CHAP. XIII.

Of the Calculation *of* Interest *.

540. We are accustomed to express the interest of any principal by *per cents*, signifying how much interest is annually paid for the sum of 100 pounds. And it is very usual to put out the principal sum at 5 *per cent*, that is, on such terms, that we receive 5 pounds interest for every 100 pounds principal. Nothing therefore is more easy than to calculate the interest for any sum; for we have only to say, according to the Rule of Three:

As 100 is to the principal proposed, so is the rate *per cent* to the interest required. Let the principal, for example, be 860*l.*, its annual interest is found by this proportion;

$$\text{As } 100 : 5 :: 860 : 43.$$

Therefore 43*l.* is the annual interest.

541. We shall not dwell any longer on examples of Simple Interest, but pass on immediately to the calculation of *Compound Interest;* where the chief subject of inquiry is, to what sum does a given principal amount, after a certain number of years, the interest being annually added to the principal. In order to resolve this question, we begin with the consideration, that 100*l.* placed out at 5 per cent, becomes, at the end of a year, a principal of 105*l.*: therefore let the principal be a; its amount, at the end of the year, will be found, by saying; As 100 is to a, so is 105 to the answer; which gives

* The theory of the calculation of interest owes its first improvements to Leibneitz, who delivered the principal elements of it in the *Acta Eruditorum* of Leipsic for 1683. It was afterwards the subject of several detached dissertations written in a very interesting manner. It has been most indebted to those mathematicians who have cultivated political arithmetic; in which are combined, in a manner truly useful, the calculation of interest, and the calculation of probabilities, founded on the data furnished by the bills of mortality. We are still in want of a good elementary treatise of political arithmetic, though this extensive branch of science has been much attended to in England, France, and Holland. F. T.

$$\frac{105a}{100} = \frac{21a}{20} = \tfrac{21}{20} \times a, = a + \tfrac{1}{20} \times a.$$

542. So that, when we add to the original principal its twentieth part, we obtain the amount of the principal at the end of the first year: and adding to this its twentieth part, we know the amount of the given principal at the end of two years, and so on. It is easy, therefore, to compute the successive and annual increases of the principal, and to continue this calculation to any length.

543. Suppose, for example, that a principal, which is at present 1000*l.*, is put out at five per cent; that the interest is added every year to the principal; and that it were required to find its amount at any time. As this calculation must lead to fractions, we shall employ decimals, but without carrying them farther than the thousandth parts of a pound, since smaller parts do not at present enter into consideration.

The given principal of 1000*l.* will be worth

after 1 year - - -	1050*l.*
	52·5,
after 2 years - - -	1102·5
	55·125,
after 3 years - - -	1157·625
	57·881,
after 4 years - - -	1215·506
	60·775,
after 5 years - - -	1276·281, &c.

which sums are formed by always adding $\frac{1}{20}$ of the preceding principal.

544. We may continue the same method, for any number of years; but when this number is very great, the calculation becomes long and tedious; but it may always be abridged, in the following manner:

Let the present principal be a, and since a principal of 20*l.* amounts to 21*l.* at the end of a year, the principal a will amount to $\frac{21}{20} \cdot a$ at the end of a year: and the same principal will amount, the following year, to $\frac{21^2}{20^2} \cdot a = (\frac{21}{20})^2 \cdot a$.

Also, this principal of two years will amount to $(\frac{21}{20})^3 \cdot a$, the year after: which will therefore be the principal of three years; and still increasing in the same manner, the given principal will amount to $(\frac{21}{20})^4 \cdot a$ at the end of four years; to $(\frac{21}{20})^5 \cdot a$, at the end of five years; and after a century, it will amount to $(\frac{21}{20})^{100} \cdot a$; so that, in general, $(\frac{21}{20})^n \cdot a$ will be the amount of this principal, after n years; and this formula will serve to determine the amount of the principal, after any number of years.

545. The fraction $\frac{21}{20}$, which is used in this calculation, depends on the interest having been reckoned at 5 per cent., and on $\frac{21}{20}$ being equal to $\frac{105}{100}$. But if the interest were estimated at 6 per cent, the principal a would amount to $\frac{106}{100} \cdot a$, at the end of a year; to $(\frac{106}{100})^2 \cdot a$, at the end of two years; and to $\frac{106^n}{100} \cdot a$, at the end of n years.

