which result is accurate, and confirms the rule. But if we had supposed $a = 1$, $b = 1$, and $c = -1$, we should have found for the cube of $1 + 1 - 1$, that is of 1,

$1 + 3 - 3 + 3 - 6 + 3 + 1 - 3 + 3 - 1 = 1$, which is a still farther confirmation of the rule.

CHAP. XII.

Of the Expression *of* Irrational Powers *by* Infinite Series.

361. As we have shewn the method of finding any power of the root $a + b$, however great the exponent may be, we are able to express, generally, the power of $a + b$, whose exponent is undetermined; for it is evident that if we represent that exponent by n, we shall have by the rule already given (Art. 348 and the following):

$$(a + b)^n = a^n + \frac{n}{1} a^{-1}b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2}b^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 + \&c.$$

362. If the same power of the root $a - b$ were required, we need only change the signs of the second, fourth, sixth, &c. terms, and should have

$$(a - b)^n = a^n - \frac{n}{1} a^{n-1}b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2}b^2 - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 - \&c.$$

363. These formulas are remarkably useful, since they serve also to express all kinds of radicals; for we have shewn that all irrational quantities may assume the form of powers whose exponents are fractional, and that $\sqrt[2]{a} = a^{\frac{1}{2}}$, $\sqrt[3]{a} = a^{\frac{1}{3}}$, and $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c.: we have, therefore,

$$\sqrt[2]{(a + b)} = (a + b)^{\frac{1}{2}};\ \sqrt[3]{(a + b)} = (a + b)^{\frac{1}{3}};$$
$$\text{and } \sqrt[4]{(a + b)} = (a + b)^{\frac{1}{4}}, \&c.$$

Consequently, if we wish to find the square root of $a + b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula, Art. 361, and we shall have first, for the coefficients,

$\frac{n}{1} = \frac{1}{2}$; $\frac{n-1}{2} = -\frac{1}{4}$; $\frac{n-2}{3} = -\frac{3}{6}$; $\frac{n-3}{4} = -\frac{5}{8}$; $\frac{n-4}{5} = -\frac{7}{10}$; $\frac{n-5}{6} = -\frac{9}{12}$. Then, $a^n = a^{\frac{1}{2}} = \sqrt{a}$ and $a^{n-1} = \frac{1}{\sqrt{a}}$; $a^{n-2} = \frac{1}{a\sqrt{a}}$; $a^{n-3} = \frac{1}{a^2\sqrt{a}}$, &c. or we might express those powers of a in the following manner: $a^n = \sqrt{a}$; $a^{n-1} = \frac{\sqrt{a}}{a}$; $a^{n-2} = \frac{a^n}{a^2} = \frac{\sqrt{a}}{a^2}$; $a^{n-3} = \frac{a^n}{a^3} = \frac{\sqrt{a}}{a^3}$; $a^{n-4} = \frac{a^n}{a^4} = \frac{\sqrt{a}}{a^4}$, &c.

364. This being laid down, the square root of $a + b$ may be expressed in the following manner:

$\sqrt{(a+b)} = \sqrt{a} + \frac{1}{2}b\frac{\sqrt{a}}{a} - \frac{1}{2}\cdot\frac{1}{4}b^2\frac{\sqrt{a}}{aa} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}b^3\frac{\sqrt{a}}{a^3} - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}b^4\frac{\sqrt{a}}{a^4}$, &c.

365. If a therefore be a square number, we may assign the value of $\sqrt{a}$, and, consequently, the square root of $a + b$ may be expressed by an infinite series, without any radical sign.

Let, for example, $a = c^2$, we shall have $\sqrt{a} = c$; then $\sqrt{(c^2 + b)} = c + \frac{1}{2}\cdot\frac{b}{c} - \frac{1}{8}\cdot\frac{b^2}{c^3} + \frac{1}{16}\cdot\frac{b^3}{c^5} - \frac{5}{128}\cdot\frac{b^4}{c^7}$, &c.

We see, therefore, that there is no number, whose square root we may not extract in this manner; since every number may be resolved into two parts, one of which is a square represented by c^2. If, for example, the square root of 6 be required, we make $6 = 4 + 2$, consequently, $c^2 = 4$, $c = 2$, $b = 2$, whence results

$$\sqrt{6} = 2 + \tfrac{1}{2} - \tfrac{1}{16} + \tfrac{1}{64} - \tfrac{5}{1024}, \text{ \&c.}$$

If we take only the two leading terms of this series, we shall have $2\frac{1}{2} = \frac{5}{2}$, the square of which, $\frac{25}{4}$, is $\frac{1}{4}$ greater than 6; but if we consider three terms, we have $2\frac{7}{16} = \frac{39}{16}$, the square of which, $\frac{1521}{256}$, is still $\frac{15}{256}$ too small.

366. Since, in this example, $\frac{5}{2}$ approaches very nearly to the true value of $\sqrt{6}$, we shall take for 6 the equivalent quantity $\frac{25}{4} - \frac{1}{4}$; thus $c^2 = \frac{25}{4}$; $c = \frac{5}{2}$; $b = \frac{1}{4}$; and calculating only the two leading terms, we find $\sqrt{6} = \frac{5}{2} + \frac{1}{2}\cdot\frac{-\frac{1}{4}}{\frac{5}{2}} = \frac{5}{2} - \frac{1}{2}\cdot\frac{\frac{1}{4}}{\frac{5}{2}} = \frac{5}{2} - \frac{1}{20} = \frac{49}{20}$; the square of which

fraction being $\frac{2401}{400}$, it exceeds the square of $\sqrt{6}$ only by $\frac{1}{400}$.

