gression, having 10 for its first term, $\frac{1}{3}$ for the common difference, and the number of terms 21. *Ans.* 140.

3. Required the number of all the strokes of a clock in twelve hours, that is, a complete revolution of the index. *Ans.* 78.

4. The clocks of Italy go on to 24 hours; how many strokes do they strike in a complete revolution of the index? *Ans.* 300.

5. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them one by one to a basket, which is placed one yard from the first stone? *Ans.* 5 *miles* and 1300 *yards*.

CHAP. V.

Of Figurate *, *or* Polygonal Numbers.

425. The summation of arithmetical progressions, which begin by 1, and the difference of which is 1, 2, 3, or any

* The French translator has justly observed, in his note at the conclusion of this chapter, that algebraists make a distinction between figurate and polygonal numbers; but as he has not entered far upon this subject, the following illustration may not be unacceptable.

It will be immediately perceived in the following Table, that each series is derived immediately from the foregoing one, being the sum of all its terms from the beginning to that place; and hence also the law of continuation, and the general term of each series, will be readily discovered.

Natural	1, 2, 3, 4, 5 - -	n general term
Triangular	1, 3, 6, 10, 15 - -	$\frac{n.(n+1)}{2}$
Pyramidal	1, 4, 10, 20, 35 - -	$\frac{n.(n+1).(n+2)}{2.3}$
Triangular-pyramidal	1, 5, 15, 35, 70 - -	$\frac{n.(n+1).(n+2).(n+3)}{2.3.4}$

And, in general, the figurate number of any order m will be expressed by the formula,

$$\frac{n.(n+1)}{1.2} \cdot \frac{(n+2)}{3} \cdot \frac{(n+3)}{4} - - \frac{(n+m-1)}{m}.$$

Now, one of the principal properties of these numbers, and other integer, leads to the theory of *polygonal numbers*, which are formed by adding together the terms of any such progression.

426. Suppose the difference to be 1; then, since the first term is 1 also, we shall have the arithmetical progression, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c. and if in this progression we take the sum of one, of two, of three, &c. terms, the following series of numbers will arise:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, &c.

for $1=1$, $1+2=3$, $1+2+3=6$, $1+2+3+4=10$, &c.

Which numbers are called *triangular*, or *trigonal* numbers, because we may always arrange as many points in the form of a triangle as they contain units, thus:

1	3	6	10	15
.	. .	. . .	. . . .	. . . . .
	.	. .	. . .	. . . .
		.	. .	. . .
			.	. .
				.

427. In all these triangles, we see how many points each side contains. In the first triangle there is only one point; in the second there are two; in the third there are three; in the fourth there are four, &c.: so that the triangular numbers, or the number of points, which is simply called the *triangle*, are arranged according to the number of points which the side contains, which number is called the *side*; that is, the third triangular number, or the third triangle, is that whose side has three points; the fourth, that whose side has four; and so on; which may be represented thus:

which Fermat considered as very interesting, (*see his notes on Diophantus, page* 16), is this: that if from the *n*th term of any series the $(n-1)$ term of the same series be subtracted, the remainder will be the *n*th term of the preceding series. Thus, in the third series above given, the *n*th term is $\frac{n.(n+1).(n+2)}{2.3}$; consequently, the $(n-1)$ term, by substituting $(n-1)$ instead of n, is $\frac{(n-1).n.(n+1)}{2.3}$; and if the latter be subtracted from the former, the remainder is $\frac{n.(n-1)}{2}$, which is the *n*th term of the preceding order of numbers. The same law will be observed between two consecutive terms of any one of these sums.

428. A question therefore presents itself here, which is, how to determine the triangle when the side is given? and, after what has been said, this may be easily resolved.

For if the side be n, the triangle will be $1+2+3+4+\text{---}n$. Now, the sum of this progression is $\frac{n^2+n}{2}$; consequently the value of the triangle is $\frac{n^2+n}{2}$ *.

$$\text{Thus, if } \begin{cases} n=1, \\ n=2, \\ n=3, \\ n=4, \end{cases} \text{the triangle is } \begin{cases} 1, \\ 3, \\ 6, \\ 10, \end{cases}$$

and so on: and when $n=100$, the triangle will be 5050.

429. This formula $\frac{n^2+n}{2}$ is called the general formula of triangular numbers; because by it we find the triangular number, or the triangle, which answers to any side indicated by n.

This may be transformed into $\frac{n(n+1)}{2}$; which serves also to facilitate the calculation; since one of the two numbers n, or $n+1$, must always be an even number, and consequently divisible by 2.

