CHAP. IV.

Of the Summation *of* Arithmetical Progressions.

412. It is often necessary also to find the sum of an arithmetical progression. This might be done by adding all the terms together; but as the addition would be very tedious, when the progression consisted of a great number of terms, a rule has been devised, by which the sum may be more readily obtained.

413. We shall first consider a particular given progression, such that the first term is 2, the difference 3, the last term 29, and the number of terms 10;

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2, & 5, & 8, & 11, & 14, & 17, & 20, & 23, & 26, & 29. \end{matrix}$$

In this progression we see that the sum of the first and last term is 31; the sum of the second and the last but one 31; the sum of the third and the last but two 31, and so on: hence we conclude, that the sum of any two terms equally distant, the one from the first, and the other from the last term, is always equal to the sum of the first and the last term.

414. The reason of this may be easily traced; for if we suppose the first to be a, the last z, and the difference d, the sum of the first and the last term is $a + z$; and the second term being $a + d$, and the last but one $z - d$, the sum of these two terms is also $a + z$. Farther, the third time being $a + 2d$, and the last but two $z - 2d$, it is evident that these two terms also, when added together, make $a + z$; and the demonstration may be easily extended to any other two terms equally distant from the first and last.

415. To determine, therefore, the sum of the progression proposed, let us write the same progression term by term, inverted, and add the corresponding terms together, as follows:

$$\begin{array}{c} 2+5+8+11+14+17+20+23+26+29 \\ 29+26+23+20+17+14+11+8+5+2 \\ \hline 31+31+31+31+31+31+31+31+31+31 \end{array}$$

This series of equal terms is evidently equal to twice the sum of the given progression: now, the number of those equal terms is 10, as in the progression, and their sum consequently is equal to $10 \times 31 = 310$. Hence, as this sum is twice the sum of the arithmetical progression, the sum required must be 155.

416. If we proceed in the same manner with respect to any arithmetical progression, the first term of which is a, the last z, and the number of terms n; writing under the given progression the same progression inverted, and adding term to term, we shall have a series of n terms, each of which will be expressed by $a + z$; therefore the sum of this series will be $n(a + z)$, which is twice the sum of the proposed arithmetical progression; the latter, therefore, will be represented by $\frac{n(a+z)}{2}$.

417. This result furnishes an easy method of finding the sum of any arithmetical progression; and may be reduced to the following rule:

Multiply the sum of the first and the last term by the number of terms, and half the product will be the sum of the whole progression. Or, which amounts to the same, multiply the sum of the first and the last term by half the number of terms. Or, multiply half the sum of the first and the last term by the whole number of terms.

418. It will be necessary to illustrate this rule by some examples.

First, let it be required to find the sum of the progression of the natural numbers, 1, 2, 3, &c. to 100. This will be, by the first rule, $\frac{100 \times 101}{2} = \frac{10100}{2} = 5050$.

If it were required to tell how many strokes a clock strikes in twelve hours; we must add together the numbers 1, 2, 3, as far as 12; now this sum is found immediately to be $\frac{12 \times 13}{2} = 6 \times 13 = 78$. If we wished to know the sum of the same progression continued to 1000, we should find it to be 500500; and the sum of this progression, continued to 10000, would be 50005000.

419. Suppose a person buys a horse, on condition that for the first nail he shall pay 5 pence, for the second 8 pence, for the third 11 pence, and so on, always increasing 3 pence more for each nail, the whole number of which is 32; required the purchase of the horse?

In this question it is required to find the sum of an arithmetical progression, the first term of which is 5, the difference 3, and the number of terms 32; we must therefore begin by determining the last term; which is found by the rule, in Articles 406 and 411, to be $5 + (31 \times 3) = 98$; after which the sum required is easily found to be $\frac{103 \times 32}{2} = 103 \times 16$; whence we conclude that the horse costs 1648 pence, or 6*l.* 17*s.* 4*d.*

420. Generally, let the first term be a, the difference d, and the number of terms n; and let it be required to find, by means of these data, the sum of the whole progression. As the last term must be $a \pm (n - 1)d$, the sum of the first and the last will be $2a \pm (n - 1)d$; and multiplying this sum by the number of terms n, we have $2na \pm n(n - 1)d$; the sum required therefore will be $na \pm \frac{n(n-1)d}{2}$.

Now, this formula, if applied to the preceding example, or to $a = 5$, $d = 3$, and $n = 32$, gives $5 \times 32 + \frac{32 . 31 . 3}{2} = 160 + 1488 = 1648$; the same sum that we obtained before.

421. If it be required to add together all the natural numbers from 1 to n, we have, for finding this sum, the first term 1, the last term n, and the number of terms n; therefore the sum required is $\frac{n^2+n}{2} = \frac{n(n+1)}{2}$. If we make $n = 1766$, the sum of all the numbers, from 1 to 1766, will be 883, or half the number of terms, multiplied by 1767 = 1560261.

