feet broader and 2 feet longer, it would then have been 68 square feet larger: required the length and breadth of the floor. *Ans.* Length 14 feet, and breadth 10 feet.

25. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but two of the greyhound's leaps are as much as three of the hare's: how many leaps must the greyhound take to catch the hare? *Ans.* 300.

CHAP. IV.

Of the Resolution *of two or more* Equations *of the* First Degree.

605. It frequently happens that we are obliged to introduce into algebraic calculations two or more unknown quantities, represented by the letters x, y, z: and if the question is determinate, we are brought to the same number of equations as there are unknown quantities; from which it is then required to deduce those quantities. As we consider, at present, those equations only, which contain no powers of an unknown quantity higher than the first, and no products of two or more unknown quantities, it is evident that all those equations have the form

$$az + by + cx = d.$$

606. Beginning therefore with two equations, we shall endeavour to find from them the value of x and y: and, in order that we may consider this case in a general manner, let the two equations be,

$$ax + by = c; \text{ and } fx + gy = h;$$

in which, a, b, c, and f, g, h, are known numbers. It is required, therefore, to obtain, from these two equations, the two unknown quantities x and y.

607. The most natural method of proceeding will readily present itself to the mind; which is, to determine, from both equations, the value of one of the unknown quantities, as for example x, and to consider the equality of those two values; for then we shall have an equation, in which the unknown quantity y will be found by itself, and may be determined by the rules already given. Then, knowing y, we shall have only to substitute its value in one of the quantities that express x.

608. According to this rule, we obtain from the first equation, $x = \frac{c-by}{a}$, and from the second, $x = \frac{h-gy}{f}$: then putting these values equal to each other, we have this new equation:

$$\frac{c-by}{a} = \frac{h-gy}{f};$$

multiplying by a, the product is $c - by = \frac{ah-agy}{f}$; and then by f, the product is $fc - fby = ah - agy$; adding agy, we have $fc - fby + agy = ah$; subtracting fc, gives $-fby + agy = ah - fc$; or $(ag - bf)y = ah - fc$; lastly, dividing by $ag - bf$, we have

$$y = \frac{ah-fc}{ag-bf}.$$

In order now to substitute this value of y in one of the two values which we have found of x, as in the first, where $x = \frac{c-by}{a}$, we shall first have

$-by = -\frac{abh-bcf}{ag-bf}$; whence $c - by = c - \frac{abh-bcf}{ag-bf}$,

$= \frac{acg-bcf-abh+bcf}{ag-bf} = \frac{acg+abh}{ag-bf}$; and dividing by a,

$x = \frac{c-by}{a} = \frac{cg-bh}{ag-bf}$.

609. *Question* 1. To illustrate this method by examples, let it be proposed to find two numbers, whose sum may be 15, and difference 7.

Let us call the greater number x, and the less y: then we shall have

$$x + y = 15, \text{ and } x - y = 7.$$

The first equation gives

$$x = 15 - y$$

and the second, $x = 7 + y$;

whence results this equation, $15 - y = 7 + y$. So that $15 = 7 + 2y$; $2y = 8$, and $y = 4$; by which means we find $x = 11$.

So that the less number is 4, and the greater is 11.

610. *Question* 2. We may also generalise the preceding

question, by requiring two numbers, whose sum may be a, and the difference b.

Let the greater of the two numbers be expressed by x, and the less by y; we shall then have $x + y = a$, and $x - y = b$.

Here the first equation gives $x = a - y$, and the second $x = b + y$.

Therefore, $a - y = b + y$; $a = b + 2y$; $2y = a - b$; lastly, $y = \frac{a-b}{2}$, and, consequently,

$$x = a - y = a - \frac{a-b}{2} = \frac{a+b}{2}.$$

Thus, we find the greater number, or x, is $\frac{a+b}{2}$, and the less, or y, is $\frac{a-b}{2}$; or, which comes to the same, $x = \frac{1}{2}a + \frac{1}{2}b$, and $y = \frac{1}{2}a - \frac{1}{2}b$. Hence we derive the following theorem: When the sum of any two numbers is a, and their difference is b, the greater of the two numbers will be equal to half the sum *plus* half the difference; and the less of the two numbers will be equal to half the sum *minus* half the difference.

