last two are $x = 2 \pm \sqrt{8}$; so that the four roots sought will be,

$$1.\ x = 1 + \sqrt{5}, \qquad 2.\ x = 1 - \sqrt{5},$$
$$3.\ x = 2 + \sqrt{8}, \qquad 4.\ x = 2 - \sqrt{8}.$$

Consequently, the four factors of our equation will be $(x - 1 - \sqrt{5}) \times (x - 1 + \sqrt{5}) \times (x - 2 - \sqrt{8}) \times (x - 2 + \sqrt{8})$, and their actual multiplication produces the given equation; for the first two being multiplied together, give $x^2 - 2x - 4$, and the other two give $x^2 - 4x - 4$: now, these products multiplied together, make $x^4 - 6x^3 + 24x + 16$, which is the same equation that was proposed.

CHAP. XIV.

Of the Rule *of* Bombelli *for reducing the* Resolution *of* Equations *of the* Fourth Degree *to that of* Equations *of the* Third Degree.

765. We have already shewn how equations of the third degree are resolved by the rule of Cardan; so that the principal object, with regard to equations of the fourth degree, is to reduce them to equations of the third degree. For it is impossible to resolve, generally, equations of the fourth degree, without the aid of those of the third; since, when we have determined one of the roots, the others always depend on an equation of the third degree. And hence we may conclude, that the resolution of equations of higher dimensions presupposes the resolution of all equations of lower degrees.

766. It is now some centuries since Bombelli, an Italian, gave a rule for this purpose, which we shall explain in this chapter *.

Let there be given the general equation of the fourth degree, $x^4 + ax^3 + bx^2 + cx + d = 0$, in which the letters a, b, c, d, represent any possible numbers; and let us suppose that this equation is the same as $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, in which it is required to determine the letters p, q, and r, in order that we may obtain the equation

* This rule rather belongs to Louis Ferrari. It is improperly called the Rule of Bombelli, in the same manner as the rule discovered by Scipio Ferreo has been ascribed to Cardan. F. T.

proposed. By squaring, and ordering this new equation, we shall have

$$\begin{aligned} x^4 + ax^3 + \tfrac{1}{4}a^2x^2 &+ apx + p^2 \\ 2px^2 &- 2qrx - r^2 \\ - q^2x^2&. \end{aligned}$$

Now, the first two terms are already the same here as in the given equation; the third term requires us to make $\frac{1}{4}a^2 + 2p - q^2 = b$, which gives $q^2 = \frac{1}{4}a^2 + 2p - b$; the fourth term shews that we must make $ap - 2qr = c$, or $2qr = ap - c$; and, lastly, we have for the last term $p^2 - r^2 = d$, or $r^2 = p^2 - d$. We have therefore three equations which will give the values of p, q, and r.

767. The easiest method of deriving those values from them is the following: if we take the first equation four times, we shall have $4q^2 = a^2 + 8p - 4b$; which equation, multiplied by the last, $r^2 = p^2 - d$, gives

$$4q^2r^2 = 8p^3 + (a^2 - 4b)p^2 - 8dp - d(a^2 - 4b).$$

Farther, if we square the second equation, we have $4q^2r^2 = a^2p^2 - 2acp + c^2$. So that we have two values of $4q^2r^2$, which, being made equal, will furnish the equation

$$8p^3 + (a^2 - 4b)p^2 - 8dp - d(a^2 - 4b) = a^2p^2 - 2acp + c^2,$$

or, bringing all the terms to one side, and arranging,

$$8p^3 - 4bp^2 + (2ac - 8d)p - a^2d + 4bd - c^2 = 0,$$

an equation of the third degree, which will always give the value of p by the rules already explained.

768. Having therefore determined three values of p by the given quantities a, b, c, d, when it was required to find only one of those values, we shall also have the values of the two other letters q and r; for the first equation will give $q = \sqrt{(\frac{1}{4}a^2 + 2p - b)}$, and the second gives $r = \dfrac{ap - c}{2q}$.

Now, these three values being determined for each given case, the four roots of the proposed equation may be found in the following manner:

This equation having been reduced to the form $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, we shall have

$$(x^2 + \tfrac{1}{2}ax + p)^2 = (qx + r)^2,$$

and, extracting the root, $x^2 + \frac{1}{2}ax + p = qx + r$, or $x^2 + \frac{1}{2}ax + p = -qx - r$. The first equation gives $x^2 = (q - \frac{1}{2}a)x - p + r$, from which we may find two roots; and the second equation, to which we may give the form $x^2 = -(q + \frac{1}{2}a)x - p - r$, will furnish the two other roots.

769. Let us illustrate this rule by an example, and suppose that the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$$

was given. If we compare it with our general formula (at the end of Art. 767.), we have $a = -10$, $b = 35$, $c = -50$, $d = 24$; and, consequently, the equation which must give the value of p is

$$8p^3 - 140p^2 + 808p - 1540 = 0, \text{ or}$$
$$2p^3 - 35p^2 + 202p - 385 = 0.$$

The divisors of the last term are 1, 5, 7, 11, &c.; the first of which does not answer; but making $p = 5$, we get $250 - 875 + 1010 - 385 = 0$, so that $p = 5$; and if we farther suppose $p = 7$, we get $686 - 1715 + 1414 - 385 = 0$, a proof that $p = 7$ is the second root. It remains now to find the third root; let us therefore divide the equation by 2, in order to have $p^3 - \frac{35}{2}p^2 + 101p - \frac{385}{2} = 0$, and let us consider that the coefficient of the second term, or $\frac{35}{2}$, being the sum of all the three roots, and the first two making together 12, the third must necessarily be $\frac{11}{2}$.

