CHAP. XVI.

Of the Resolution *of* Equations *by* Approximation.

784. When the roots of an equation are not rational, whether they may be expressed by radical quantities, or even if we have not that resource, as is the case with equations which exceed the fourth degree, we must be satisfied with determining their values by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at last the error being very small, it may be neglected. Different methods of this kind have been proposed, the chief of which we shall explain.

785. The first method which we shall mention, supposes that we have already determined, with tolerable exactness, the value of one root *; that we know, for example, that such a value exceeds 4, and that it is less than 5. In this case, if we suppose this value $= 4 + p$, we are certain that p expresses a fraction. Now, as p is a fraction, and consequently less than unity, the square of p, its cube, and, in general, all the higher powers of p, will be much less with respect to unity; and, for this reason, since we require only an approximation, they may be neglected in the calculation. When we have, therefore, nearly determined the fraction p, we shall know more exactly the root $4 + p$; from that we proceed to determine a new value still more exact, and continue the same process till we come as near the truth as we desire.

786. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2 = 20$.

Here we perceive, that x is greater than 4 and less than 5; making, therefore, $x = 4 + p$, we shall have $x^2 = 16 + 8p + p^2 = 20$; but as p^2 must be very small, we shall neglect it, in order that we may have only the equation 16 +

* This is the method given by Sir Is. Newton at the beginning of his Method of Fluxions. When investigated, it is found subject to different imperfections; for which reason we may with advantage substitute the method given by M. de la Grange, in the Memoirs of Berlin for 1767 and 1768. F. T.

This method has since been published by De la Grange, in a separate Treatise, where the subject is discussed in the usual masterly style of this author.

$8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p'$; we are sure that p' expresses a fraction much smaller than before, and that we may neglect p'^2 with greater propriety. We have, therefore, $x^2 = 20\frac{1}{4} + 9p' = 20$, or $9p' = -\frac{1}{4}$; and consequently, $p' = -\frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.

And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p''$, and should thus have $x^2 = 20\frac{1}{1296} + 8\frac{34}{36}p'' = 20$; so that $8\frac{34}{36}p'' = -\frac{1}{1296}$, $322p' = -\frac{36}{1296} = -\frac{1}{36}$, and

$$p = -\frac{1}{36 \times 322} = -\frac{1}{11592}:$$

therefore $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

787. Now, in order to generalise what we have here laid down, let us suppose the given equation to be $x^2 = a$, and that we previously know x to be greater than n, but less than $n + 1$. If we now make $x = n + p$, p must be a fraction, and p^2 may be neglected as a very small quantity, so that we shall have $x^2 = n^2 + 2np = a$; or $2np = a - n^2$, and $p = \frac{a-n^2}{2n}$; consequently, $x = n + \frac{a-n^2}{2n} = \frac{n^2+a}{2n}$.

Now, if n approximated towards the true value, this new value $\frac{n^2+a}{2n}$ will approximate much nearer; and, by substituting it for n, we shall find the result much nearer the truth; that is, we shall obtain a new value, which may again be substituted, in order to approach still nearer; and the same operation may be continued as long as we please.

For example, let $x^2 = 2$; that is to say, let the square root of 2 be required; and as we already know a value sufficiently near, which is expressed by n, we shall have a still nearer value of the root expressed by $\frac{n^2+2}{2n}$. Let therefore,

1. $n = 1$, and we shall have $x = \frac{3}{2}$,
2. $n = \frac{3}{2}$, and we shall have $x = \frac{17}{12}$,
3. $n = \frac{17}{12}$, and we shall have $x = \frac{577}{408}$.

This last value approaches so near $\sqrt{2}$, that its square $\frac{332929}{166464}$ differs from the number 2 only by the small quantity $\frac{1}{166464}$, by which it exceeds it.

788. We may proceed in the same manner, when it is

required to find by approximation cube roots, biquadrate roots, &c.

Let there be given the equation of the third degree, $x^3 = a$; or let it be proposed to find the value of $\sqrt[3]{a}$.