611. We may resolve the same question in the following manner:

Since the two equations are,

$$x + y = a, \text{ and}$$
$$x - y = b;$$

if we add the one to the other, we have $2x = a + b$.

$$\text{Therefore } x = \frac{a+b}{2}.$$

Lastly, subtracting the same equations from each other, we have $2y = a - b$; and therefore

$$y = \frac{a-b}{2}.$$

612. *Question* 3. A mule and an ass were carrying burdens amounting to several hundred weight. The ass complained of his, and said to the mule, I need only one hundred weight of your load, to make mine twice as heavy as yours; to which the mule answered, But if you give me a hundred weight of yours, I shall be loaded three times as much you will be. How many hundred weight did each carry?

Suppose the mule's load to be x hundred weight, and that of the ass to be y hundred weight. If the mule gives one hundred weight to the ass, the one will have $y + 1$, and there will remain for the other $x - 1$; and since, in this case, the ass is loaded twice as much as the mule, we have $y + 1 = 2x - 2$.

Farther, if the ass gives a hundred weight to the mule, the latter has $x + 1$, and the ass retains $y - 1$; but the burden of the former being now three times that of the latter, we have $x + 1 = 3y - 3$.

Consequently our two equations will be,

$$y + 1 = 2x - 2, \text{ and } x + 1 = 3y - 3.$$

From the first, $x = \frac{y+3}{2}$, and the second gives $x = 3y - 4$; whence we have the new equation $\frac{y+3}{2} = 3y - 4$, which gives $y = \frac{11}{5}$,: this also determines the value of x, which becomes $2\frac{3}{5}$.

The mule therefore carried $2\frac{3}{5}$ hundred weight, and the ass $2\frac{1}{5}$ hundred weight.

613. When there are three unknown numbers, and as many equations; as, for example,

$$x + y - z = 8,$$
$$x + z - y = 9,$$
$$y + z - x = 10;$$

we begin, as before, by deducing a value of x from each, and have, from the

1st $x = 8 + z - y$;
2d $x = 9 + y - z$;
3d $x = y + z - 10$.

Comparing the first of these values with the second, and after that with the third, we have the following equations:

$$8 + z - y = 9 + y - z,$$
$$8 + z - y = y + z - 10.$$

Now, the first gives $2z - 2y = 1$, and, by the second, $2y = 18$, or $y = 9$; if therefore we substitute this value of y in $2z - 2y = 1$, we have $2z - 18 = 1$, or $2z = 19$, so that $z = 9\frac{1}{2}$; it remains, therefore, only to determine x, which is easily found $= 8\frac{1}{2}$.

Here it happens, that the letter z vanishes in the last equation, and that the value of y is found immediately; but if this had not been the case, we should have had two equations between z and y, to be resolved by the preceding rule.

614. Suppose we had found the three following equations:

$$3x + 5y - 4z = 25,$$
$$5x - 2y + 3z = 46,$$
$$3y + 5z - x = 62.$$

If we deduce from each the value of x, we shall have from the

$$1\text{st } x = \frac{25-5y+4z}{3},$$

$$2\text{d } x = \frac{46+2y-3z}{5},$$

$$3\text{d } x = 3y + 5z - 62.$$

Comparing these three values together, and first the third with the first,

we have $3y + 5z - 62 = \dfrac{25-5y+4z}{3}$;

multiplying by 3, gives $9y + 15z - 186 = 25 - 5y + 4z$;
so that $9y + 15z = 211 - 5y + 4z$,
and $14y + 11z = 211$.

Comparing also the third with the second,

we have $3y + 5z - 62 = \dfrac{46+2y-3z}{5}$,

or $46 + 2y - 3z = 15y + 25z - 310$,
which, when reduced, becomes $356 = 13y + 28z$.

We shall now deduce, from these two new equations, the value of y:

1st $14y + 11z = 211$; or $14y = 211 - 11z$,

$$\text{and } y = \frac{211-11z}{14}.$$

2d $13y + 28z = 356$; or $13y = 356 - 28z$,

$$\text{and } y = \frac{356-28z}{13}.$$

These two values form the new equation

$$\frac{211-11z}{14} = \frac{356-28z}{13}, \text{ whence,}$$

$2743 - 143z = 4984 - 392z$, or $249z = 2241$, and $z = 9$.

