of x is then altogether impossible and imaginary; and this result proves that the question, which leads to such an equation, is in itself impossible.

631. We shall also observe, before proceeding farther, that whenever it is required to extract the square root of a number, that root, as we have already remarked, has always two values, the one positive and the other negative. Suppose, for example, we have the equation $x^2 = 49$, the value

of x will be not only $+7$, but also -7, which is expressed by $x = \pm 7$. So that all those questions admit of a double answer; but it will be easily perceived that in several cases, as those which relate to a certain number of men, the negative value cannot exist.

632. In such equations, also, as $ax^2 = bx$, where the known quantity c is wanting, there may be two values of x, though we find only one if we divide by x. In the equation $x^2 = 3x$, for example, in which it is required to assign such a value of x, that x^2 may become equal to $3x$, this is done by supposing $x = 3$, a value which is found by dividing the equation by x; but, beside this value, there is also another, which is equally satisfactory, namely, $x = 0$; for then $x^2 = 0$, and $3x = 0$. Equations therefore of the second degree, in general, admit of two solutions, whilst simple equations admit only of one.

We shall now illustrate what we have said with regard to pure equations of the second degree by some examples.

633. *Question* 1. Required a number, the half of which multiplied by the third, may produce 24.

Let this number be x; then by the question $\frac{1}{2}x$, multiplied by $\frac{1}{3}x$, must give 24; we shall therefore have the equation $\frac{1}{6}x^2 = 24$.

Multiplying by 6, we have $x^2 = 144$; and the extraction of the root gives $x = \pm 12$. We put $\pm$; for if $x = +12$, we have $\frac{1}{2}x = 6$, and $\frac{1}{3}x = 4$: now, the product of these two numbers is 24; and if $x = -12$, we have $\frac{1}{2}x = -6$, and $\frac{1}{3}x = -4$, the product of which is likewise 24.

634. *Question* 2. Required a number such, that being increased by 5, and diminished by 5, the product of the sum by the difference may be 96.

Let this number be x, then $x + 5$, multiplied by $x - 5$, must give 96; whence results the equation,

$$x^2 - 25 = 96.$$

Adding 25, we have $x^2 = 121$; and extracting the root, we have $x = 11$. Thus $x + 5 = 16$, also $x - 5 = 6$; and, lastly, $6 \times 16 = 96$.

635. *Question* 3. Required a number such, that by adding it to 10, and subtracting it from 10, the sum, multiplied by the difference, will give 51.

Let x be this number; then $10 + x$, multiplied by $10 - x$, must make 51, so that $100 - x^2 = 51$. Adding x^2, and subtracting 51, we have $x^2 = 49$, the square root of which gives $x = 7$.

636. *Question* 4. Three persons, who had been playing, leave off; the first, with as many times 7 crowns, as the second has three crowns; and the second, with as many

times 17 crowns, as the third has 5 crowns. Farther, if we multiply the money of the first by the money of the second, and the money of the second by the money of the third, and, lastly, the money of the third by that of the first, the sum of these three products will be $3830\frac{2}{3}$. How much money has each?

Suppose that the first player has x crowns; and since he has as many times 7 crowns, as the second has 3 crowns, we know that his money is to that of the second, in the ratio of 7 : 3.

We shall therefore have $7 : 3 :: x : \frac{3}{7}x$, the money of the second player.

Also, as the money of the second player is to that of the third in the ratio of 17 : 5, we shall have $17 : 5 :: \frac{3}{7}x : \frac{15}{119}x$, the money of the third player.

Multiplying x, or the money of the first player, by $\frac{3}{7}x$, the money of the second, we have the product $\frac{3}{7}x^2$: then, $\frac{3}{7}x$, the money of the second, multiplied by the money of the third, or by $\frac{15}{119}x$, gives $\frac{45}{833}x^2$; and, lastly, the money of the third, or $\frac{15}{119}x$, multiplied by x, or the money of the first, gives $\frac{15}{119}x^2$. Now, the sum of these three products is $\frac{3}{7}x^2 + \frac{45}{833}x^2 + \frac{15}{119}x^2$; and reducing these fractions to the same denominator, we find their sum $\frac{507}{833}x^2$, which must be equal to the number $3830\frac{2}{3}$.

We have therefore, $\frac{507}{833}x^2 = 3830\frac{2}{3}$.

So that $\frac{1521}{833}x^2 = 11492$, and $1521x^2$ being equal to 9572836, dividing by 1521, we have $x^2 = \frac{9572836}{1521}$; and taking its root, we find $x = \frac{3094}{39}$. This fraction is reducible to lower terms, if we divide by 13; so that $x = \frac{238}{3} = 79\frac{1}{3}$; and hence we conclude, that $\frac{3}{7}x = 34$, and $\frac{15}{119}x = 10$.