Knowing that it is nearly n, we shall suppose $x = n + p$; neglecting p^2 and p^3, we shall have $x^3 = n^3 + 3n^2p = a$; so that $3n^2p = a - n^3$, and $p = \frac{a - n^3}{3n^2}$; whence $x = \frac{2n^3 + a}{3n^2}$. If, therefore, n is nearly $= \sqrt[3]{a}$, the quantity which we have now found will be much nearer it. But for still greater exactness, we may again substitute this new value for n, and so on.

For example, let $x^3 = 2$; and let it be required to determine $\sqrt[3]{2}$. Here, if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly; let us therefore make

1. $n = 1$, and we shall have $x = \frac{4}{3}$,
2. $n = \frac{4}{3}$, and we shall have $x = \frac{91}{72}$,
3. $n = \frac{91}{72}$, and we shall have $x = \frac{162130896}{128634294}$.

789. This method of approximation may be employed, with the same success, in finding the roots of all equations.

To shew this, suppose we have the general equation of the third degree, $x^3 + ax^2 + bx + c = 0$, in which n is very nearly the value of one of the roots. Let us make $x = n - p$; and, since p will be a fraction, neglecting the powers of this letter, which are higher than the first degree, we shall have $x^2 = n^2 - 2np$, and $x^3 = n^3 - 3n^2p$; whence we have the equation $n^3 - 3n^2p + an^2 - 2anp + bn - bp + c = 0$, or $n^3 + an^2 + bn + c = 3n^2p + 2anp + bp = (3n^2 + 2an + b)p$; so that $p = \frac{n^3 + an^2b + n - c}{3n^2 + 2an + b}$, and $x = n - (\frac{n^3 + an^2 + bn + c}{3n^2 + 2an + b}) = \frac{2n^3 + an^2 - c}{3n^2 + 2an + b}$. This value, which is more exact than the first, being substituted for n, will furnish a new value still more accurate.

790. In order to apply this operation to an example, let $x^3 + 2x^2 + 3x - 50 = 0$, in which $a = 2$, $b = 3$, and $c = -50$. If n is supposed to be nearly the value of one of the roots, $x = \frac{2n^3 + 2n^2 + 50}{3n^2 + 4n + 3}$, will be a value still nearer the truth.

Now, the assumed value of $x = 3$ not being far from the true one, we shall suppose $n = 3$, which gives us $x = \frac{62}{21}$; and if we were to substitute this new value instead of n, we should find another still more exact.

791. We shall give only the following example, for equations of higher dimensions than the third.

Let $x^5 = 6x + 10$, or $x^5 - 6x - 10 = 0$, where we readily perceive that 1 is too small, and that 2 is too great. Now, if $x = n$ is a value not far from the true one, and we make $x = n + p$, we shall have $x^5 = n^5 + 5n^4p$; and, consequently,

$$n^5 + 5n^4p = 6n + 6p + 10;\ \text{or}$$
$$p(5n^4 - 6) = 6n + 10 - n^5.$$

Wherefore $p = \frac{6n+10-n^5}{5n^4-6}$, and $x = \frac{4n^5+10}{5n^4-6}$. If we suppose $n = 1$, we shall have $x = \frac{14}{-1} = -14$; this value is altogether inapplicable, a circumstance which arises from the approximated value of n having been taken by much too small. We shall therefore make $n = 2$, and shall thus obtain $x = \frac{138}{74} = \frac{69}{37}$, a value which is much nearer the truth. And if we were now to substitute for n, the fraction $\frac{69}{37}$, we should obtain a still more exact value of the root x.

792. Such is the most usual method of finding the roots of an equation by approximation, and it applies successfully to all cases.

We shall however explain another method *, which deserves attention, on account of the facility of the calculation. The foundation of this method consists in determining for each equation a series of numbers, as a, b, c, &c. such, that each term of the series, divided by the preceding one, may express the value of the root with so much the more exactness, according as this series of numbers is carried to a greater length.

Suppose we have already got the terms p, q, r, s, t, &c.