This value being substituted in one of the two equations of y and z, we find $y = 8$; and, lastly, a similar substitution in one of the three values of x, will give $x = 7$.

615. If there were more than three unknown quantities to determine, and as many equations to resolve, we should proceed in the same manner; but the calculations would often prove very tedious.

It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating the solution; which consist in introducing into the calculation, beside the principal unknown quantities, a new unknown quantity arbitrarily assumed, such as, for example, the sum of all the rest; and when a person is a little accustomed to such calculations, he easily perceives what is most proper to be done *. The following examples may serve to facilitate the application of these artifices.

616. *Question* 4. Three persons, A, B, and C, play together; and, in the first game, A loses to each of the other two, as much money as each of them has. In the next game, B loses to each of the other two, as much money as they then had. Lastly, in the third game, A and B gain each, from C, as much money as they had before. On leaving off, they find that each has an equal sum, namely, 24 guineas. Required, with how much money each sat down to play?

Suppose that the stake of the first person was x, that of the second y, and that of the third z: also, let us make the sum of all the stakes, or $x + y + z, = s$. Now, A losing in the first game as much money as the other two have, he loses $s - x$ (for he himself having had x, the two others must have had $s - x$); therefore there will remain to him $2x - s$; also B will have $2y$, and C will have $2z$.

So that, after the first game, each will have as follows: $A = 2x - s$, $B = 2y$, and $C = 2z$.

In the second game, B, who has now $2y$, loses as much money as the other two have, that is to say, $s - 2y$; so that he has left $4y - s$. With regard to the other two, they will each have double what they had; so that after the second game, the three persons have as follows: $A = 4x - 2s$, $B = 4y - s$, and $C = 4z$.

In the third game, C, who has now $4z$, is the loser; he loses to A, $4x - 2s$, and to B, $4y - s$; consequently, after this game, they will have:

* M. Cramer has given, at the end of his Introduction to the Analysis of Curve Lines, a very excellent rule for determining immediately, and without the necessity of passing through the ordinary operations, the value of the unknown quantities of such equations, to any number. F. T.

$$A = 8x - 4s,\ B = 8y - 2s, \text{ and } C = 8z - s.$$

Now, each having at the end of this game 24 guineas, we have three equations, the first of which immediately gives x, the second y, and the third z; farther, s is known to be 72, since the three persons have in all 72 guineas at the end of the last game; but it is not necessary to attend to this at first; since we have,

1st $8x - 4s = 24$, or $8x = 24 + 4s$, or $x = 3 + \frac{1}{2}s$;
2d $8y - 2s = 24$, or $8y = 24 + 2s$, or $y = 3 + \frac{1}{4}s$;
3d $8z - s = 24$, or $8z = 24 + s$, or $z = 3 + \frac{1}{8}s$;

and adding these three values, we have

$$x + y + z = 9 + \tfrac{7}{8}s.$$

So that, since $x + y + z = s$, we have $s = 9 + \frac{7}{8}s$; and, consequently, $\frac{1}{8}s = 9$, and $s = 72$.

If we now substitute this value of s in the expressions which we have found for x, y, and z, we shall find that, before they began to play, A had 39 guineas, B 21, and C 12.

This solution shews, that, by means of an expression for the sum of the three unknown quantities, we may overcome the difficulties which occur in the ordinary method.

617. Although the preceding question appears difficult at first, it may be resolved even without algebra, by proceeding inversely. For since the players, when they left off, had each 24 guineas, and, in the third game, A and B doubled their money, they must have had before that last game, as follows:

$$A = 12,\ B = 12, \text{ and } C = 48.$$

In the second game, A and C doubled their money; so that before that game they had;

$$A = 6,\ B = 42, \text{ and } C = 24.$$

Lastly, in the first game, A and C gained each as much money as they began with; so that at first the three persons had:

$$A = 39,\ B = 21,\ C = 12.$$

The same result as we obtained by the former solution.

