same quantity; as in the numbers 4, 7, 10; and the decreasing progression is that in which the terms go on always diminishing by the same quantity, such as the numbers 9, 5, 1.

399. Let us suppose the numbers a, b, c, to be in arithmetical progression; then $a - b = b - c$, whence it follows, from the equality between the sum of the extremes and that of the means, that $2b = a + c$; and if we subtract a from both, we have $2b - a = c$.

400. So that when the first two terms a, b, of an arithmetical progression are given, the third is found by taking the first from twice the second. Let 1 and 3 be the first two terms of an arithmetical progression, the third will be $2 \times 3 - 1 = 5$; and these three numbers 1, 3, 5, give the proportion

$$1 - 3 = 3 - 5.$$

401. By following the same method, we may pursue the arithmetical progression as far as we please; we have only to find the fourth term by means of the second and third, in the same manner as we determined the third by means of the first and second, and so on. Let a be the first term, and b the second, the third will be $2b - a$, the fourth $4b - 2a - b = 3b - 2a$, the fifth $6b - 4a - 2b + a = 4b - 3a$, the sixth $8b - 6a - 3b + 2a = 5b - 4a$, the seventh $10b - 8a - 4b + 3a = 6b - 5a$, &c.

CHAP. III.

Of Arithmetical Progressions.

402. We have already remarked, that a series of numbers composed of any number of terms, which always increase, or decrease, by the same quantity, is called an *arithmetical progression.*

Thus, the natural numbers written in their order, as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

403. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the

difference; so that when the first term and the difference are given, we may continue the arithmetical progression to any length.

For example, let the first term be 2, and the difference 3, and we shall have the following increasing progression: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, &c. in which each term is found by adding the difference to the preceding one.

404. It is usual to write the natural numbers, 1, 2, 3, 4, 5, &c. above the terms of such an arithmetical progression, in order that we may immediately perceive the rank which any term holds in the progression, which numbers, when written above the terms, are called *indices;* thus, the above example will be written as follows:

Indices.	1	2	3	4	5	6	7	8	9	10	
Arith. Prog.	2,	5,	8,	11,	14,	17,	20,	23,	26,	29,	&c.

where we see that 29 is the tenth term.

405. Let a be the first term, and d the difference, the arithmetical progression will go on in the following order:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ a, & a \pm d, & a \pm 2d, & a \pm 3d, & a \pm 4d, & a \pm 5d, & a \pm 6d, & \&c. \end{array}$$

according as the series is increasing, or decreasing, whence it appears that any term of the progression might be easily found, without the necessity of finding all the preceding ones, by means only of the first term a and the difference d; thus, for example, the tenth term will be $a \pm 9d$, the hundredth term $a \pm 99d$, and, generally, the nth term will be $a \pm (n-1)d$.

406. When we stop at any point of the progression, it is of importance to attend to the first and the last term, since the index of the last term will represent the number of terms. If, therefore, the first term be a, the difference d, and the number of terms n, we shall have for the last term $a \pm (n-1)d$, according as the series is increasing or decreasing, which is consequently found by multiplying the difference by the number of terms *minus* one, and adding, or subtracting, that product from the first term. Suppose, for example, in an ascending arithmetical progression of a hundred terms, the first term is 4, and the difference 3; then the last term will be $99 \times 3 + 4 = 301$.

407. When we know the first term a, and the last z, with the number of terms n, we can find the difference d; for, since the last term $z = a \pm (n-1)d$, if we subtract a from both sides, we obtain $z - a = (n-1)d$. So that by taking the difference between the first and last term, we have the product of the difference multiplied by the number of terms *minus* 1; we have therefore only to divide $z - a$ by $n - 1$

in order to obtain the required value of the difference d, which will be $\frac{z-a}{n-1}$. This result furnishes the following rule: Subtract the first term from the last, divide the remainder by the number of terms *minus* 1, and the quotient will be the common difference: by means of which we may write the whole progression.