* The theory of approximation here given, is founded on the theory of what are called *recurring* series, invented by M. de Moivre. This method was given by Daniel Bernoulli, in vol. iii. of the Ancient Commentaries of Petersburg. But Euler has here presented it in rather a different point of view. Those who wish to investigate these matters, may consult chapters 13 and 17 of vol. i. of our author's Introd. in Anal. Infin.; an excellent work, in which several subjects treated of in this first part, beside others equally connected with pure mathematics, are profoundly analysed and clearly explained. F. T.

$\frac{q}{p}$ must express the root x with tolerable exactness; that is to say, we have $\frac{q}{p} = x$ nearly. We shall have also $\frac{r}{q} = x$ *, and the multiplication of the two values will give $\frac{r}{p} = x^2$. Farther as $\frac{s}{r} = x$, we shall also have $\frac{s}{p} = x^3$; then, since $\frac{t}{s} = x$, we shall have $\frac{t}{p} = x^4$, and so on.

793. For the better explanation of this method, we shall begin with an equation of the second degree, $x^2 = x + 1$, and shall suppose that in the above series we have found the terms p, q, r, s, t, &c. Now, as $\frac{q}{p} = x$, and $\frac{r}{p} = x^2$, we shall have the equation $\frac{r}{p} = \frac{q}{p} + 1$, or $q + p = r$. And as we find, in the same manner, that $s = r + q$, and $t = s + r$; we conclude that each term of our series is the sum of the two preceding terms; so that having the first two terms, we can easily continue the series to any length. With regard to the first two terms, they may be taken at pleasure: if we therefore suppose them to be 0, 1, our series will be 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c. and such, that if we divide any term by that which immediately precedes it, we shall have a value of x so much nearer the true one, according as we have chosen a term more distant. The error, indeed, is very great at first, but it diminishes as we advance. The series of those values of x, in the order in which they are always approximating towards the true one, is as follows:

$$x = \tfrac{1}{0}, \tfrac{1}{1}, \tfrac{2}{1}, \tfrac{3}{2}, \tfrac{5}{3}, \tfrac{8}{5}, \tfrac{13}{8}, \tfrac{21}{13}, \tfrac{34}{21}, \tfrac{55}{34}, \tfrac{89}{55}, \tfrac{144}{89}, \text{\&c.}$$

If, for example, we make $x = \frac{21}{13}$, we have $\frac{441}{169} = \frac{21}{13} + 1 = \frac{442}{169}$, in which the error is only $\frac{1}{169}$. Any of the succeeding terms will render it still less.

794. Let us also consider the equation $x^2 = 2x + 1$; and since, in all cases, $x = \frac{q}{p}$, and $x^2 = \frac{r}{p}$, we shall have

* It must only be understood here that $\frac{r}{q}$ is nearly equal to x.

$\frac{r}{p} = \frac{2q}{p} + 1$, or $r = 2q + p$; whence we infer that the double of each term, added to the preceding term, will give the succeeding one. If, therefore, we begin again with 0, 1, we shall have the series,

0, 1, 2, 5, 12, 29, 70, 169, 408, &c.

Whence it follows, that the value of x will be expressed still more accurately by the following fractions:

$$x = \tfrac{1}{0}, \tfrac{2}{1}, \tfrac{5}{2}, \tfrac{12}{5}, \tfrac{29}{12}, \tfrac{70}{29}, \tfrac{169}{70}, \tfrac{408}{169}, \&c.$$

which, consequently, will always approximate nearer and nearer the true value of $x = 1 + \sqrt{2}$; so that if we take unity from these fractions, the value of $\sqrt{2}$ will be expressed more and more exactly by the succeeding fractions:

$$\tfrac{1}{0}, \tfrac{1}{1}, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}, \tfrac{41}{29}, \tfrac{99}{70}, \tfrac{239}{169}, \&c.$$

For example, $\frac{99}{70}$ has for its square $\frac{9801}{4900}$, which differs only by $\frac{1}{4900}$ from the number 2.

