not a square; and, consequently, that its root cannot be assigned. In such cases, the radical sign, which we before employed, is made use of. This is written before the quantity, and the quantity itself is placed between parentheses, or under a line: thus, the square root of $a^2 + b^2$ is represented by $\sqrt{}(a^2 + b^2)$, or by $\sqrt{a^2 + b^2}$; and $\sqrt{}(1 - x^2)$, or $\sqrt{1 - x^2}$, expresses the square root of $1 - x^2$. Instead of this radical sign, we may use the fractional exponent $\frac{1}{2}$, and represent the square root of $a^2 + b^2$, for instance, by $(a^2 + b^2)^{\frac{1}{2}}$, or by $\overline{a^2 + b^2}|^{\frac{1}{2}}$

CHAP. VIII.

Of the Calculation *of* Irrational Quantities.

326. When it is required to add together two or more irrational quantities, this is to be done, according to the method before laid down, by writing all the terms in succession, each with its proper sign: and, with regard to abbreviations, we must remark that, instead of $\sqrt{a} + \sqrt{a}$, for example, we may write $2\sqrt{a}$; and that $\sqrt{a} - \sqrt{a} = 0$, because these two terms destroy one another. Thus, the quantities $3 + \sqrt{2}$ and $1 + \sqrt{2}$, added together, make $4 + 2\sqrt{2}$, or $4 + \sqrt{8}$; the sum of $5 + \sqrt{3}$ and $4 - \sqrt{3}$, is 9; and that of $2\sqrt{3} + 3\sqrt{2}$ and $\sqrt{3} - \sqrt{2}$, is $3\sqrt{3} + 2\sqrt{2}$.

327. Subtraction also is very easy, since we have only to add the proposed numbers, after having changed their signs; as will be readily seen in the following example, by subtracting the lower line from the upper.

$$\begin{array}{l} 4-\sqrt{2}+2\sqrt{3}-3\sqrt{5}+4\sqrt{6} \\ 1+2\sqrt{2}-2\sqrt{3}-5\sqrt{5}+6\sqrt{6} \\ \hline 3-3\sqrt{2}+4\sqrt{3}+2\sqrt{5}-2\sqrt{6}. \end{array}$$

328. In multiplication, we must recollect that $\sqrt{a}$ multiplied by $\sqrt{a}$ produces a; and that if the numbers which follow the sign $\sqrt{}$ are different, as a and b, we have $\sqrt{ab}$ for the product of $\sqrt{a}$ multiplied by $\sqrt{b}$. After this, it will be easy to calculate the following examples:

$$\begin{array}{ll} \begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad\sqrt{2}+2 \\ \hline 1+2\sqrt{2}+2=3+2\sqrt{2}. \end{array} & \qquad \begin{array}{l} 4+2\sqrt{2} \\ 2-\ \sqrt{2} \\ \hline 8+4\sqrt{2} \\ \quad -4\sqrt{2}-4 \\ \hline 8-4=4. \end{array} \end{array}$$

329. What we have said applies also to imaginary quantities; we shall only observe farther, that $\sqrt{-a}$ multiplied by $\sqrt{-a}$ produces $-a$. If it were required to find the cube of $-1+\sqrt{-3}$, we should take the square of that number, and then multiply that square by the same number; as in the following operation:

$$\begin{array}{l} -1+\sqrt{-3} \\ -1+\sqrt{-3} \\ \hline \quad 1-\sqrt{-3} \\ \quad\ -\sqrt{-3}-3 \\ \hline 1-2\sqrt{-3}-3=-2-2\sqrt{-3} \\ \qquad\qquad\qquad\ \ -1+\ \sqrt{-3} \\ \hline \qquad\qquad\qquad\quad 2+2\sqrt{-3} \\ \qquad\qquad\qquad\qquad -2\sqrt{-3}+6 \\ \hline \qquad\qquad\qquad 2+6=8. \end{array}$$