If the interest is only at 4 per cent. the principal a will amount only to $(\frac{104}{100})^n \cdot a$, after n years.

546. When the principal a, as well as the number of years, is given, it is easy to resolve these formulæ by logarithms. For if the question be according to our first supposition, we shall take the logarithm of $(\frac{21}{20})^n \cdot a$, which is $= log.\ (\frac{21}{20})^n + log.\ a$; because the given formula is the product of $(\frac{21}{20})^n$ and a. Also, as $(\frac{21}{20})^n$ is a power, we shall have $log.\ (\frac{21}{20})^n = n\ log.\ \frac{21}{20}$: so that the logarithm of the amount required is $n\ log.\ \frac{21}{20} + log.\ a$; and farther, the logarithm of the fraction $\frac{21}{20} = log.\ 21 - log.\ 20$.

547. Let now the principal be 1000*l.* and let it be required to find how much this principal will amount to at the end of 100 years, reckoning the interest at 5 per cent.

Here we have $n = 100$; and, consequently, the logarithm of the amount required will be $100\ log.\ \frac{21}{20} + log.\ 1000$, which is calculated thus:

$$\begin{array}{rr}
log.\ 21 = & 1{\cdot}3222193 \\
\text{subtracting } log.\ 20 = & 1{\cdot}3010300 \\
\hline
log.\ \frac{21}{20} = & 0{\cdot}0211893 \\
\text{multiplying by} & 100 \\
\hline
100\ log.\ \frac{21}{20} = & 2{\cdot}1189300 \\
\text{adding } log.\ 1000 = & 3{\cdot}0000000 \\
\hline
\text{gives} & 5{\cdot}1189300
\end{array}$$

which is the logarithm of the principal required.

We perceive, from the characteristic of this logarithm, that the principal required will be a number consisting of six figures, and it is found to be 131501*l.*

548. Again, suppose a principal of 3452*l.* were put out at 6 per cent, what would it amount to at the end of 64 years?

We have here $a = 3452$, and $n = 64$. Wherefore the logarithm of the amount sought is

$64\ log.\ \frac{53}{50} + log.\ 3452$, which is calculated thus:

$$
\begin{array}{lr}
log.\ 53 = & 1{\cdot}7242759 \\
\text{subtracting } log.\ 50 = & 1{\cdot}6989700 \\
\hline
log.\ \frac{53}{50} = & 0{\cdot}0253059 \\
\text{multiplying by} & 64 \\
\hline
64\ log.\ \frac{53}{50} = & 1{\cdot}6195776 \\
log.\ 3452 = & 3{\cdot}5380708 \\
\hline
\text{which gives} & 5{\cdot}1576484
\end{array}
$$

And taking the number of this logarithm, we find the amount required equal to 143763*l.*

549. When the number of years is very great, as it is required to multiply this number by the logarithm of a fraction, a considerable error might arise from the logarithms in the Tables not being calculated beyond 7 figures of decimals; for which reason it will be necessary to employ logarithms carried to a greater number of figures, as in the following example.

A principal of 1*l.* being placed at 5 per cent., compound interest, for 500 years, it is required to find to what sum this principal will amount, at the end of that period.

We have here $a = 1$ and $n = 500$; consequently, the logarithm of the principal sought is equal to $500\ log.\ \frac{21}{20} + log.\ 1$, which produces this calculation:

$$
\begin{array}{lr}
log.\ 21 = & 1{\cdot}322219294733919 \\
\text{subtracting } log.\ 20 = & 1{\cdot}301029995663981 \\
\hline
log.\ \frac{21}{20} = & 0{\cdot}021189299069938 \\
\text{multiply by} & 500 \\
\hline
\end{array}
$$

$500\ log.\ \frac{21}{20} = 10{\cdot}594649534969000$, the logarithm of the amount required; which will be found equal to 39323200000*l.*

550. If we not only add the interest annually to the principal, but also increase it every year by a new sum b, the original principal, which we call a, would increase each year in the following manner:

after 1 year, $\frac{21}{20}a + b$,
after 2 years, $(\frac{21}{20})^2a + \frac{21}{20}b + b$,
after 3 years, $(\frac{21}{20})^3a + (\frac{21}{20})^2b + \frac{21}{20}b + b$,

after 4 years, $(\frac{21}{20})^4a + (\frac{21}{20})^3b + (\frac{21}{20})^2b + \frac{21}{20}b + b$,
after n years, $(\frac{21}{20})^na + (\frac{21}{20})^{n-1}b + (\frac{21}{20})^{n-2}b + \frac{21}{20}b$, &c.