The first player therefore has $79\frac{1}{3}$ crowns, the second has 34 crowns, and the third 10 crowns.

Remark. This calculation may be performed in an easier manner; namely, by taking the factors of the numbers which present themselves, and attending chiefly to the squares of those factors.

It is evident, that $507 = 3 \times 169$, and that 169 is the square of 13; then, that $833 = 7 \times 119$, and $119 = 7 \times 17$: therefore $\frac{3 \times 169}{17 \times 49}x^2 = 3830\frac{2}{3}$, and if we multiply by 3, we have $\frac{9 \times 169}{17 \times 49}x^2 = 11492$. Let us resolve this number also into its factors; and we readily perceive, that the first is 4, that is to say, that $11492 = 4 \times 2873$; farther, 2873 is divisible by 17, so that $2873 = 17 \times 169$.

Consequently, our equation will assume the following form, $\frac{9\times169}{17\times49}x^2 = 4 \times 17 \times 169$, which, divided by 169, is reduced to $\frac{9}{17\times49}x^2 = 4 \times 17$; multiplying also by 17×49, and dividing by 9, we have $x^2 = \frac{4\times289\times49}{9}$, in which all the factors are squares; whence we have, without any further calculation, the root $x = \frac{2\times17\times7}{3} = \frac{238}{3} = 79\frac{1}{3}$, as before.

637. *Question* 5. A company of merchants appoint a factor at Archangel. Each of them contributes for the trade, which they have in view, ten times as many crowns as there are partners; and the profit of the factor is fixed at twice as many crowns, *per cent*, as there are partners. Also, if $\frac{1}{100}$ part of his total gain be multiplied by $2\frac{2}{9}$, it will give the number of partners. That number is required.

Let it be x; and since, each partner has contributed $10x$, the whole capital is $10x^2$. Now, for every hundred crowns, the factor gains $2x$, so that with the capital of $10x^2$ his profit will be $\frac{1}{5}x^3$. The $\frac{1}{100}$ part of his gain is $\frac{1}{500}x^3$; multiplying by $2\frac{2}{9}$, or by $\frac{20}{9}$, we have $\frac{20}{4500}x^3$, or $\frac{1}{225}x^3$, and this must be equal to the number of partners, or x.

We have, therefore, the equation $\frac{1}{225}x^3 = x$, or $x^3 = 225x$; which appears, at first, to be of the third degree; but as we may divide by x, it is reduced to the quadratic $x^2 = 225$; whence $x = 15$.

So that there are fifteen partners, and each contributed 150 crowns.

QUESTIONS FOR PRACTICE.

1. To find a number, to which 20 being added, and from which 10 being subtracted, the square of the sum, added to twice the square of the remainder, shall be 17475.
Ans. 75.

2. What two numbers are those, which are to one another in the ratio of 3 to 5, and whose squares, added together, make 1666? *Ans.* 21 and 35.

3. The sum $2a$, and the sum of the squares $2b$, of two numbers being given; to find the numbers.
Ans. $a - \sqrt{(b - a^2)}$ and $a + \sqrt{(b - a^2)}$.

4. To divide the number 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

5. To find three such numbers, that the sum of the first and second multiplied into the third, may be equal to 63; and the sum of the second and third, multiplied into the first, may be equal to 28; also, that the sum of the first and third, multiplied into the second, may be equal to 55.

Ans. 2, 5, 9.

6. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132? *Ans.* 14 and 8.

CHAP. VI.

Of the Resolution *of* Mixt Equations *of the* Second Degree.

638. An equation of the second degree is said to be *mixt*, or complete, when three terms are found in it, namely, that which contains the square of the unknown quantity, as ax^2; that, in which the unknown quantity is found only in the first power, as bx; and, lastly, the term which is composed of only known quantities. And since we may unite two or more terms of the same kind into one, and bring all the terms to one side of the sign $=$, the general form of a mixt equation of the second degree will be

$$ax^2 \pm bx \pm c = 0.$$

In this chapter, we shall shew how the value of x may be derived from such equations: and it will be seen, that there are two methods of obtaining it.

639. An equation of the kind that we are now considering may be reduced, by division, to such a form, that the first term may contain only the square, x^2, of the unknown quantity x. We shall leave the second term on the same side with x, and transpose the known term to the other side of the sign $=$. By these means our equation will assume the form of $x^2 \pm px = \pm q$, in which p and q represent any known numbers, positive or negative; and the whole is at present reduced to determining the true value of x. We shall begin by remarking, that if $x^2 + px$ were a real square, the resolution would be attended with no difficulty, because it would only be required to take the square root of both sides.