422. Let the progression of uneven numbers be proposed, 1, 3, 5, 7, &c. continued to n terms, and let the sum of it be required. Here the first term is 1, the difference 2, the number of terms n; the last term will therefore be $1 + (n - 1)2 = 2n - 1$, and consequently the sum required $= n^2$.

The whole therefore consists in multiplying the number of terms by itself; so that whatever number of terms of this progression we add together, the sum will be always a square, namely, the square of the number of terms; which we shall exemplify as follows:

Indices,	1	2	3	4	5	6	7	8	9	10, &c.
Progress.	1,	3,	5,	7,	9,	11,	13,	15,	17,	19, &c.
Sum.	1,	4,	9,	16,	25,	36,	49,	64,	81,	100, &c.

423. Let the first term be 1, the difference 3, and the number of terms n; we shall have the progression 1, 4, 7, 10, &c. the last term of which will be $1+(n-1)3=3n-2$;

wherefore the sum of the first and the last term is $3n - 1$, and consequently the sum of this progression is equal to $\frac{n(3n-1)}{2} = \frac{3n^2-n}{2}$; and if we suppose $n = 20$, the sum will be $10 \times 59 = 590$.

424. Again, let the first term be 1, the difference d, and the number of terms n; then the last term will be $1 + (n - 1)d$; to which adding the first, we have $2 + (n - 1)d$, and multiplying by the number of terms, we have $2n + n(n - 1)d$; whence we deduce the sum of the progression $n + \frac{n(n-1)d}{2}$.

And by making d successively equal to 1, 2, 3, 4, &c., we obtain the following particular values, as shewn in the subjoined Table.

If $d = 1$, the sum is $n + \frac{n(n-1)}{2} = \frac{n^2+n}{2}$

$d = 2$, - - - $n + \frac{2n(n-1)}{2} = n^2$

$d = 3$, - - - $n + \frac{3n(n-1)}{2} = \frac{3n^2-n}{2}$

$d = 4$, - - - $n + \frac{4n(n-1)}{2} = 2n^2 - n$

$d = 5$, - - - $n + \frac{5n(n-1)}{2} = \frac{5n^2-3n}{2}$

$d = 6$, - - - $n + \frac{6n(n-1)}{2} = 3n^2 - 2n$

$d = 7$, - - - $n + \frac{7n(n-1)}{2} = \frac{7n^2-5n}{2}$

$d = 8$, - - - $n + \frac{8n(n-1)}{2} = 4n^2 - 3n$

$d = 9$, - - - $n + \frac{9n(n-1)}{2} = \frac{9n^2-7n}{2}$

$d = 10$, - - - $n + \frac{10n(n-1)}{2} = 5n^2 - 4n$

QUESTIONS FOR PRACTICE.

1. Required the sum of an increasing arithmetical progression, having 3 for its first term, 2 for the common difference, and the number of terms 20. *Ans.* 440.

2. Required the sum of a decreasing arithmetical pro-

gression, having 10 for its first term, $\frac{1}{3}$ for the common difference, and the number of terms 21. *Ans.* 140.

3. Required the number of all the strokes of a clock in twelve hours, that is, a complete revolution of the index. *Ans.* 78.

4. The clocks of Italy go on to 24 hours; how many strokes do they strike in a complete revolution of the index? *Ans.* 300.

5. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them one by one to a basket, which is placed one yard from the first stone? *Ans.* 5 *miles* and 1300 *yards*.

CHAP. V.

Of Figurate *, *or* Polygonal Numbers.

425. The summation of arithmetical progressions, which begin by 1, and the difference of which is 1, 2, 3, or any

* The French translator has justly observed, in his note at the conclusion of this chapter, that algebraists make a distinction between figurate and polygonal numbers; but as he has not entered far upon this subject, the following illustration may not be unacceptable.

It will be immediately perceived in the following Table, that each series is derived immediately from the foregoing one, being the sum of all its terms from the beginning to that place; and hence also the law of continuation, and the general term of each series, will be readily discovered.

Natural	1, 2, 3, 4, 5 - -	n general term
Triangular	1, 3, 6, 10, 15 - -	$\frac{n.(n+1)}{2}$
Pyramidal	1, 4, 10, 20, 35 - -	$\frac{n.(n+1).(n+2)}{2.3}$
Triangular-pyramidal	1, 5, 15, 35, 70 - -	$\frac{n.(n+1).(n+2).(n+3)}{2.3.4}$

And, in general, the figurate number of any order m will be expressed by the formula,

$$\frac{n.(n+1).(n+2).(n+3) - - (n+m-1)}{1.2 \;.\; 3 \;.\; 4 \;-\,-\; m}.$$

Now, one of the principal properties of these numbers, and