This amount evidently consists of two parts, of which the first is $(\frac{21}{20})^na$; and the other, taken inversely, forms the series $b + \frac{21}{20}b + (\frac{21}{20})^2b + (\frac{21}{20})^3b + \ldots . (\frac{21}{20})^{n-1}b$; which series is evidently a geometrical progression, the ratio of which is equal to $\frac{21}{20}$, and we shall therefore find its sum, by first multiplying the last term $(\frac{21}{20})^{n-1}b$ by the exponent $\frac{21}{20}$; which gives $(\frac{21}{20})^nb$. Then, subtracting the first term b, there remains $(\frac{21}{20})^nb - b$; and, lastly, dividing by the exponent *minus* 1, that is to say by $\frac{1}{20}$, we shall find the sum required to be $20(\frac{21}{20})^nb - 20b$; therefore the amount sought is, $(\frac{21}{20})^na + 20(\frac{21}{20})^nb - 20b = (\frac{21}{20})^n \times (a + 20b) - 20b$.

551. The resolution of this formula requires us to calculate, separately, its first term $(\frac{21}{20})^n \times (a + 20b)$, which is $n\ log.\ \frac{21}{20} + log.\ (a + 20b)$; for the number which answers to this logarithm in the Tables, will be the first term; and if from this we subtract $20b$, we shall have the amount sought.

552. A person has a principal of 1000*l.* placed out at five per cent, compound interest, to which he adds annually 100*l.* beside the interest: what will be the amount of this principal at the end of twenty-five years?

We have here $a = 1000$; $b = 100$; $n = 25$; the operation is therefore as follows:

$$log.\ \tfrac{21}{20} = 0{\cdot}021189299$$

Multiplying by 25, we have

$$25\ log.\ \tfrac{21}{20} = 0{\cdot}5297324750$$
$$log.\ (a + 20b) = 3{\cdot}4771213135$$

And the sum $= 4{\cdot}0068537885$.

So that the first part, or the number which answers to this logarithm, is 10159·1, and if we subtract $20b = 2000$, we find that the principal in question, after twenty-five years, will amount to 8159·1*l.*

553. Since then this principal of 1000*l.* is always increasing, and after twenty-five years amounts to $8159\frac{1}{10}$*l.* we may require, in how many years it will amount to 1000000*l.*

Let n be the number of years required: and, since $a = 1000$, $b = 100$, the principal will be, at the end of n years, $(\frac{21}{20})^n . (3000) - 2000$, which sum must make 1000000; from it therefore results this equation;

$$3000 . (\tfrac{21}{20})^n - 2000 = 1000000;$$

And adding 2000 to both sides, we have

$$3000 \, . \, (\tfrac{21}{20})^n = 1002000.$$

Then dividing both sides by 3000, we have $(\frac{21}{20})^n = 334$.

Taking the logarithms, $n \; log. \frac{21}{20} = log. \; 334$; and dividing by $log. \frac{21}{20}$, we obtain $n = \dfrac{log. \; 334}{log. \frac{21}{20}}$. Now, $log. \; 334 = 2{\cdot}5237465$, and $log. \frac{21}{20} = 0{\cdot}0211893$; therefore $n = \dfrac{2{\cdot}5237465}{0{\cdot}0211893}$; and, lastly, if we multiply the two terms of this fraction by 10000000, we shall have $n = \frac{25237465}{211893}$, = 119 years, 1 month, 7 days; and this is the time, in which the principal of 1000*l.* will be increased to 1000000*l.*

554. But if we supposed that a person, instead of annually increasing his principal by a certain fixed sum, diminished it, by spending a certain sum every year, we should have the following gradations, as the values of that principal a, year after year, supposing it put out at 5 per cent, compound interest, and representing the sum which is annually taken from it by b:

after 1 year, it would be $\frac{21}{20}a - b$,
after 2 years, $(\frac{21}{20})^2 a - \frac{21}{20}b - b$,
after 3 years, $(\frac{21}{20})^3 a - (\frac{21}{20})^2 b - \frac{21}{20}b - b$,
after n years, $(\frac{21}{20})^n a - (\frac{21}{20})^{n-1} b - (\frac{21}{20})^{n-2} b \ldots - (\frac{21}{20})b - b$.