618. *Question* 5. Two persons owe conjointly 29 pistoles; they have both money, but neither of them enough to enable him, singly, to discharge this common debt: the first debtor says therefore to the second, If you give me $\frac{2}{3}$ of your money, I can immediately pay the debt; and the second answers, that he also could discharge the debt, if the other would give him $\frac{3}{4}$ of his money. Required, how many pistoles each had?

Suppose that the first has x pistoles, and that the second has y pistoles.

Then we shall first have, $x + \frac{2}{3}y = 29$;

and also, $y + \frac{3}{4}x = 29$.

The first equation gives $x = 29 - \frac{2}{3}y$,

and the second $x = \frac{116-4y}{3}$;

so that $29 - \frac{2}{3}y = \frac{116-4y}{3}$.

From which equation, we obtain $y = 14\frac{1}{2}$;

Therefore $x = 19\frac{1}{3}$.

Hence the first person had $19\frac{1}{3}$ pistoles, and the second had $14\frac{1}{2}$ pistoles.

619. *Question* 6. Three brothers bought a vineyard for a hundred guineas. The youngest says, that he could pay for it alone, if the second gave him half the money which he had; the second says, that if the eldest would give him only the third of his money, he could pay for the vineyard singly; lastly, the eldest asks only a fourth part of the money of the youngest, to pay for the vineyard himself. How much money had each?

Suppose the first had x guineas; the second, y guineas; the third, z guineas; we shall then have the three following equations:

$$x + \tfrac{1}{2}y = 100;$$
$$y + \tfrac{1}{3}z = 100;$$
$$z + \tfrac{1}{4}x = 100;$$

two of which only give the value of x, namely,

$$\text{1st } x = 100 - \tfrac{1}{2}y,$$
$$\text{3d } x = 400 - 4z.$$

So that we have the equation,

$100 - \frac{1}{2}y = 400 - 4z$, or $4z - \frac{1}{2}y = 300$, which must be combined with the second, in order to determine y and z. Now, the second equation was, $y + \frac{1}{3}z = 100$: we therefore deduce from it $y = 100 - \frac{1}{3}z$; and the equation found last being $4z - \frac{1}{2}y = 300$, we have $y = 8z - 600$. The final equation, therefore, becomes

$100 - \frac{1}{3}z = 8z - 600$; so that $8\frac{1}{3}z = 700$, or $\frac{25}{3}z = 700$, and $z = 84$. Consequently,

$$y = 100 - 28 = 72, \text{ and } x = 64.$$

The youngest therefore had 64 guineas, the second had 72 guineas, and the eldest had 84 guineas.

620. As, in this example, each equation contains only two unknown quantities, we may obtain the solution required in an easier way.

The first equation gives $y = 200 - 2x$, so that y is determined by x; and if we substitute this value in the second equation, we have

$$200 - 2x + \tfrac{1}{3}z = 100;\ \text{therefore } \tfrac{1}{3}z = 2x - 100,$$
$$\text{and } z = 6x - 300.$$

So that z is also determined by x; and if we introduce this value into the third equation, we obtain $6x - 300 + \frac{1}{4}x = 100$, in which x stands alone, and which, when reduced to $25x - 1600 = 0$, gives $x = 64$. Consequently,

$$y = 200 - 128 = 72, \text{ and } z = 384 - 300 = 84.$$

621. We may follow the same method, when we have a greater number of equations. Suppose, for example, that we have in general;

$$1.\ u + \frac{x}{a} = n, \qquad 2.\ x + \frac{y}{b} = n,$$

$$3.\ y + \frac{z}{c} = n, \qquad 4.\ z + \frac{u}{d} = n;$$

or, destroying the fractions, these equations become,

$$1.\ au + x = an, \qquad 2.\ bx + y = bn,$$
$$3.\ cy + z = cn, \qquad 4.\ dz + u = dn.$$

Here, the first gives immediately $x = an - au$, and, this value being substituted in the second, we have $abn - abu + y = bn$; so that $y = bn - abn + abu$; and the substitution of this value, in the third equation, gives $bcn - abcn + abcu + z = cn$; therefore

$$z = cn - bcn + abcn - abcu.$$

Substituting this in the fourth equation, we have

$$cdn - bcdn + abcdn - abcdu + u = dn.$$

So that $dn - cdn + bcdn - abcdn = abcdu - u$,

or $(abcd - 1) . u = abcdn - bcdn + cdn - dn$; whence we have

$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = \frac{n . (abcd - bcd + cd - d)}{abcd - 1}.$$