Now, making $6 = \frac{2401}{400} - \frac{1}{400}$, so that $c = \frac{49}{20}$ and $b = -\frac{1}{400}$; and still taking only the two leading terms, we have $\sqrt{6} = \frac{49}{20} + \frac{1}{2} \cdot \frac{-\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{2} \cdot \frac{\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{1960}$ $= \frac{4801}{1960}$, the square of which is $\frac{23049601}{3841600}$; and 6, when reduced to the same denominator, is $= \frac{23049600}{3841600}$; the error therefore is only $\frac{1}{3841600}$.

367. In the same manner, we may express the cube root of $a+b$ by an infinite series; for since $\sqrt[3]{(a+b)} = (a+b)^{\frac{1}{3}}$, we shall have in the general formula, $n = \frac{1}{3}$, and for the coefficients, $\frac{n}{1} = \frac{1}{3}$; $\frac{n-1}{2} = -\frac{1}{3}$; $\frac{n-2}{3} = -\frac{5}{9}$; $\frac{n-3}{4} = -\frac{2}{3}$; $\frac{n-4}{5} = -\frac{11}{15}$, &c. and, with regard to the powers of a, we shall have $a^n = \sqrt[3]{a}$; $a^{n-1} = \frac{\sqrt[3]{a}}{a}$; $a^{n-2} = \frac{\sqrt[3]{a}}{a^2}$; $a^{n-3} = \frac{\sqrt[3]{a}}{a^3}$, &c. then

$$\sqrt[3]{(a+b)} = \sqrt[3]{a} + \tfrac{1}{3} \cdot b\frac{\sqrt[3]{a}}{a} - \tfrac{1}{9} \cdot b^2\frac{\sqrt[3]{a}}{a^2} + \tfrac{5}{81} \cdot b^3\frac{\sqrt[3]{a}}{a^3} - \tfrac{10}{243} \cdot b^4\frac{\sqrt[3]{a}}{a^4}, \text{ \&c.}$$

368. If a therefore be a cube, or $a = c^3$, we have $\sqrt[3]{a} = c$, and the radical signs will vanish; for we shall have

$$\sqrt[3]{(c^3+b)} = c + \tfrac{1}{3} \cdot \frac{b}{c^2} - \tfrac{1}{9} \cdot \frac{b^2}{c^5} + \tfrac{5}{81} \cdot \frac{b^3}{c^8} - \tfrac{10}{243} \cdot \frac{b^4}{c^{11}} + \text{, \&c.}$$

369. We have therefore arrived at a formula, which will enable us to find, *by approximation*, the cube root of any number; since every number may be resolved into two parts, as $c^3 + b$, the first of which is a cube.

If we wish, for example, to determine the cube root of 2, we represent 2 by $1 + 1$, so that $c = 1$ and $b = 1$; consequently, $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81}$, &c. The two leading terms of this series make $1\frac{1}{3} = \frac{4}{3}$, the cube of which $\frac{64}{27}$ is too great by $\frac{10}{27}$: let us therefore make $2 = \frac{64}{27} - \frac{10}{27}$, we have $c = \frac{4}{3}$ and $b = -\frac{10}{27}$, and consequently $\sqrt[3]{2} = \frac{4}{3} + \frac{1}{3} \cdot \frac{-\frac{10}{27}}{\frac{16}{9}}$: these two terms give $\frac{4}{3} - \frac{5}{72} = \frac{91}{72}$, the cube of which is $\frac{753571}{373248}$: but, $2 = \frac{746496}{373248}$, so that the error is $\frac{7075}{373248}$; and in this way we might still approximate, the faster in proportion as we take a greater number of terms *.

* In the Philosophical Transactions for 1694, Dr. Halley has given a very elegant and general method for extracting roots of

CHAP. XIII.

Of the Resolution *of* Negative Powers.

370. We have already shewn, that $\frac{1}{a}$ may be expressed by a^{-1}; we may therefore express $\frac{1}{a+b}$ also by $(a+b)^{-1}$; so that the fraction $\frac{1}{a+b}$ may be considered as a power of $a+b$, namely, that power whose exponent is -1; from which it follows, that the series already found as the value of $(a+b)^n$ extends also to this case.

371. Since, therefore $\frac{1}{a+b}$ is the same as $(a+b)^{-1}$, let us suppose, in the general formula, [Art. 361.] $n=-1$; and we shall first have, for the coefficients, $\frac{n}{1}=-1$; $\frac{n-1}{2}=-1$; $\frac{n-2}{3}=-1$; $\frac{n-3}{4}=-1$, &c. And, for the powers of a, we have $a^n=a^{-1}=\frac{1}{a}$; $a^{n-1}=a^{-2}=\frac{1}{a^2}$; $a^{n-2}=\frac{1}{a^3}$; $a^{n-3}=\frac{1}{a^4}$ &c.: so that $(a+b)^{-1}=\frac{1}{a+b}$ $=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\frac{b^5}{a^6}$, &c. which is the same series that we found before by division.

372. Farther, $\frac{1}{(a+b)^2}$ being the same with $(a+b)^{-2}$, let

any degree whatever by approximation; where he demonstrates this general formula,

$$\sqrt[m]{(a^m \pm b)}=\frac{m-2}{m-1}a+\sqrt{\left(\frac{a^2}{(m-1)^2}\pm\frac{2b}{(m^2-m)a^{m-1}}\right)}.$$

Those who have not an opportunity of consulting the Philosophical Transactions, will find the formation and the use of this formula explained in the new edition of Leçons Elementaires de Mathematiques by M. D'Abbé de la Caille, published by M. L'Abbé Marie. F. T. See also Dr. Hutton's Math. Dictionary.