555. This principal consists of two parts, one of which is $(\frac{21}{20})^n \, . \, a$, and the other, which must be subtracted from it, taking the terms inversely, forms the following geometrical progression:

$$b + (\tfrac{21}{20})b + (\tfrac{21}{20})^2 b + (\tfrac{21}{20})^3 b + \ldots . \, (\tfrac{21}{20})^{n-1} b.$$

Now we have already found (Art. 550.) that the sum of this progression is $20(\frac{21}{20})^n b - 20b$; if, therefore, we subtract this quantity from $(\frac{21}{20})^n . \, a$, we shall have for the principal required, after n years =

$$(\tfrac{21}{20})^n \, . \, (a - 20b) + 20b.$$

556. We might have deduced this formula immediately from that of Art. 550. For, in the same manner as we annually added the sum b, in the former supposition; so, in the present, we subtract the same sum b every year. We have therefore only to put in the former formula, $-b$ every where, instead of $+b$. But it must here be particularly remarked, that if $20b$ is greater than a, the first part becomes negative, and, consequently, the principal will continually diminish. This will be easily perceived; for if we annually take away from the principal more than is added to it by the interest, it is evident that this principal must continually be-

come less, and at last it will be absolutely reduced to nothing; as will appear from the following example:

557. A person puts out a principal of 100000*l.* at 5 per cent interest; but he spends annually 6000*l.*; which is more than the interest of his principal, the latter being only 5000*l.*; consequently, the principal will continually diminish; and it is required to determine, in what time it will be all spent.

Let us suppose the number of years to be n, and since $a = 100000$, and $b = 6000$, we know that after n years the amount of the principal will be $-20000 (\frac{21}{20})^n + 120000$, or $120000 - 20000(\frac{21}{20})^n$, where the factor, -20000, is the result of $a - 20b$; or $100000 - 120000$.

So that the principal will become nothing, when $20000(\frac{21}{20})^n$ amounts to 120000; or when $20000(\frac{21}{20})^n = 120000$. Now, dividing both sides by 20000, we have $(\frac{21}{20})^n = 6$; and taking the logarithm, we have $n \, log. (\frac{21}{20}) = log. 6$; then dividing by $log. \frac{21}{20}$, we have $n = \frac{log. 6}{log. \frac{21}{20}}$, or $n = - - - \frac{0{\cdot}7781513}{0{\cdot}0211893}$; and, consequently, $n = 36$ years, 8 months, 22 days; at the end of which time, no part of the principal will remain.

558. It will here be proper also to shew how, from the same principles, we may calculate interest for times shorter than whole years. For this purpose, we make use of the formula $(\frac{21}{20})^n . a$ already found, which expresses the amount of a principal, at 5 per cent, compound interest, at the end of n years; for if the time be less than a year, the exponent n becomes a fraction, and the calculation is performed by logarithms as before. If, for example, the amount of a principal at the end of one day were required, we should make $n = \frac{1}{365}$; if after two days, $n = \frac{2}{365}$, and so on.

559. Suppose the amount of 100000*l.* for 8 days were required, the interest being at 5 per cent.

Here $a = 100000$, and $n = \frac{8}{365}$, consequently, the amount sought is $(\frac{21}{20})^{\frac{8}{365}} \times 100000$; the logarithm of which quantity is $log. (\frac{21}{20})^{\frac{8}{365}} + log. 100000 = \frac{8}{365} \, log. \frac{20}{21} + log. 100000$. Now, $log. \frac{21}{20} = 0{\cdot}0211893$, which, multiplied by $\frac{8}{365}$, gives $0{\cdot}0004644$, to which adding

$$log. 100000 = 5{\cdot}0000000$$

the sum is $5{\cdot}0004644$,

The natural number of this logarithm is found to be 100107. So that, subtracting the principal, 100000 from this amount, the interest, for eight days, is 107*l.*

560. To this subject belong also the calculation of the present value of a sum of money, which is payable only after a term of years. For as 20*l.*, in ready money, amounts to 21*l.* in a year; so, reciprocally, a sum of 21*l.*, which cannot be received till the end of one year, is really worth only 20*l.* If, therefore, we express, by a, a sum whose payment is due at the end of a year, the present value of this sum is $\frac{20}{21}a$; and therefore to find the present worth of a principal a, payable a year hence, we must multiply it by $\frac{20}{21}$; to find its value two years before the time of payment, we multiply it by $(\frac{20}{21})^2a$; and in general, its value, n years before the time of payment, will be expressed by $(\frac{20}{21})^na$.