330. In the division of surds, we have only to express the proposed quantities in the form of a fraction; which may be then changed into another expression having a rational denominator; for if the denominator be $a+\sqrt{b}$, for example, and we multiply both this and the numerator by $a-\sqrt{b}$, the new denominator will be a^2-b, in which there is no radical sign. Let it be proposed, for example, to divide $3+2\sqrt{2}$ by $1+\sqrt{2}$: we shall first have $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; then multiplying the two terms of the fraction by $1-\sqrt{2}$, we shall have for the numerator:

$$\begin{array}{l} 3+2\sqrt{2} \\ 1-\ \sqrt{2} \\ \hline 3+2\sqrt{2} \\ \quad -3\sqrt{2}-4 \\ \hline 3-\sqrt{2}-4=-\sqrt{2}-1; \end{array}$$

and for the denominator:

$$\begin{array}{l} 1+\sqrt{2} \\ 1-\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad -\sqrt{2}-2 \\ \hline 1-2=-1. \end{array}$$

Our new fraction therefore is $\frac{-\sqrt{2}-1}{-1}$; and if we again multiply the two terms by -1, we shall have for the numerator $\sqrt{2}+1$, and for the denominator $+1$. Now, it is easy to shew that $\sqrt{2}+1$ is equal to the proposed fraction $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; for $\sqrt{2}+1$ being multiplied by the divisor $1+\sqrt{2}$, thus,

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad \sqrt{2}+2 \\ \hline \end{array}$$

we have $1+2\sqrt{2}+2=3+2\sqrt{2}$.

Another example. Let $8-5\sqrt{2}$ be divided by $3-2\sqrt{2}$. This, in the first instance, is $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$; and multiplying the two terms of this fraction by $3+2\sqrt{2}$, we have for the numerator,

$$\begin{array}{l} 8-5\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 24-15\sqrt{2} \\ \quad 16\sqrt{2}-20 \\ \hline 24+\sqrt{2}-20=4+\sqrt{2}; \end{array}$$

and for the denominator,

$$\begin{array}{l} 3-2\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 9-6\sqrt{2} \\ \quad 6\sqrt{2}-8 \\ \hline 9-8=1. \end{array}$$

Consequently, the quotient will be $4+\sqrt{2}$. The truth of this may be proved, as before, by multiplication; thus,

$$\begin{array}{l} 4+\ \sqrt{2} \\ 3-2\sqrt{2} \\ \hline 12+3\sqrt{2} \\ \quad -8\sqrt{2}-4 \\ \hline 12-5\sqrt{2}-4=8-5\sqrt{2}. \end{array}$$

331. In the same manner, we may transform irrational fractions into others, that have rational denominators. If we have, for example, the fraction $\frac{1}{5-2\sqrt{6}}$, and multiply its numerator and denominator by $5+2\sqrt{6}$, we transform it into this, $\frac{5+2\sqrt{6}}{1}=5+2\sqrt{6}$; in like manner, the fraction $\frac{2}{-1+\sqrt{-3}}$ assumes this form, $\frac{2+2\sqrt{-3}}{-4}=\frac{1+\sqrt{-3}}{-2}$; also $\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}-\sqrt{5}}=\frac{11+2\sqrt{30}}{1}=11+2\sqrt{30}$.

332. When the denominator contains several terms, we may, in the same manner, make the radical signs in it vanish one by one. Thus, if the fraction $\frac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$ be proposed, we first multiply these two terms by $\sqrt{10}+\sqrt{2}+\sqrt{3}$, and obtain the fraction $\frac{\sqrt{10}+\sqrt{2}+\sqrt{3}}{5-2\sqrt{6}}$; then multiplying its numerator and denominator by $5+2\sqrt{6}$, we have $5\sqrt{10}+11\sqrt{2}+9\sqrt{3}+2\sqrt{60}$.

CHAP. IX.

Of Cubes, *and of the* Extraction *of* Cube Roots.

333. To find the cube of $a+b$, we have only to multiply its square, $a^2+2ab+b^2$, again by $a+b$, thus;

$$\begin{array}{l} a^2+2ab+b^2 \\ a\ +b \\ \hline a^3+2a^2b+ab^2 \\ \qquad a^2b+2ab^2+b^3 \\ \hline \end{array}$$

and the cube will be $a^3+3a^2b+3ab^2+b^3$.