795. This method is no less applicable to equations, which have a greater number of dimensions. If, for example, we have the equation of the third degree $x^3 = x^2 + 2x + 1$, we must make $x = \frac{q}{p}$, $x^2 = \frac{r}{p}$, and $x^3 = \frac{s}{p}$; we shall then have $s = r + 2q + p$; which shews how, by means of the three terms p, q, and r, we are to determine the succeeding one, s; and, as the beginning is always arbitrary, we may form the following series:

0, 0, 1, 1, 3, 6, 13, 28, 60, 129, &c.

from which result the following fractions for the approximate values of x:

$$x = \tfrac{0}{0}, \tfrac{1}{0}, \tfrac{1}{1}, \tfrac{3}{1}, \tfrac{6}{3}, \tfrac{13}{6}, \tfrac{28}{13}, \tfrac{60}{28}, \tfrac{129}{60}, \&c.$$

The first of these values would be very far from the truth; but if we substitute in the equation $\frac{60}{28}$, or $\frac{15}{7}$, instead of x, we obtain

$$\tfrac{3375}{343} = \tfrac{225}{49} + \tfrac{30}{7} + 1 = \tfrac{3388}{343},$$

in which the error is only $\frac{13}{343}$.

796. It must be observed, however, that all equations are not of such a nature as to admit the application of this method; and, particularly, when the second term is wanting, it cannot be made use of. For example, let $x^2 = 2$; if we wished to make $x = \frac{q}{p}$, and $x^2 = \frac{r}{p}$, we should have

$\frac{r}{p} = 2$, or $r = 2p$, that is to say, $r = 0q + 2p$, whence would result the series

$$1, 1, 2, 2, 4, 4, 8, 8, 16, 16, 32, 32, \&c.$$

from which we can draw no conclusion, because each term, divided by the preceding, gives always $x = 1$, or $x = 2$. But we may obviate this inconvenience, by making $x = y - 1$; for by these means we have $y^2 - 2y + 1 = 2$; and if we now make $y = \frac{q}{p}$, and $y^2 = \frac{r}{p}$, we shall obtain the same approximation that has been already given.

797. It would be the same with the equation $x^3 = 2$. This method would not furnish such a series of numbers as would express the value of $\sqrt[3]{2}$. But we have only to suppose $x = y - 1$, in order to have the equation $y^3 - 3y^2 + 3y - 1 = 2$, or $y^3 = 3y^2 - 3y + 3$; and then making $y = \frac{q}{p}$, $y^2 = \frac{r}{p}$, and $y^3 = \frac{s}{p}$, we have $s = 3r - 3q + 3p$, by means of which we see how three given terms determine the succeeding one.

Assuming then any three terms for the first, for example 0, 0, 1, we have the following series:

$$0, 0, 1, 3, 6, 12, 27, 63, 144, 324, \&c.$$

The last two terms of this series give $y = \frac{324}{144}$ and $x = \frac{5}{4}$. This fraction approaches sufficiently near the cube root of 2; for the cube of $\frac{5}{4}$ is $\frac{125}{64}$, and $2 = \frac{128}{64}$.

798. We must farther observe, with regard to this method, that when the equation has a rational root, and the beginning of the period is chosen such, that this root may result from it, each term of the series, divided by the preceding term, will give the root with equal accuracy.

To shew this, let there be given the equation $x^2 = x + 2$, one of the roots of which is $x = 2$; as we have here, for the series, the formula $r = q + 2p$, if we take 1, 2, for the first two terms, we have the series 1, 2, 4, 8, 16, 32, 64, &c. a geometrical progression, whose exponent $= 2$. The same property is proved by the equation of the third degree $x^3 = x^2 + 3x + 9$, which has $x = 3$ for one of the roots. If we suppose the first terms to be 1, 3, 9, we shall find, by the formula, $s = r + 3q + 9p$, and the series 1, 3, 9, 27, 81, 243, &c. which is likewise a geometrical progression.

799. But if the beginning of the series exceed the root, we shall not approximate towards that root at all; for when the equation has more than one root, the series gives by approximation only the greatest: and we do not find one of the less roots, unless the first terms have been properly chosen for that purpose. This will be illustrated by the following example.