561. Suppose, for example, a man has to receive for five successive years, an annual rent of 100*l.* and that he wishes to give it up for ready money, the interest being at 5 per cent; it is required to find how much he is to receive. Here the calculations may be made in the following manner:

For 100*l.* due

after 1 year, he receives	95·239
after 2 years - - -	90·704
after 3 years - - -	86·385
after 4 years - - -	82·272
after 5 years - - -	78·355
Sum of the 5 terms =	432·955

So that the possessor of the rent can claim, in ready money, only 432·955*l.*

562. If such a rent were to last a greater number of years, the calculation, in the manner we have performed it, would become very tedious; but in that case it may be facilitated as follows:

Let the annual rent be a, commencing at present and lasting n years, it will be actually worth

$$a + (\tfrac{20}{21})a + (\tfrac{20}{21})^2a + (\tfrac{20}{21})^3a + (\tfrac{20}{21})^4a \ldots + (\tfrac{20}{21})^na.$$

Which is a geometrical progression, and the whole is reduced to finding its sum. We therefore multiply the last term by the exponent, the product of which is $(\frac{20}{21})^{n+1}a$; then, subtracting the first term, there remains $(\frac{20}{21})^{n+1}a - a$; and, lastly, dividing by the exponent *minus* 1, that is, by $-\frac{1}{21}$, or, which amounts to the same, multiplying by -21, we shall have the sum required,

$$-21 \cdot (\tfrac{20}{21})^{n+1} \cdot a + 21a, \text{ or, } 21a - 21 \cdot (\tfrac{20}{21})^{n+1} \cdot a;$$

and the value of the second term, which it is required to subtract, is easily calculated by logarithms.

QUESTIONS FOR PRACTICE.

1. What will 375*l.* 10*s.* amount to in 9 years at 6 *per cent.* compound interest? *Ans.* 634*l.* 8*s.*

2. What is the interest of 1*l.* for one day, at the rate of 5 *per cent.*? *Ans.* 0·0001369863 parts of a pound.

3. What will 365*l.* amount to in 875 days, at the rate of 4 *per cent.*? *Ans.* 400*l.*

4. What will 256*l.* 10*s.* amount to in seven years, at the rate of 6 *per cent.* compound interest? *Ans.* 385*l.* 13*s.* 7½*d.*

5. What will 563*l.* amount to in 7 years and 99 days, at the rate of 6 *per cent.* compound interest? *Ans.* 860*l.*

6. What is the amount of 400*l.* at the end of 3½ years, at 6 *per cent.* compound interest? *Ans.* 490*l.* 11*s.* 7½*d.*

7. What will 320*l.* 10*s.* amount to in four years, at 5 *per cent.* compound interest? *Ans.* 389*l.* 11*s.* 4¼*d.*

8. What will 650*l.* amount to in 5 years, at 5 *per cent.* compound interest? *Ans.* 829*l.* 11*s.* 7½*d.*

9. What will 550*l.* 10*s.* amount to in 3 years and 6 months, at 6 *per cent.* compound interest? *Ans.* 675*l.* 6*s.* 5*d.*

10. What will 15*l.* 10*s.* amount to in 9 years, at 3½ *per cent.* compound interest? *Ans.* 21*l.* 2*s.* 4¼*d.*

11. What is the amount of 550*l.* at 4 *per cent.* in seven months? *Ans.* 562*l.* 16*s.* 8*d.*

12. What is the amount of 100*l.* at 7·37 *per cent.* in nine years and nine months? *Ans.* 200*l.*

13. If a principal x be put out at compound interest for x years, at x *per cent.* required the time in which it will gain x. *Ans.* 8·49824 years.

14. What sum, in ready money, is equivalent to 600*l.* due nine months hence, reckoning the interest at 5 *per cent.*? *Ans.* 578*l.* 6*s.* 3¾*d.*

15. What sum, in ready money, is equivalent to an annuity of 70*l.* to commence 6 years hence, and then to continue for 21 years at 5 *per cent*? *Ans.* 669*l.* 14*s.* 0¾*d.*

16. A man puts out a sum of money, at 6 *per cent.*, to continue 40 years; and then both principal and interest are to sink. What is that *per cent.* to continue for ever? *Ans.* 52 *per cent.*