CHAP. X.

Of the higher Powers *of* Compound Quantities.

340. After squares and cubes, we must consider higher powers, or powers of a greater number of degrees; which are generally represented by exponents in the manner which we before explained: we have only to remember, when the root is compound, to enclose it in a parenthesis: thus, $(a + b)^5$ means that $a + b$ is raised to the fifth power, and $(a - b)^6$ represents the sixth power of $a - b$, and so on. We shall in this chapter explain the nature of these powers.

341. Let $a + b$ be the root, or the first power, and the higher powers will be found, by multiplication, in the following manner:

$$(a+b)^1 = a+b$$
$$a+b$$
$$\overline{a^2+ab}$$
$$ab+b^2$$
$$\overline{(a+b)^2 = a^2+2ab+b^2}$$
$$a+b$$
$$\overline{a^3+2a^2b+ab^2}$$
$$a^2b+2ab^2+b^3$$
$$\overline{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$$
$$a+b$$
$$\overline{a^4+3a^3b+3a^2b^2+ab^3}$$
$$a^3b+3a^2b^2+3ab^3+b^4$$
$$\overline{(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4}$$
$$a+b$$
$$\overline{a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4}$$
$$a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$$
$$\overline{a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5}$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$a + b$$
$$a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + a^5b$$
$$a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6, \text{ \&c.}$$

342. The powers of the root $a - b$ are found in the same manner; and we shall immediately perceive that they do not differ from the preceding, excepting that the 2d, 4th, 6th, &c. terms are affected by the sign *minus*.

$$(a-b)^1 = a - b$$
$$a - b$$
$$a^2 - ab$$
$$-ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a - b$$
$$a^3 - 2a^2b + ab^2$$
$$- a^2b + 2ab^2 - b^3$$
$$(a-b)^3 = a^3 + 3a^2b + 3ab^2 - b^3$$
$$a - b$$
$$a^4 - 3a^3b + 3a^2b^2 - ab^3$$
$$- a^3b + 3a^2b^2 - 3ab^3 + b^4$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$
$$a - b$$
$$a^5 - 4a^4b + 6a^3b^2 - 4a^2b^3 + ab^4$$
$$- a^4b + 4a^3b^2 - 6a^2b^3 + 4ab^4 - b^5$$
$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$$
$$a - b$$
$$a^6 - 5a^5b + 10a^4b^2 - 10a^3b^3 + 5a^2b^4 - ab^5$$
$$- a^5b + 5a^4b^2 - 10a^3b^3 + 10a^2b^4 - 5ab^5 + b^6$$
$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6, \text{ \&c.}$$

Here we see that all the odd powers of b have the sign $-$, while the even powers retain the sign $+$. The reason of this is evident; for since $-b$ is a term of the root, the powers of that letter will ascend in the following series, $-b$, $+b^2$, $-b^3$, $+b^4$, $-b^5$, $+b^6$, &c. which clearly shews that the even powers must be affected by the sign $+$, and the odd ones by the contrary sign $-$.

343. An important question occurs in this place; namely, how we may find, without being obliged to perform the same calculation, all the powers either of $a+b$, or $a-b$.

We must remark, in the first place, that if we can assign all the powers of $a+b$, those of $a-b$ are also found; since we have only to change the signs of the even terms, that is to say, of the second, the fourth, the sixth, &c. The business then is to establish a rule, by which any power of $a+b$, however high, may be determined without the necessity of calculating all the preceding powers.

344. Now, if from the powers which we have already determined we take away the numbers that precede each term, which are called the *coefficients*, we observe in all the terms a singular order: first, we see the first term a of the root raised to the power which is required; in the following terms, the powers of a diminish continually by unity, and the powers of b increase in the same proportion; so that the sum of the exponents of a and of b is always the same, and always equal to the exponent of the power required; and, lastly, we find the term b by itself raised to the same power. If therefore the tenth power of $a+b$ were required, we are certain that the terms, without their coefficients, would succeed each other in the following order; a^{10}, a^9b, a^8b^2, a^7b^3, a^6b^4, a^5b^5, a^4b^6, a^3b^7, a^2b^8, ab^9, b^{10}.

345. It remains therefore to shew how we are to determine the coefficients, which belong to those terms, or the numbers by which they are to be multiplied. Now, with respect to the first term, its coefficient is always unity; and, as to the second, its coefficient is constantly the exponent of the power. With regard to the other terms, it is not so easy to observe any order in their coefficients; but, if we continue those coefficients, we shall not fail to discover the law by which they are formed; as will appear from the following Table.

Powers.	Coefficients.
1st	1, 1
2d	1, 2, 1
3d	1, 3, 3, 1
4th	1, 4, 6, 4, 1
5th	1, 5, 10, 10, 5, 1
6th	1, 6, 15, 20, 15, 6, 1
7th	1, 7, 21, 35, 35, 21, 7, 1
8th	1, 8, 28, 56, 70, 56, 28, 8, 1
9th	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
10th	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, &c.

We see then that the tenth power of $a + b$ will be $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$.

346. Now, with regard to the coefficients, it must be observed, that for each power their sum must be equal to the number 2 raised to the same power; for let $a = 1$ and $b = 1$, then each term, without the coefficients, will be 1; consequently, the value of the power will be simply the sum of the coefficients. This sum, in the preceding example, is 1024, and accordingly $(1 + 1)^{10} = 2^{10} = 1024$. It is the same with respect to all other powers; thus, we have for the

1st $1+1=2=2^1$,
2d $1+2+1=4=2^2$,
3d $1+3+3+1=8=2^3$,
4th $1+4+6+4+1=16=2^4$,
5th $1+5+10+10+5+1=32=2^5$,
6th $1+6+15+20+15+6+1=64=2^6$,
7th $1+7+21+35+35+21+7+1=128=2^7$, &c.