Let there be given the equation $x^2 = 4x - 3$, whose two roots are $x = 1$, and $x = 3$. The formula for the series is $r = 4q - 3p$, and if we take 1, 1, for the first two terms of the series, which consequently expresses the least root, we have for the whole series, 1, 1, 1, 1, 1, 1, 1, 1, &c. but assuming for the leading terms the numbers 1, 3, which contain the greatest root, we have the series, 1, 3, 9, 27, 81, 243, 729, &c. in which all the terms express precisely the root 3. Lastly, if we assume any other beginning, provided it be such that the least term is not comprised in it, the series will continually approximate towards the greatest root 3; which may be seen by the following series:

Beginning,

0, 1, 4, 13, 40, 121, 364, &c.
1, 2, 5, 14, 41, 122, 365, &c.
2, 3, 6, 15, 42, 123, 366, 1095, &c.
2, 1, −2, −11, −38, −118, −362, −1091, −3278, &c.

in which the quotients of the division of the last terms by the preceding always approximate towards the greater root 3, and never towards the less.

800. We may even apply this method to equations which go on to infinity. The following will furnish an example:

$$x^\infty = x^{\infty-1} + x^{\infty-2} + x^{\infty-3} + x^{\infty-4} +, \text{\&c.}$$

The series for this equation must be such, that each term may be equal to the sum of all the preceding; that is, we must have

1, 1, 2, 4, 8, 16, 32, 64, 128, &c.

whence we see that the greater root of the given equation is exactly $x = 2$; and this may be shewn in the following manner. If we divide the equation by x^∞, we shall have

$$1 = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} +, \text{\&c.}$$

a geometrical progression, whose sum is found $= \frac{1}{x-1}$; so that $1 = \frac{1}{x-1}$; multiplying therefore by $x = 1$, we have $x - 1 = 1$, and $x = 2$.

801. Beside these methods of determining the roots of an equation by approximation, some others have been invented, but they are all either too tedious, or not sufficiently general*. The method which deserves the preference to all others, is

* This remark does not apply to the method of finding the roots of equations of all degrees, and however affected, by The Rule of Double Position. In order, therefore, that this chapter might be more complete, we shall explain this method as briefly as possible.

Substitute in the given equation two numbers, as near the true root as possible, and observe the separate results. Then, as the difference of these results is to the difference of the two numbers; so is the difference between the true result, and either of the former, to the respective correction of each. This being added to the number, when too small, or substracted from it, when too great, will give the true root nearly.

The number thus found, with any other that may be supposed to approach still nearer to the true root, may be assumed for another operation, which may be repeated, till the root shall be determined to any degree of exactness that may be required.

Example. Given $x^3 + x^2 + x = 100$.

Having ascertained by a few trials, or by inspecting a Table of roots and powers, that x is more than 4, and less than 5, let us substitute these two numbers in the given equation, and calculate the results.

By the first supposition	$x = 4$, $x^2 = 16$, $x^3 = 64$	By the second supposition	$x = 5$, $x^2 = 25$, $x^3 = 125$	
	84	. . . Results	155	
	155	5	100	true result.
	84	4	84	
Differences	71	1	16	
Then,	As 71 :	1	:: 16 : ·2253 +	

Therefore 4 + ·2253, or 4·2253 approximates nearly to the true root.

If now 4·2 and 4·3 were taken as the assumed numbers, and substituted in the given equation, we should obtain the value of $x = 4{\cdot}264$ very nearly.

that which we explained first; for it applies successfully to to all kinds of equations: whereas the other often requires the equation to be prepared in a certain manner, without which it cannot be employed; and of this we have seen a proof in different examples.

QUESTIONS FOR PRACTICE.

1. Given $x^3 + 2x^2 - 23x - 70 = 0$, to find x.
Ans. $x = 5{\cdot}13450$.
2. Given $x^3 - 15x^2 + 63x - 50 = 0$, to find x.
Ans. $x = 1{\cdot}028039$.
3. Given $x^4 - 3x^2 - 75x = 10000$, to find x.
Ans. $x = 10{\cdot}2615$.
4. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find x.
Ans. $x = 8{\cdot}4144$.
5. Let $120x^3 + 3657x^2 - 38059x = 8007115$, to find x.
Ans. $x = 34{\cdot}6532$.

END OF PART I.