And, consequently, by substituting this value of u in the equation, $x = an - au$, we have

$$x = \frac{abcdn - acdn + adn - an}{abcd - 1} = \frac{n . (abcd - acd + ad - a)}{abcd - 1}.$$

$$y = \frac{abcdn - abdn + abn - bn}{abcd - 1} = \frac{n.(abcd - abd + ab - b)}{abcd - 1}.$$

$$z = \frac{abcdn - abcn + bcn - cn}{abcd - 1} = \frac{n.(abcd - abc + bc - c)}{abcd - 1}.$$

$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = \frac{n.(abcd - bcd + cd - d)}{abcd - 1}.$$

622. *Question* 7. A captain has three companies, one of Swiss, another of Swabians, and a third of Saxons. He wishes to storm with part of these troops, and he promises a reward of 901 crowns, on the following condition; namely, that each soldier of the company, which assaults, shall receive 1 crown, and that the rest of the money shall be equally distributed among the two other companies. Now, it is found, that if the Swiss make the assault, each soldier of the other companies will receive half-a-crown; that, if the Swabians assault, each of the others will receive $\frac{1}{3}$ of a crown; and, lastly, if the Saxons make the assault, each of the others will receive $\frac{1}{4}$ of a crown. Required the number of men in each company?

Let us suppose the number of Swiss to be x, that of Swabians y, and that of Saxons z. And let us also make $x + y + z = s$, because it is easy to see, that, by this, we abridge the calculation considerably. If, therefore, the Swiss make the assault, their number being x, that of the other will be $s - x$: now, the former receive 1 crown, and the latter half-a-crown; so that we shall have,

$$x + \tfrac{1}{2}s - \tfrac{1}{2}x = 901.$$

In the same manner, if the Swabians make the assault, we have

$$y + \tfrac{1}{3}s - \tfrac{1}{3}y = 901.$$

And lastly, if the Saxons make the assault, we have,

$$z + \tfrac{1}{4}s - \tfrac{1}{4}z = 901.$$

Each of these three equations will enable us to determine one of the unknown quantities, x, y, and z;

For the first gives $x = 1802 - s$,
the second $2y = 2703 - s$,
the third $3z = 3604 - s$.

And if we now take the values of $6x$, $6y$, and $6z$, and write those values one above the other, we shall have

$$6x = 10812 - 6s,$$
$$6y = 8109 - 3s,$$
$$6z = 7208 - 2s,$$

and, by addition, $6s = 26129 - 11s$; or $17s = 26129$;

so that $s = 1537$; which is the whole number of soldiers. By this means we find,

$$x = 1802 - 1537 = 265;$$
$$2y = 2703 - 1537 = 1166, \text{ or } y = 583;$$
$$3x = 3604 - 1537 = 2067, \text{ or } z = 689.$$

The company of Swiss therefore has 265 men; that of Swabians 583; and that of Saxons 689.

CHAP. V.

Of the Resolution *of* Pure Quadratic Equations.

623. An equation is said to be of the second degree, when it contains the square, or the second power, of the unknown quantity, without any of its higher powers; and an equation, containing likewise the third power of the unknown quantity, belongs to cubic equations, and its resolution requires particular rules.

624. There are, therefore, only three kinds of terms in an equation of the second degree:

1. The terms in which the unknown quantity is not found at all, or which is composed only of known numbers.
2. The terms in which we find only the first power of the unknown quantity.
3. The terms which contain the square, or the second power, of the unknown quantity.

So that x representing an unknown quantity, and the letters a, b, c, d, &c. the known quantities, the terms of the first kind will have the form a, the terms of the second kind will have the form bx, and the terms of the third kind will have the form cx^2.

625. We have already seen, how two or more terms of the same kind may be united together, and considered as a single term.

For example, we may consider the formula
$ax^2 - bx^2 + cx^2$ as a single term, representing it thus,
$(a - b + c)x^2$; since, in fact, $(a - b + c)$ is a known quantity.

And also, when such terms are found on both sides of the sign $=$, we have seen how they may be brought to one side,