So, if $n=12$, the triangle is $\frac{12\times13}{2}=6\times13=78$; and if $n=15$, the triangle is $\frac{15\times16}{2}=15\times8=120$, &c.

430. Let us now suppose the difference to be 2, and we shall have the following arithmetical progression:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

the sums of which, taking successively one, two, three, four terms, &c. form the following series:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, &c.

* M. de Joncourt published at the Hague, in 1762, a Table of trigonal numbers answering to all the natural numbers from 1 to 20000; which Tables are found useful in facilitating a great number of arithmetical operations, as the author shews in a very long introduction. F. T.

the terms of which are called *quadrangular* numbers, or *squares;* since they represent the squares of the natural numbers, as we have already seen; and this denomination is the more suitable from this circumstance, that we can always form a square with the number of points which those terms indicate, thus:

1, 4, 9, 16, 25,

431. We see here, that the side of any square contains precisely the number of points which the square root indicates. Thus, for example, the side of the square 16 consists of 4 points; that of the square 25 consists of 5 points; and, in general, if the side be n, that is, if the number of the terms of the progression, 1, 3, 5, 7, &c. which we have taken, be expressed by n, the square, or the quadrangular number, will be equal to the sum of those terms; that is to n^2, as we have already seen, Article 422; but it is unnecessary to extend our consideration of square numbers any farther, having already treated of them at length.

432. If now we call the difference 3, and take the sums in the same manner as before, we obtain numbers which are called *pentagons,* or *pentagonal* numbers, though they cannot be so well represented by points *.

* It is not, however, that we are unable to represent, by points, polygons of any number of sides; but the rule which I am going to explain for this purpose, seems to have escaped all the writers on algebra whom I have consulted.

I begin with drawing a small polygon that has the number of sides required; this number remains constant for one and the same series of polygonal numbers, and it is equal to 2 *plus* the difference of the arithmetical progression from which the series is produced. I then choose one of its angles, in order to draw from the angular point all the diagonals of this polygon, which, with the two sides containing the angle that has been taken, are to be indefinitely produced; after that, I take these two sides, and the diagonals of the first polygon on the indefinite lines, each as often as I choose; and draw, from the corresponding points marked by the compass, lines parallel to the sides of the first polygon; and divide them into as many equal parts, or by as many points as there are actually in the diagonals and the two sides produced. This rule is general, from the triangle up to the polygon of an infinite number of sides: and the division

Indices,	1	2	3	4	5	6	7	8	9	&c.
Arith. Prog.	1,	4,	7,	10,	13,	16,	19,	22,	25,	&c.
Pentagon,	1,	5,	12,	22,	35,	51,	70,	92,	117,	&c.

the indices shewing the side of each pentagon.

433. It follows from this, that if we make the side n, the pentagonal number will be $\frac{3n^2-n}{2}=\frac{n(3n-1)}{2}$.

Let, for example, $n=7$, the pentagon will be 70; and if the pentagon, whose side is 100, be required, we make $n=100$, and obtain 14950 for the number sought.

434. If we suppose the difference to be 4, we arrive at *hexagonal* numbers, as we see by the following progressions:

Indices,	1	2	3	4	5	6	7	8	9	&c.
Arith. Prog.	1,	5,	9,	13,	17,	21,	25,	29,	33,	&c.
Hexagon,	1,	6,	15,	28,	45,	66,	91,	120,	153,	&c.

where the indices still shew the side of each hexagon.

435. So that when the side is n, the hexagonal number is $2n^2-n=n(2n-1)$; and we have farther to remark, that all the hexagonal numbers are also triangular; since, if we take of these last the first, the third, the fifth, &c. we have precisely the series of hexagons.

436. In the same manner, we may find the numbers which are heptagonal, octagonal, &c. It will be sufficient therefore to exhibit the following Table of formulæ for all numbers that are comprehended under the general name of *polygonal* numbers.

Supposing the side to be represented by n, we have for the

Triangle $\frac{n^2+n}{2}=\frac{n(n+1)}{2}$.

Square - $\frac{2n^2+0n}{2}=n^2$.

vgon - $\frac{3n^2-n}{2}=\frac{n(3n-1)}{2}$.

vigon - $\frac{4n^2-2n}{2}=2n^2-n=n(2n-1)$.

viigon - $\frac{5n^2-3n}{2}=\frac{n(5n-3)}{2}$.

of these figures into triangles might furnish matter for many curious considerations, and for elegant transformations of the general formulæ, by which the polygonal numbers are expressed in this chapter; but it is unnecessary to dwell on them at present. F. T.