We consequently know the three roots required. But it may be observed that one would have been sufficient, because each gives the same four roots for our equation of the fourth degree.

770. To prove this, let $p = 5$; we shall then have, by the formula, $\sqrt{(\frac{1}{4}a^2 + 2p - b)}$, $q = \sqrt{(25 + 10 - 35)} = 0$, and $r = \dfrac{-50+50}{0} = \frac{0}{0}$. Now, nothing being determined by this, let us take the third equation,

$$r^2 = p^2 - d = 25 - 24 = 1,$$

so that $r = 1$; our two equations of the second degree will then be:

$$1.\ x^2 = 5x - 4, \qquad 2.\ x^2 = 5x - 6.$$

The first gives the two roots

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{9}{4}}, \text{ or } x = \frac{5 \pm 3}{2},$$

that is to say, $x = 4$ and $x = 1$.

The second equation gives

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{1}{4}} = \frac{5 \pm 1}{2},$$

that is to say, $x = 3$, and $x = 2$.

But suppose now $p = 7$, we shall have

$$q = \sqrt{(25 + 14 - 35)} = 2, \text{ and } r = \frac{-70+50}{4} = -5,$$

whence result the two equations of the second degree,

$$1.\ x^2 = 7x - 12, \qquad 2.\ x^2 = 3x - 2;$$

the first gives $x = \frac{7}{2} \pm \sqrt{\frac{1}{4}}$, or $x = \frac{7 \pm 1}{2}$,

so that $x = 4$, and $x = 3$; the second furnishes the root

$$x = \tfrac{3}{2} \pm \sqrt{\tfrac{1}{4}} = \frac{3 \pm 1}{2},$$

and, consequently, $x = 2$, and $x = 1$; therefore by this second supposition the same four roots are found as by the first.

Lastly, the same roots are found, by the third value of p, $= \frac{11}{2}$: for, in this case, we have

$$q = \sqrt{(25 + 11 - 35)} = 1, \text{ and } r = \frac{-55+50}{2} = -\tfrac{5}{2};$$

so that the two equations of the second degree become,

$$1.\ x^2 = 6x, \qquad 2.\ x^2 = 4x - 3.$$

Whence we obtain from the first, $x = 3 \pm \sqrt{1}$, that is to say, $x = 4$, and $x = 2$; and from the second, $x = 2 \pm \sqrt{1}$, that is to say, $x = 3$, and $x = 1$, which are the same roots that we originally obtained.

771. Let there now be proposed the equation

$$x^4 - 16x - 12 = 0,$$

in which $a = 0$, $b = 0$, $c = -16$, $d = -12$; and our equation of the third degree will be

$$8p^3 + 96p - 256 = 0, \text{ or } p^3 + 12p - 32 = 0,$$

and we may make this equation still more simple, by writing $p = 2t$; for we have then

$$8t^3 + 24t - 32 = 0, \text{ or } t^3 + 3t - 4 = 0.$$

The divisors of the last term are 1, 2, 4; whence one of the roots is found to be $t = 1$; therefore $p = 2$, $q = \sqrt{4} = 2$, and $r = \frac{16}{4} = 4$. Consequently, the two equations of the second degree are

$$x^2 = 2x + 2, \text{ and } x^2 = -2x - 6;$$

which give the roots

$$x = 1 \pm \sqrt{3}, \text{ and } x = -1 \pm \sqrt{-5}.$$

772. We shall endeavour to render this resolution still more familiar, by a repetition of it in the following example. Suppose there were given the equation

$$x^4 - 6x^3 + 12x^2 - 12x + 4 = 0,$$

which must be contained in the formula

$$(x^2 - 3x + p)^2 - (qx + r)^2 = 0,$$

in the former part of which we have put $-3x$, because -3 is half the coefficient -6, of the given equation. This formula being expanded, gives

$$x^4 - 6x^3 + (2p + 9 - q^2)x^2 - (6p + 2qr)x + p^2 - r^2 = 0;$$

which, compared with our equation, there will result from that comparison the following equations:

1. $2p + 9 - q^2 = 12$,
2. $6p + 2qr = 12$,
3. $p^2 - r^2 = 4$.

The first gives $q^2 = 2p - 3$;
the second, $2qr = 12 - 6p$, or $qr = 6 - 3p$;
the third, $r^2 = p^2 - 4$.

Multiplying r^2 by q^2, and $p^2 - 4$ by $2p - 3$, we have

$$q^2r^2 = 2p^3 - 3p^2 - 8p + 12;$$

and if we square the value of qr, we have

$$q^2r^2 = 36 - 36p + 9p^2;$$

so that we have the equation

$$2p^3 - 3p^2 - 8p + 12 = 9p^2 - 36p + 36, \text{ or}$$
$$2p^3 - 12p^2 + 28p - 24 = 0, \text{ or}$$
$$p^3 - 6p^2 + 14p - 12 = 0,$$

one of the roots of which is $p = 2$; and it follows that $q^2 = 1$, $q = 1$, and $qr - r = 0$. Therefore our equation will be $(x^2 - 3x + 2)^2 = x^2$, and its square root will be $x^2 - 3x + 2 = \pm x$. If we take the upper sign, we have $x^2 = 4x - 2$; and taking the lower sign, we obtain $x^2 = 2x - 2$, whence we derive the four roots $x = 2 \pm \sqrt{2}$, and $x = 1 \pm \sqrt{-1}$.

CHAP. XV.

Of a new Method *of resolving* Equations *of the* Fourth Degree.

773. The rule of Bombelli, as we have seen, resolves equations of the fourth degree by means of an equation of the third degree; but since the invention of that Rule,