408. Suppose, for example, that we have an increasing arithmetical progression of nine terms, whose first is 2, and last 26, and that it is required to find the difference. We must subtract the first term 2 from the last 26, and divide the remainder, which is 24, by $9 - 1$, that is, by 8; the quotient 3 will be equal to the difference required, and the whole progression will be:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2, & 5, & 8, & 11, & 14, & 17, & 20, & 23, & 26. \end{array}$$

To give another example, let us suppose that the first term is 1, the last 2, the number of terms 10, and that the arithmetical progression, answering to these suppositions, is required; we shall immediately have for the difference $\frac{2-1}{10-1} = \frac{1}{9}$, and thence conclude that the progression is:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1, & 1\frac{1}{9}, & 1\frac{2}{9}, & 1\frac{3}{9}, & 1\frac{4}{9}, & 1\frac{5}{9}, & 1\frac{6}{9}, & 1\frac{7}{9}, & 1\frac{8}{9}, & 2. \end{array}$$

Another example. Let the first term be $2\frac{1}{3}$, the last term $12\frac{1}{2}$, and the number of terms 7; the difference will be $\frac{12\frac{1}{2}-2\frac{1}{3}}{7-1} = \frac{10\frac{1}{6}}{6} = \frac{61}{36} = 1\frac{25}{36}$, and consequently the progression:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2\frac{1}{3}, & 4\frac{1}{36}, & 5\frac{13}{18}, & 7\frac{5}{12}, & 9\frac{1}{9}, & 10\frac{29}{36}, & 12\frac{1}{2}. \end{array}$$

409. If now the first term a, the last term z, and the difference d, are given, we may from them find the number of terms n; for since $z - a = (n - 1)d$, by dividing both sides by d, we have $\frac{z-a}{d} = n - 1$; also n being greater by 1 than $n - 1$, we have $n = \frac{z-a}{d} + 1$; consequently the number of terms is found by dividing the difference between the first and the last term, or $z - a$, by the difference of the progression, and adding unity to the quotient.

For example, let the first term be 4, the last 100, and the difference 12, the number of terms will be $\frac{100-4}{12} + 1 = 9$;

and these nine terms will be,

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4, & 16, & 28, & 40, & 52, & 64, & 76, & 88, & 100. \end{array}$$

If the first term be 2, the last 6, and the difference $1\frac{1}{3}$, the number of terms will be $\frac{4}{1\frac{1}{3}} + 1 = 4$; and these four terms will be,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2, & 3\frac{1}{3}, & 4\frac{2}{3}, & 6. \end{array}$$

Again, let the first term be $3\frac{1}{3}$, the last $7\frac{2}{3}$, and the difference $1\frac{4}{9}$, the number of terms will be $\frac{7\frac{2}{3} - 3\frac{1}{3}}{1\frac{4}{9}} + 1 = 4$; which are,

$$3\frac{1}{3},\ 4\frac{7}{9},\ 6\frac{2}{9},\ 7\frac{2}{3}.$$

410. It must be observed, however, that as the number of terms is necessarily an integer, if we had not obtained such a number for n, in the examples of the preceding article, the questions would have been absurd.

Whenever we do not obtain an integer number for the value of $\frac{z-a}{d}$, it will be impossible to resolve the question; and consequently, in order that questions of this kind may be possible, $z - a$ must be divisible by d.

411. From what has been said, it may be concluded, that we have always four quantities, or things, to consider in an arithmetical progression:

1st. The first term, a; 2d. The last term, z;

3d. The difference, d; and 4th. The number of terms, n.

The relations of these quantities to each other are such, that if we know three of them, we are able to determine the fourth; for,

1. If a, d, and n, are known, we have $z = a \pm (n - 1)d$.
2. If z, d, and n, are known, we have

$$a = z - (n - 1)d.$$

3. If a, z, and n, are known, we have $d = \frac{z-a}{n-1}$.
4. If a, z, and d, are known, we have $n = \frac{z-a}{d} + 1$.