347. Another necessary remark, with regard to the coefficients, is, that they increase from the beginning to the middle, and then decrease in the same order. In the even powers, the greatest coefficient is exactly in the middle; but in the odd powers, two coefficients, equal and greater than the others, are found in the middle, belonging to the mean terms.

The order of the coefficients likewise deserves particular attention; for it is in this order that we discover the means of determining them for any power whatever, without calculating all the preceding powers. We shall here explain this method, reserving the demonstration however for the next chapter.

348. In order to find the coefficients of any power proposed, the seventh for example, let us write the following fractions one after the other:

$$\frac{7}{1}, \frac{6}{2}, \frac{5}{3}, \frac{4}{4}, \frac{3}{5}, \frac{2}{6}, \frac{1}{7}.$$

In this arrangement, we perceive that the numerators begin by the exponent of the power required, and that they diminish successively by unity; while the denominators follow in the natural order of the numbers, 1, 2, 3, 4, &c. Now, the first coefficient being always 1, the first fraction gives the second coefficient; the product of the first two fractions, multiplied together, represents the third coefficient; the product of the three first fractions represents the fourth coefficient, and so on. Thus, the

$$\begin{array}{lll}
\text{1st coefficient is } 1 & & = 1 \\
\text{2d - - - -} & \dfrac{7}{1} & = 7 \\
\text{3d - - - -} & \dfrac{7 \cdot 6}{1 \cdot 2} & = 21 \\
\text{4th - - - -} & \dfrac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} & = 35 \\
\text{5th - - - -} & \dfrac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} & = 35 \\
\text{6th - - - -} & \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} & = 21 \\
\text{7th - - - -} & \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} & = 7 \\
\text{8th - - - -} & \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} & = 1
\end{array}$$

349. So that we have, for the second power, the fractions $\frac{2}{1}$, $\frac{1}{2}$; whence the first coefficient is 1, the second $\frac{2}{1} = 2$, and the third $2 \times \frac{1}{2} = 1$.

The third power furnishes the fractions $\frac{3}{1}$, $\frac{2}{2}$, $\frac{1}{3}$; wherefore the

1st coefficient $= 1$; 2d $= \frac{3}{1} = 3$;
3d $= 3 \cdot \frac{2}{2} = 3$; and 4th $= \frac{3}{1} \cdot \frac{2}{2} \cdot \frac{1}{3} = 1$.

We have, for the fourth power, the fractions $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, consequently, the

1st coefficient $= 1$;
2d $\frac{4}{1} = 4$; 3d $\frac{4}{1} \cdot \frac{3}{2} = 6$;
4th $\frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3} = 4$; and 5th $\frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = 1$.

350. This rule evidently renders it unnecessary to find the coefficients of the preceding powers, as it enables us to discover immediately the coefficients which belong to any one proposed. Thus, for the tenth power, we write the fractions $\frac{10}{1}$, $\frac{9}{2}$, $\frac{8}{3}$, $\frac{7}{4}$, $\frac{6}{5}$, $\frac{5}{6}$, $\frac{4}{7}$, $\frac{3}{8}$, $\frac{2}{9}$, $\frac{1}{10}$, by means of which we find the

1st coefficient $= 1$;

2d $= \frac{10}{1} = 10$; 7th $= 252 . \frac{5}{6} = 210$;
3d $= 10 . \frac{9}{2} = 45$; 8th $= 210 . \frac{4}{7} = 120$;
4th $= 45 . \frac{8}{3} = 120$; 9th $= 120 . \frac{3}{8} = 45$;
5th $= 120 . \frac{7}{4} = 210$; 10th $= 45 . \frac{2}{9} = 10$;
6th $= 210 . \frac{6}{5} = 252$; and 11th $= 10 . \frac{1}{10} = 1$.

351. We may also write these fractions as they are, without computing their value; and in this manner it is easy to express any power of $a + b$. Thus, $(a + b)^{100} = a^{100} + \frac{100}{1} . a^{99}b + \frac{100 . 99}{1 . 2} + a^{98}b^2 + \frac{100 . 99 . 98}{1 . 2 . 3} a^{97}b^3 + \frac{100 . 99 . 98 . 97}{1 . 2 . 3 . 4} a^{96}b^4 +$, &c.* Whence the law of the succeeding terms may be easily deduced.

CHAP. XI.

Of the Transposition *of the* Letters, *on which the demonstration of the preceding* Rule *is founded.*

352. If we trace back the origin of the coefficients which we have been considering, we shall find, that each term is presented, as many times as it is possible to transpose the letters, of which that term is composed; or, to express the same thing differently, the coefficient of each term is equal to the number of transpositions which the letters composing that term admit of. In the second power, for example, the term ab is taken twice, that is to say, its coefficient is 2; and in fact we may change the order of the letters which compose that term twice, since we may write ab and ba.

* Or, which is a more general mode of expression,

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n . (n-1)}{1 . 2} a^{n-2} b^2 + \frac{n . (n-1) . (n-2)}{1 . 2 . 3} a^{n-3} b^3 + \frac{n . (n-1) . (n-2) . (n-3)}{1 . 2 . 3 . 4} a^{n-4} b^4 \text{ \&c. - - } \frac{n . (n-1) . (n-2) . (n-3) \text{ - - - - } 1}{1 . 2 . 3 . 4 \text{ - - - - - } n}.$$

This elegant theorem for the involution of a compound quantity of two terms, evidently includes all powers whatever; and we shall afterwards shew how the same may be applied to the extraction of roots.