VIIIgon $\frac{6n^2-4n}{2} = 3n^2 - 2n = n(3n - 2)$.

IXgon - $\frac{7n^2-5n}{2} = \frac{n(7n-5)}{2}$.

Xgon - $\frac{8n^2-6n}{2} = 4n^2 - 3n = n(4n - 3)$.

XIgon - $\frac{9n^2-7n}{2} = \frac{n(9n-7)}{2}$.

XIIgon - $\frac{10n^2-8n}{2} = 5n^2 - 4n = n(5n - 4)$.

XXgon - $\frac{18n^2-16n}{2} = 9n^2 - 8n = n(9n - 8)$.

XXVgon - $\frac{23n^2-21n}{2} = \frac{n(23n-21)}{2}$.

*m*gon - $\frac{(m-2)n^2-(m-4)n}{2}$ *.

437. So that the side being n, the m-gonal number will be represented by $\frac{(m-2)n^2-(m-4)n}{2}$; whence we may deduce all the possible polygonal numbers which have the side n. Thus, for example, if the bigonal numbers were required, we should have $m = 2$, and consequently the number sought $= n$; that is to say, the bigonal numbers are the natural numbers, 1, 2, 3, &c.*

If we make $m = 3$, we have $\frac{n^2+n}{2}$ for the triangular number required.

If we make $m = 4$, we have the square number n^2, &c.

438. To illustrate this rule by examples, suppose that the XXV-gonal number, whose side is 36, were required; we

* The general expression for the m-gonal number is easily derived from the summation of an arithmetical progression, whose first term is 1, common difference d, and number of terms n; as in the following series; viz. $1+(1+d)+(1+2d)+$, &c. $(1+(n-1).d)$, the sum of which is expressed by $\frac{(2+(n-1).d)n}{2}$; but in all cases $d=m-2$, therefore substituting this value for d, the expression becomes $\frac{2n+(n^2-n).(m-2)}{2} = \frac{(m-2)n^2-(m-4)n}{2}$ as in the formula.

look first in the Table for the xxv-gonal number, whose side is n, and it is found to be $\frac{23n^2-21n}{2}$. Then making $n = 36$, we find 14526 for the number sought.

439. *Question.* A person bought a house, and he is asked how much he paid for it. He answers, that the 365th gonal number of 12 is the number of crowns which it cost him.

In order to find this number, we make $m = 365$, and $n = 12$; and substituting these values in the general formula, we find for the price of the house 23970 crowns *.

* This chapter is entitled "Of Figurate or Polygonal Numbers." It is not however without foundation that some algebraists make a distinction between *figurate* numbers and *polygonal* numbers. For the numbers commonly called *figurate* are all derived from a single arithmetical progression, and each series of numbers is formed from it by adding together the terms of the series which goes before. On the other hand, every series of *polygonal* numbers is produced from a different arithmetical progression. Hence, in strictness, we cannot speak of a single series of figurate numbers, as being at the same time a series of polygonal numbers. This will be made more evident by the following Tables.

TABLE OF FIGURATE NUMBERS.

Constant numbers - -	1.	1.	1.	1.	1.	1.	&c.
Natural - - - - - -	1.	2.	3.	4.	5.	6.	&c.
Triangular - - - - -	1.	3.	6.	10.	15.	21.	&c.
Pyramidal - - - - -	1.	4.	10.	20.	35.	56.	&c.
Triangular-pyramidal -	1.	5.	15.	35.	70.	126.	&c.

TABLE OF POLYGONAL NUMBERS.

Diff. of the progr.	Numbers						
1	triangular	1.	3.	6.	10.	15.	&c.
2	square - -	1.	4.	9.	16.	25.	&c.
3	pentagon -	1.	5.	12.	22.	35.	&c.
4	hexagon -	1.	6.	15.	28.	45.	&c.

Powers likewise form particular series of numbers. The first two are to be found among the figurate numbers, and the third among the polygonal; which will appear by successively substituting for a the numbers 1, 2, 3, &c.

TABLE OF POWERS.

a^0 - - - - -	1.	1.	1.	1.	1.	&c.
a^1 - - - - -	1.	2.	3.	4.	5.	&c.
a^2 - - - - -	1.	4.	9.	16.	25.	&c.
a^3 - - - - -	1.	8.	27.	64.	125.	&c.
a^4 - - - - -	1.	16.	81.	256.	625.	&c.